Question 1 The differential equation is
\[
\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{\sqrt{e(p_1, p_2, u)p_2}}{\sqrt{p_1(p_1 + p_2)}}.
\]

Question 2 Since \(u_0 < u^1\) we have \(e(p, u^0) < e(p, u^1)\) for every \(p\). Thus, because good 1 is inferior we get \(x_1(p_1, \tilde{p} - 1, e(p_1, \tilde{p} - 1, u^0)) > x_1(p_1, \tilde{p} - 1, e(p_1, \tilde{p} - 1, u^1))\) for every \(p_1\). This implies \(h_1(p_1, \tilde{p} - 1, u^0)) > h_1(p_1, \tilde{p} - 1, u^1))\) for every \(p_1\). Next, note that
\[
EV = \int_{p_1^0}^{p_1^1} h_1(p_1, \tilde{p} - 1, u^1) \, dp_1, \quad CV = \int_{p_1^0}^{p_1^1} h_1(p_1, \tilde{p} - 1, u^0) \, dp_1.
\]
Thus, \(CV > EV\).

Question 3

In the graph, the aggregate budget lines are blue, and the aggregate demand is indicated by the red circles. It follows immediately that the weak Axiom is violated. Hence, aggregate demand cannot be rationalized by preferences.
Question 4 Suppose that $Y$ satisfies free disposal (i.e., $y \in Y$ and $y' \leq y$ implies $y' \in Y$).

(a) $Y$ has constant returns to scale if and only if $y \in Y$ implies $\alpha y \in Y$ for any $\alpha \geq 0$. Let $z$ be arbitrary. Then $(-z, f(z)) \in Y$. Thus, $(-\alpha z, \alpha f(z)) \in Y$, which implies $\alpha f(z) \leq f(\alpha z)$ for any $\alpha \geq 0$. $\alpha f(z) \geq f(\alpha z)$ for any $\alpha \geq 0$.

We now show that Let $z' = \alpha z$ and $\alpha' = 1/\alpha$. Then $\alpha f(z) \leq f(\alpha z)$ implies $\alpha' f(\beta z') \leq f(z')$. Thus, $f(\alpha' z') \leq \alpha' f(z')$. Because, $\alpha \geq 0$ was arbitrary. Thus, $f(\alpha z) = f(z)$.

To prove the reverse, let $f$ be homogeneous of degree 1. Let $y \in Y$. Then $y \leq (y_{L-1}, f(-y_{L-1}))$. Thus, $\alpha y \leq (\alpha y_{L-1}, \alpha f(-y_{L-1})) = (\alpha y_{L-1}, f(-\alpha y_{L-1}))$. Because $Y$ satisfies free disposal, $\alpha z \in Y$.

(b) Let $y \in Y$ and $0 \leq \alpha < 1$. Then $y \leq (y_{L-1}, f(-y_{L-1}))$. Concavity implies

$$f(\alpha(-y_{L-1})) = f(\alpha(-y_{L-1}) + (1 - \alpha)0) \geq \alpha f(-y_{L-1}) + (1 - \alpha) f(0).$$

Because $0 \in Y$, it follows that $f(0) \geq 0$. Thus,

$$f(\alpha(-y_{L-1})) \geq \alpha f(-y_{L-1}).$$

$\alpha y \leq (\alpha y_{L-1}, \alpha f(-y_{L-1})) \leq (\alpha y_{L-1}, f(-\alpha y_{L-1}))$. Because $Y$ satisfies free disposal, $\alpha z \in Y$.

Question 5 Proof: Assume by way of contradiction that the costs of producing $q$ units of output are minimized when $q_1 > 0$ units are produced with the first technique, and $q_2 > 0$ with the second technique, where $q = q_1 + q_2$. Total production costs are therefore $w_1\phi(q_1) + w_2\phi(q_2)$. Then concavity implies

$$\phi_i(q_1) = \phi_i\left(\frac{q_i}{q_1 + q_2}q\right) = \phi_i\left(\frac{q_i}{q_1 + q_2}q + \frac{q_i}{q_1 + q_2}0\right) > \frac{q_i}{q_1 + q_1}\phi_i(q) + \frac{q_i}{q_1 + q_1}\phi_i(0) = \frac{q_i}{q_1 + q_1}\phi_i(q). \quad (1)$$

Without loss of generality we can assume that $w_1\phi_1(q) \leq w_2\phi_2(q)$. Thus, (1) implies

$$\frac{q_i}{q_1 + q_1}w_1\phi_1(q) < w_1\phi_1(q_1), \quad (2)$$

for $i = 1, 2$. Adding (2) for $i = 1, 2$, we get $w_1\phi_1(q) < w_1\phi_1(q_1) + w_2\phi_2(q_2)$, a contradiction to the assumption that $q_1, q_2$ minimize costs.

Question 6 A production function is given by $f(z_1, z_2) = \sqrt{2z_1 + z_2}$. The firm will choose always the relatively less expensive input. If the firm choose only input 1 then $z = q^2/2$ to get $q$ units of output. The firm’s costs are $q^2w_1/2$. Otherwise, if the firm selects only input 2, then $z = q^2$, and the firm’s cost are $q^2w_2$. The firm’s cost function is therefore $c(w, q) = q^2 \min \left\{ \frac{w_1}{2}, w_2 \right\}$. Thus, $MC = 2q \min \left\{ \frac{w_1}{2}, w_2 \right\}$. At the profit maximizing choice $p = MC = 2q \min \left\{ \frac{w_1}{2}, w_2 \right\}$. Thus, the firm’s supply is given by

$$y(w, p) = \frac{p}{2 \min \left\{ \frac{w_1}{2}, w_2 \right\}} = \frac{p}{\min \{w_1, 2w_2\}}.$$
Question 7 Note that

\[ \mathcal{L} = \max_{z_1, z_2} pf(z_1, z_2) + \lambda [pf(z_1, z_2) - w_1z_1 - w_2z_2 - \pi]. \]

The envelope theorem therefore implies

\[ \frac{\partial S(p, w, \pi)}{\partial p} = f(z_1, z_2) + \lambda f(z_1, z_2) = (1 + \lambda) f(z_1, z_2). \]  \hfill (3)

\[ \frac{\partial S(p, w, \pi)}{\partial w_i} = -\lambda \]  \hfill (4)

\[ \frac{\partial S(p, w, \pi)}{\partial \pi} = -\lambda \]  \hfill (5)

Thus, (3) and (5) imply

\[ y(p, w, \pi) = \frac{\partial S(p, w, \pi)}{\partial p} = \frac{\partial S(p, w, \pi)}{1 + \lambda} = \frac{\partial S(p, w, \pi)}{\partial \pi}. \]

Similarly, (4) and (5) imply

\[ z_i(p, w, \pi) = \frac{\partial S(p, w, \pi)}{\partial w_i} = \frac{\partial S(p, w, \pi)}{-\lambda} = \frac{\partial S(p, w, \pi)}{\partial \pi}. \]

Question 8 Expected utility is given by \( \alpha E[X] - E[X^2]. \) Recall that the variance is given by \( E[X^2] - E[X]^2. \)

For the first lottery \( E[X] = E[X^2] = 1. \) Thus, expected utility is \( \alpha - 1. \)

For the second lottery \( 0.5 = E[X^2] - 1.4^2 \) implies \( E[X^2] = 2.46. \) The expected utility is therefore \( -2.46 + 1.4\alpha. \) Thus, \( \alpha - 1 = -2.46 + 1.4\alpha, \) which implies \( \alpha = 3.65. \)

Question 9 Note that \( \tilde{u}'(x) = ku'(x) + v'(x) \) and \( \tilde{u}''(x) = ku''(x) + v''(x). \) Since \( v \) is nonincreasing, we get \( v' \leq 0. \) Since \( v \) is concave we get \( v'' \leq 0. \) Therefore,

\[ r_A(x, \tilde{u}) = -\frac{\tilde{u}''(x)}{u'(x)} = -\frac{ku''(x) + v''(x)}{ku'(x) + v'(x)} \geq -\frac{ku''(x)}{ku'(x)} = r_A(x, u). \]