Economics 478  
Topics in Survival Analysis  

Lecture 1  
Foundations for Counting Process Formulation of Survival Analysis  

1. Instant Measure Theory  

Our first question, which is not strictly necessary, or relevant, but is a useful bit of general education is: What is the difference between Riemann and Lebesgue integral? A straightforward answer may be provided with the aid of a couple of pictures. Riemann integral works by dividing up the domain and approximating via mean value theorem like this 

\[ RS_m = \sum_{i=1}^{m} f(x^*_m)(x_{mi} - x_{mi-1}). \]

This works well for nonnegative, continuous functions, but fails in the sense that the approximation may fail to converge, i.e., there exist sequences \( f_n \rightarrow f \) such that 

\[ \int_{a}^{b} f_n(x)dx \rightarrow \int_{a}^{b} f(x)dx \]

fails.

The Lebesgue integral proceeds by subdividing the range rather than the domain, 

\[ LS_m = \sum_{k=1}^{m^{2n}} \frac{k-1}{2^n} \times \mu(\{x: \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}) \]

so rather than having fixed-width intervals of the domain and multiplying by heights as in elementary calculus, we have fixed width division of the range and we weight by the width, or length, or measure of the set of \( x \) such that \( f(x) \) lies in these intervals.

So the Lebesgue approximation is the area below the red curve. To see this start with the top block we get all area below this block. Then we get the two neighboring strips, etc. etc.

Note that we can perturb \( f \) at a few isolated points and this doesn’t affect the Lebesgue sum, since these heights are multiplied by zero.
Set Theory in Brief

When about brothers?

This leads us to the concept of 0-measures.

If we can assume outer measures that these sets are measurable, then we are set.

\[ \left\{ \frac{\text{me}}{\gamma} > (x) \right\} \geq \frac{\text{me}}{1 - \gamma} : x \}

measure of the sets

work better. For our function, \( f \), we need to be able to find the length.

So why does this work better than the Riemann sum? Oh when does it

piece 1. Riemann approximation divides the domain into

\( x \)

\( f(x) \)
Remark $\text{o-fields are also closed under intersections since } V \cap B = B \cap V$.

so $\forall i \in \mathbb{N} \cup \{0\}$ and $\forall V \subseteq \mathbb{R}$

Def. $A$ is a $\gamma$-field if it is closed under complements and countable unions.

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Def. $A$ is a $\gamma$-field if it is closed under complements and countable unions.

$A, B \subseteq V \implies A \cap B \subseteq V$.

\[ x \]

\[ f(x) \]

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
0.00 & 0.00 \\
0.02 & 0.00 \\
0.04 & 0.00 \\
0.06 & 0.00 \\
0.08 & 0.00 \\
0.10 & 0.00 \\
0.12 & 0.00 \\
\hline
\end{array}
\]
Def. If \( \mathcal{A} \) is a \( \sigma \)-field and \( \mu : \mathcal{A} \to [0, 1] \) is a set function that is countably additive in the sense that for disjoint \( A_i \)

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i),
\]

then \( \mu \) is called a measure, or a countably additive measure on \((\Omega, \mathcal{A})\).

Examples:

1) Lebesgue: length of the set \( A \)

2) Counting: \( \#(A) \) cardinality of \( A \), i.e. number of elements of \( A \)

3) Indicator: \( \delta_{\omega_0}(A) = I_{\omega_0}(A) = \begin{cases} 
1 & \text{if } \omega_0 \in A \\
0 & \text{otherwise.}
\end{cases} \)

Borel Sets

It is convenient to have a minimal \( \sigma \)-field containing a specified class of sets, \( \xi \). We say \( \xi \) is the generator of a \( \sigma \)-field and write,

\[
\sigma[\xi] \equiv \bigcap \{ \mathcal{F}_a : \mathcal{F}_a \text{ is a } \sigma \text{-field of subsets of } \Omega \text{ for which } \xi \subset \mathcal{F}_a \}
\]

Since it is the intersection it has to be minimal.

Suppose \( \Omega = \mathbb{R} \), and \( \xi \) consists of all finite disjoint unions of intervals of the form

\[ (a, b], (-\infty, b], \text{ and } (a, +\infty) \]

\( \xi \) is a field, but not a \( \sigma \)-field since we can’t get \( (a, b) \) by finite unions but we can by countable unions: take \( (a, b_n] \) with \( b_n \nearrow b \). Then,

\[ B = \sigma[\xi] \]

is called the Borel subsets of \( \mathbb{R} \). And \( \mu(A) \) is the countably additive measure assigning the length of the intervals composing \( A \).

This can be extended to general metric spaces. Let \((\Omega, d)\) be a metric space and

\[ U = \{ \text{ all } d\text{-open supsets of } \Omega \} \]

then \( B = \sigma[U] \) are the Borel sets of \((\Omega, d)\) or the Borel \( \sigma \)-field of \((\Omega, d)\).

Expectations and the Lebesgue Integral

Let \((\Omega, \mathcal{A}, \mu)\) be a probability space and suppose that \( X, X_1, X_2, \ldots \) are measurable functions from \((\Omega, \mathcal{A}, \mu)\) to \((\mathbb{R}, \mathcal{B})\). If the sets \( \{A_i : i = 1, \ldots, n\} \) are disjoint, it is convenient to write

\[
\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i
\]

and if, further, \( \sum A_i = \Omega \) we say that the \( A_i \) are a partition of \( \Omega \).
We may define the Lebesgue integral in a sequence of steps beginning with
the simplest cases and building up from these.

(1) If $X = \sum_{i=1}^{n} x_iI_{A_i}$, we call it a simple (block) function provided
$x_i \geq 0$ and the $A_i$’s constitute a partition of $\Omega$. Then

$$\int X d\mu \equiv \sum x_i \mu(A_i).$$

This is the simplest version of our original definition with only a
finite number of possible values for $X$.

(2) If $X \geq 0$, then

$$\int X d\mu = \sup \{ \int Y d\mu : Y \text{ is a simple function such that } 0 \leq Y \leq X \}$$

(3) For general measurable $X$,

$$\int X d\mu = \int X^+ d\mu - \int X^- d\mu$$

provided either $\int X^+ d\mu$ or $\int X^- d\mu$ is finite.

(4) For unmeasurable $X$, if $X$ equals a measurable function $Y$ on a set
$A$ such that $\mu(A^c) = 0$, has zero measure, then

$$\int X d\mu = \int Y d\mu$$

Properties

(1) Using a simple functions it is easy to show that

(a) $\int (X + Y) d\mu = \int X d\mu + \int Y d\mu$

(b) $\int cX d\mu = c \int X d\mu$

(c) $X \geq 0 \Rightarrow \int X d\mu \geq 0$.

(2) (Monotone Convergence Theorem) Suppose $X_n \nearrow X$ a.e. for mea-
surable functions $X_n \geq 0$, then

$$0 \leq \int X_n d\mu \nearrow \int X d\mu$$

(3) (Fatou’s Lemma) For measurable $X_n \geq 0$ a.e.

$$\int \liminf X_n d\mu \leq \liminf \int X_n d\mu$$

(4) (Dominated Convergence Theorem) Suppose $|X_n| \leq Y$ a.e. for some
$Y$ such that $\int |Y| d\mu < \infty$. And assume either (i) $X_n \to X$ a.e., or
(ii) $X_n \to \mu X$. Then

$$\int |X_n - X| d\mu \to 0 \quad \text{as } n \to \infty$$

Remark. Note that (4) implies that

$$\int X_n d\mu \to \int X d\mu$$
and that
\[
\sup_{A \in \mathcal{A}} \left| \int_A X_n d\mu - \int_A X d\mu \right| \to 0.
\]
This follows from the observation that
\[
\left| \int X_n - \int X \right| \leq \int |X_n - X|
\]
and thus uniformly for \( A \in \mathcal{A} \),
\[
\left| \int_A X_n - \int_A X \right| \leq \int_A |X_n - X| \leq \int |X_n - X| \to 0.
\]