The homework is due on Thursday, February 12. Each questions is worth 0.8 points. No partial credits.

For the graphic arguments, use the graphing paper that is attached. For the computer exercises, you need to attach a printout of each Excel worksheet.

**Question 1** Solve the following optimization problem graphically.

\[
\begin{align*}
\text{max } x_1 + x_2 \text{ subject to } \\
& (i) \quad 3x_1 + 4x_2 \leq 120 \\
& (ii) \quad 3x_1 + x_2 \leq 75 \\
& (iii) \quad x_1 \geq 0 \\
& (iv) \quad x_2 \geq 0.
\end{align*}
\]

**Question 2** Suppose an oil company has supplies of four crude products. In the refinery the crude products can be used to make two refined products \(x_1\) and \(x_2\), which the company can sell at prices 3 and 6, respectively. In order to produce \(x_1\) units of the first refined product one needs 1 unit of crude product 1, 3 of product 2, 4 of product 3, and 0.5 of product 4. In order to produce \(x_2\) units of the second refined product one needs 1 unit of crude product 1, 3 units of product 2, 1 of product 3, and 2 of product 4. The company has a fixed supply of the crude products. In particular supplies of the crude products are 400, 450, 480, and 150 units, respectively. The company wants to maximize the total revenue from selling the product. As a consequence, the company solves the following optimization problem.

\[
\begin{align*}
\text{max } 3x_1 + 6x_2 \text{ subject to } \\
& (i) \quad x_1 + x_2 \leq 200 \\
& (ii) \quad 3x_1 + 3x_2 \leq 450 \\
& (iii) \quad 4x_1 + x_2 \leq 480 \\
& (iv) \quad 0.5x_1 + 2x_2 \leq 150. \\
& (v) \quad x_1 \geq 0 \\
& (vi) \quad x_2 \geq 0.
\end{align*}
\]

Determine the optimum graphically. Note: When you graph the lines representing (i)–(iv) then one of the lines will be strictly to the right of the feasible set, i.e., the boundaries of the feasible set are determined by only three of the lines in addition to the conditions that \(x_1, x_2 \geq 0\).
**Question 3** A firm wishes to produce 10 units of a product at the lowest possible cost. Two inputs are needed. The costs of the inputs are 4 and 6 Dollars, respectively. In order to produce 10 units of output, the inputs must fulfill $5x_1 + 4x_2 \geq 300$. Thus, the firm solves:

$$\begin{align*}
\min_{x_1, x_2} & \quad 4x_1 + 6x_2 \\
\text{subject to} & \quad 5x_1 + 4x_2 \geq 300 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0.
\end{align*}$$

Determine the optimal choice of $x_1$ and $x_2$ graphically. *Note:* This is a minimization problem, i.e., moving down and to the left and your graph decreases costs.

**Question 4** Now assume that there are six different crude oil products and 4 different refined products. The optimization problem is given by

$$\begin{align*}
\max_{x_1, x_2, x_3, x_4} & \quad 3x_1 + 6x_2 + 2x_3 + 5x_4 \\
\text{subject to} & \quad x_1 + x_2 + 2x_3 + x_4 \leq 500 \\
& \quad 3x_1 + 8x_2 + 0.5x_3 + 3x_4 \leq 600 \\
& \quad 2x_1 + 2x_2 + 3x_3 + 8x_4 \leq 700 \\
& \quad x_1 + 2x_2 + 3x_3 + x_4 \leq 420 \\
& \quad 3x_1 + x_2 + 3.5x_3 + 2x_4 \leq 630 \\
& \quad 8x_1 + 2x_2 + 2x_3 + 3x_4 \leq 500 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0. \\
& \quad x_3 \geq 0. \\
& \quad x_4 \geq 0.
\end{align*}$$

Using Excel, determine the optimal amount of the refined product that the company should produce. Also determine which constraints bind and which constraints are slack.
Question 5 This question considers a transportation problem that is similar to that in problem 2 of our lab class. All we changed are some of the numbers and we added a fifth retail store. The transportation problem is graphed above. Again, suppose that the transportation costs are 10 cents per kilometer and let $x_{i,j}$ denote the quantity of the product shipped from store $i$ to store $j$. Thus, the company solves

$$\min 18x_{1,1} + 14x_{1,2} + 10x_{1,3} + 16x_{1,4} + 22x_{1,5} + 20x_{2,1} + 9x_{2,2} + 11x_{2,3} + 2x_{2,4} + 8x_{2,5}$$

subject to constraints. Determine the constraints and solve for the optimal values of $x_{i,j}$ by using Excel. Also determine which constraints bind and which are slack.

Question 6 A consumer has a utility function $u(x_1, x_2) = x_1^3x_2$. The prices are $p_1 = 2$, $p_2 = 4$ and the person’s wealth is 90. Thus, the consumer solves

$$\max x_1^3x_2 \text{ subject to}$$

(i) $2x_1 + 4x_2 \leq 90$

(ii) $x_1 \geq 0$

(iii) $x_2 \geq 0$.

(a) Determine the optimal consumption, i.e., the optimal values of $x_1$ and $x_2$ by using Excel. Note: This is a nonlinear optimization problem!

(b) Determine the percentage of wealth that the person spends on each good. In particular, to determine this percentage you compute

$$100\frac{2x_1}{90}, \text{ and } 100\frac{4x_2}{90}.$$
(c) Now suppose that the wealth increases to 140. This changes constraint (i) to 
\[2x_1 + 4x_2 \leq 140.\] Determine again the optimal values of \(x_1\) and \(x_2\) and the 
the percentages of income the person spends on each good. Because wealth 
is now 140, these percentages are given by 
\[\frac{2x_1}{140}\text{ and }\frac{4x_2}{140}.\]

(d) Now suppose that the wealth is again 140, but that the price of good 1 increases 
to 4. This changes constraint (i) to \(4x_1 + 4x_2 \leq 140\). Determine again the 
the optimal values of \(x_1\) and \(x_2\) and the the percentages of income the person 
spends on each good. Now the percentages are given by 
\[\frac{4x_1}{140}\text{ and }\frac{4x_2}{140}.\]

(e) Any guess what the percentage will be if you choose for example the budget 
line \(8x_1 + 2x_2 = 200\)?

**Question 7** A consumer has a utility function 
\[u(x_1, x_2) = 20x_1^{0.5} + x_2.\] The prices are 
\(p_1 = 2, p_2 = 4\). If the person’s wealth is \(m\) then the person solves 
\[
\max_{x_1, x_2} \quad 20x_1^{0.5} + x_2 \\
\text{subject to} \\
\quad (i) \ 2x_1 + 4x_2 \leq m \\
\quad (ii) \ x_1 \geq 0 \\
\quad (iii) \ x_2 \geq 0.
\]

(a) Determine the optimal consumption, i.e., the optimal values of \(x_1\) and \(x_2\) by 
using Excel if \(m = 100\). Does the person consume both goods?

(b) Now suppose wealth is \(m = 1,000\). Does the person consume both goods 
now?

(c) By inserting different values for \(m\) and optimizing, try to find a value of \(m\) at 
which the person will just start to consume both goods.