Is Mandatory Voting Better than Voluntary Voting?

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Abstract

We investigate the welfare effects of policies that increase voter turnout in costly voting models. Generalizing the costly voting model of Börgers [3], we show that if the electorate is sufficiently large, then increasing voter turnout is generically efficient. Increasing turnout in small elections is only inefficient if the electorate is evenly divided or if there is already almost complete voter participation. Finally, we argue that the effects underlying our results are robust in a large class of endogenous participation models.

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1 Introduction

Many societies encourage voter participation in elections and meetings. In the recent 2004 U.S. elections, many states expanded the opportunities for early and absentee voting. Several other countries have tried to increase turnout by making participation in elections “mandatory”.\(^1\) For example, the Australian parliament enacted mandatory voting in 1924, because voter turnout had dropped below 60 percent. By law, all Australian citizens over the age of 18 must register to vote and show up at a polling place on election day. A citizen who misses the election is subject to a $15 fine.\(^2\) The Australian mandatory voting law is successful in increasing voter turnout above 90%.\(^3\) Supporters of mandatory voting see voting as a civic duty similar to paying taxes, and argue that a higher level of participation increases the legitimacy of government. “The most important [argument] is that compulsory voting ensures that government does indeed represent the will of the whole population, not merely the section of the population that decides to express their opinions” (Wikipedia [21]). Relatedly, political scientists and political commentators often deplore participation rates in US elections as “too low” and advocate steps to increase turnout (e.g. Olbermann [16], Dean [7] and Weiner [20]). Yet, beyond the claim that higher participation would confer a greater legitimacy on the elected government, there are no solid economic arguments why increased turnout should be desirable. Providing such an argument is the main contributions of this paper.

We use a costly voting model to investigate whether policies that increase voter turnout are socially beneficial. Specifically, we address the following questions: Does fining non-voters (or, equivalently, subsidizing voters) alter election outcomes, relative to voluntary voting? If election outcomes are affected, and if subsidized voting improves social decisions, do these benefits outweigh the increased voting costs that are a consequence of higher voter turnout?

In our model, which generalizes Börgers [3], \(N\) citizens must decide between two candidates \(A\) and \(B\). Each citizen’s preference is private information and independently drawn from a common distribution that assigns probability \(\alpha\) to being an \(A\)-supporter (and \(1 - \alpha\) to being a \(B\)-supporter). In addition, each individual has private information about his voting cost \(c\). The parameter \(\alpha\) is drawn at an interim stage, so that individuals know \(\alpha\) when they decide whether or not to vote. For example, potential voters can learn \(\alpha\) from pre-election opinion polls. However, institutional choices (e.g., whether voting should be mandatory) cannot be conditioned on the realization of \(\alpha\), because institutional choices apply for a longer time period than just a single issue election, and a rule that explicitly conditions on \(\alpha\) (say, “choose \(A\) without an election if \(\alpha > \frac{1}{2}\)”) would likely lead to large controversies among \(A\) and \(B\)-supporters as to what \(\alpha\) is.

We characterize conditions when voting should be encouraged and when voting should be discouraged, and show that the former is much more likely to occur in practice. We show that voting should be encour-

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\(^1\)These include most South American countries, as well as Australia and several European and Asian countries. See, for example, Wikipedia [21] for a list.

\(^2\)All non-voters receive a letter asking them to pay the fine. Instead of paying the fine, non-voters can also provide a written excuse. Thus, the actual cost of non-voting is the minimum over the disutility of paying $15 or writing the letter (see Weiner [20] for details).

\(^3\)There were a few experiments with mandatory voting laws before 1900 in the US. In 1896, the Supreme Court of Missouri struck down a Kansas City charter provision as unconstitutional that assessed a $2.50 poll tax on every man twenty-one years of age or older who failed to vote in the general city election. See Dean [7].
aged, if the expected absolute size of the two candidates’ supporter groups are sufficiently different, which occurs generically if the number of citizens is large. We also show that the optimal policy is to pay a subsidy equal to the minimal voting cost as the size of the electorate goes to infinity. While we derive our main results in a simple costly voting model, we argue in Section 4 that our findings that voting should be encouraged applies more generally to endogenous participation models in which cost considerations are important at least for the marginal voters, or for leaders who can induce some followers to vote.

To understand the intuition for our results, suppose that $\alpha > 0.5$, i.e., it is more likely that $A$-supporters are in the majority. However, in equilibrium, the probability that candidate $B$ wins must be sufficiently large, otherwise voters would have no incentive to participate. For this to occur, the participation rate by $B$ supporters must exceed that of $A$-supporters, which, in turn, implies that the (expected) proportion of $A$-supporters among non-voters is even higher than $\alpha$. If a particular $A$-supporter votes and is pivotal, then he imposes a positive externality on non-voters who support $A$, and a negative externality on non-voters who support $B$. The net change in surplus for non-voters is approximately $0.5$ (the utility change for an individual citizen) times the difference between the number of $A$ and $B$ supporters among non-voters. This change in surplus is positive, because in expectation there are more $A$ than $B$-supporters among non-voters. In contrast, the externality on voters is negative: There are two configurations in which our $A$ supporter is pivotal. First, if the number of other $A$-voters equals the number of other $B$-voters, and second, if the number of other $A$-voters is one smaller than the number of other $B$-voters. In the first case there is no externality, while in the second case the net change in surplus is $-0.5$. Thus, our $A$ supporter increases surplus by voting, unless there are either almost no non-voters or the electorate is fairly evenly split ($\alpha$ close to 0.5). Subsidies encourage more $A$ than $B$ supporters to vote, because there are more $A$-supporters than $B$-supporters among the whole population, and in particular, among non-voters), and are therefore welfare improving.

In an important paper, Börgers [3] has analyzed optimal voter participation policies. In his model, the expected number of $A$ and $B$ supporters are equal, which corresponds to the special case $\alpha = 0.5$ (with certainty) in our model. As a consequence, only the negative externality occurs in Börgers’ model, implying that voluntary voting dominates mandatory voting.

The costly voting literature dates back to Ledyard [13] and Palfrey and Rosenthal [17], [18]. Empirical evidence confirms many comparative static predictions of the costly voting model (see Blais [2], Shachar and Nalebuff [19], Levine and Palfrey [14]): In particular, minority supporters are more active in equilibrium, and the overall participation rate is higher the closer $\alpha$ is to 0.5. A problem with the simple costly voting model, however, is the unrealistic prediction of a zero participation rate in very large elections. More sophisticated endogenous participation models combine cost considerations with “ethical” voters (Feddersen and Sandroni [9]) or introduce leaders, who can influence the participation decision of a large number of citizens (e.g., Shachar and Nalebuff [19], Herrera and Martinelli [12]). These models succeed in generating positive participation in large electorates, while preserving the nice comparative static results of the costly voting model. While we derive our results in a simple costly voting model, we argue in Section 4 that they are robust in a considerably larger class of endogenous participation models, including the ones just mentioned.

Some other recent papers study costly voting environments related to our model. Campbell [4] shows that, if voting costs of $A$ and $B$ supporters are drawn from different distributions, then the candidate with the more favorable cost distribution wins almost certainly in large elections, rather than the candidate preferred
by the majority. We assume that $A$ and $B$ supporters’ cost of voting are drawn from the same distribution and therefore Campbell’s effect is absent in our model. Goeree and Grosser [11] consider the effect of opinion polls in a model where voting cost are deterministic and equal for all voters. However, in their model, subsidies would either be ineffective or lead to full participation. Ghosal and Lockwood [10] analyze a model in which voluntary participation can be inefficiently low, but the cause of this inefficiency differs from ours. In their model, each individual has both a “private value” preference for one of the politicians, and a “common value” preference to select the politician who matches the unknown state of the world. When the common value component is sufficiently important, participation by more people leads to better information aggregation, which creates a positive externality that citizens do not internalize.

2 Model

There are $N$ citizens who have the right to vote for one of two candidates, $A$ or $B$. The probability $\alpha$ that a citizen prefers candidate $A$ to candidate $B$ is chosen by nature according to a probability density function $g(\alpha)$, and becomes public information before the election. For example, the public information about $\alpha$ could be the result of pre-election opinion polls. Preferences for candidates $A$ and $B$, respectively, are then drawn independently across individuals according to probability $\alpha$.

Participating in the election is costly. The cost $c$ a citizen pays if and only if he votes is drawn independently according to probability density function $f(c)$. We assume that $f(c)$ is strictly positive on its support $[c, \bar{c}]$, where $c > 0$. We write $F(c)$ for the corresponding cumulative distribution function. The outcome of the election is determined by majority rule. In case of a tie, each candidate wins with probability $\frac{1}{2}$. Citizen $i$ receives a benefit normalized to $1$ if his preferred candidate is elected, and $0$ if the other candidate wins. We allow the government to encourage voting via a subsidy $s$ that is paid to all voters. The subsidy $s$ can be interpreted either as actions by government that reduce citizens’ costs of electoral participation, or equivalently, as a fine that non-voters must pay. Because mandatory voting laws cannot force individuals to vote, but rather encourage participation through fines imposed on abstainers, our notion of subsidized voting corresponds to how mandatory voting laws work in practice. Formally, if $P_i \in \{A, B\}$ is $i$’s preferred candidate, if $E$ is the candidate who is elected, and if $v_i = 1$ and $v_i = 0$ is $i$’s decision whether or not to vote, then $i$’s utility is given by

$$u(E, v_i; P_i, c_i) = \begin{cases} 1 - v_i(c_i - s) & \text{if } E = P_i; \\ -v_i(c_i - s) & \text{if } E \neq P_i. \end{cases}$$

There are two interesting special cases encompassed in this framework of subsidized voting. First, if $s = 0$, we have a standard costly voting model; we call this case voluntary voting. Second, if $s \geq \bar{c}$, then all citizens vote, and we talk of compulsory voting.

3 Results

First note that if an individual votes, his weakly dominant strategy is to vote for his preferred candidate, and it is therefore sufficient to focus solely on an agent’s participation decision. In Proposition 1, we show that
there exists a symmetric equilibrium in pure strategies, which is characterized by a simple cutoff value rule for the voting costs: A-supporters choose to vote if and only if their voting costs are no higher than $c_A$, and B-supporters have an analogous cost threshold $c_B$. The proofs of all propositions are in the Appendix.

**Proposition 1** There exists a symmetric equilibrium in pure strategies that is characterized by cutoff values $c_A$ and $c_B$ such that individual $i$ votes for his preferred candidate $P_i$ if $c_i \leq c_{P_i}$, and abstains otherwise.

We now analyze the welfare impact of voting subsidies. Welfare is the sum of citizens’ payoffs from the elected candidate minus the voting costs of those citizens who vote. Formally, let $S(a, N, k_A, k_B)$ be the social surplus in a population with $N$ citizens, $a$ of whom prefer candidate $A$, and let $k_A$ and $k_B$ be the number of $A$ and $B$ voters.

$$S(a, N, k_A, k_B) = \begin{cases} a - k_A E(c|c \leq c_A) - k_B E(c|c \leq c_B) & \text{if } k_A > k_B \\ \frac{1}{2}a + \frac{1}{2}(N-a) - k_A E(c|c \leq c_A) - k_B E(c|c \leq c_B) & \text{if } k_A = k_B \\ N - a - k_A E(c|c \leq c_A) - k_B E(c|c \leq c_B) & \text{if } k_A < k_B \end{cases}$$

Ex-ante expected welfare is then given by

$$W = \sum_{a=0}^{N} \sum_{k_A=0}^{a} \sum_{k_B=0}^{N-a} \binom{N}{a} \alpha^a (1-\alpha)^{N-a} S(a, N, k_A, k_B) \times \binom{a}{k_A} F(c_A)^{k_A} [1 - F(c_A)]^{a-k_A} \binom{N-a}{k_B} (1-F(c_B))^{k_B} [1 - F(c_B)]^{N-a-k_B},$$

which is the probability that there are $a$ A-supporters, $k_A$ A-voters and $k_B$ B-voters, multiplied with the conditional expected social surplus, and summed over all possible realizations. Subsidy payments do not enter the definition of welfare directly, because they are transfers between citizens and therefore drop out. Instead, subsidies affect welfare indirectly by changing the participation choices, which may affect both the total voting costs incurred and the electoral outcome.

Proposition 2 considers small electorates in which all citizens vote as long as the numbers of $A$ and $B$ supporters are sufficiently close.

**Proposition 2** Suppose that $\tilde{c} < \left(\binom{N-1}{(N-1)/2}\right)^2$. Then there exist $\delta > 0$ such that

1. All agents vote if and only if $\alpha \in [0.5 - \delta, 0.5 + \delta]$. In this case subsidies have no effect.

2. Subsidies strictly lower welfare for all $\alpha \notin [0.5 - \delta, 0.5 + \delta]$ that are close to $0.5 - \delta$ and $0.5 + \delta$.

If all agents participate, then subsidies have no effect, because $c_A$ and $c_B$ do not change. However, once $\alpha$ becomes sufficiently large, some citizens no longer vote. In this case, voters impose a negative externality on other voters, which is essentially Börgers’ effect. In addition, because the expected number of non-voters is marginal, this is the only effect that matters. The proof of Proposition 2 indicates that the effect is stronger (of first order) when the number of citizens is odd than when it is even (where it is of second order).

While Proposition 2 characterizes conditions under which voting subsidies may be detrimental for intermediate values of $\alpha$, Proposition 3 shows that subsidies are beneficial for sufficiently high values of $\alpha.$
Proposition 3 There exists $\alpha_2 < 1$ such that for all $\alpha \in (\alpha_2, 1)$, $\frac{dW}{ds} > 0$ at $s = 0$: A small voting subsidy, starting at $s = 0$, is welfare improving. The marginal benefit per-capita benefit $\frac{d(W/N)}{ds}$ goes to 1 as $N \to \infty$.

The intuition for the positive effect of subsidies is as follows: For $\alpha$ close to 1, (almost) all individuals agree which candidate they prefer. However, every citizen would like other citizens to incur the voting cost, so that the participation is essentially equal to the problem of private provision of a public good, with the associated free-riding problem among the majority.\(^4\) To provide any incentive to participate, there must be a positive probability that nobody votes. A voting subsidy increases expected participation and therefore reduces the risk of a wrong decision.

We also show in the proposition that the per-capita welfare effect of the subsidy converges to 1 as $N \to \infty$. This means that subsidies are very effective, i.e., a small unit of subsidy provided to voters raises each citizen’s welfare by one unit. The reason is that, with many citizens, the equilibrium probability of a mistake is about $(c - s)$ (so that the lowest cost citizens just barely have an incentive to vote). A marginal increase in $s$ reduces this mistake probability (which benefits all citizens), without leading to a significantly higher per capita participation cost.

The results of Propositions 2 and 3 are illustrated in the left panel of Figure 1, which displays the marginal benefit of a subsidy when voting cost are uniformly distributed in the interval $[0.01, 0.05]$ and $N = 20$. For $\alpha$ close to 0.5, all citizens vote voluntarily, so that a subsidy (or the threat of a fine for non-voters) has no effect on welfare.\(^5\) Around $\alpha = 0.6$, some high cost citizens start to abstain under voluntary voting. It is not helpful to induce them to vote. The reason is that the most likely scenario in which an abstainer’s vote would matter is one where there is only one abstainer, in which case he is only pivotal if his vote leads to a 10-10 split of the electorate. But in that case, he imposes a net negative externality on the other.

\(^4\)Note that citizens with minority preference have a much smaller incentive to free-ride, because the expected number of other citizens with the same preference is very small.

\(^5\)The reader may wonder whether taxing voters, if feasible, would increase welfare in the full participation region to the left of $\alpha = 0.6$ of Figure 1. This may, but need not be the case. In order to have any effect on participation, the tax would have to be sufficiently large, which would deter some citizens with substantial private benefit from voting. However, because the negative externality is small (especially if the number of citizens is odd, where it is of second order), the net change in welfare is often negative.
voters. The negative impact on ex-ante expected welfare, however, is very limited because the probability that the subsidy increases voter participation is small, there is already almost full voter participation, and the negative externality impacts only one voter. For larger values of $\alpha$, the effect identified in Proposition 3 dominates, as voluntary voting leads to a significant error probability that is decreased by a subsidy. Both the small range where the derivative is negative, and the small size of the negative derivative indicates that for most distributions $G$ over $\alpha$, subsidies are helpful.

Now consider the case that $N$ is so large that the condition in Proposition 2 does not hold and some high cost agents abstain even if $\alpha = 0.5$. For the above parametrization this occurs for $N \geq 64$. At this point, Börgers’ negative externality dominates for $\alpha$ that are sufficiently close to 0.5, which is what happens in the right panel of Figure 1 for $N = 80$ and $N = 320$. For higher values of $\alpha$, Proposition 3 applies so that subsidies increase surplus.

In the examples, a marginal subsidy is helpful as long as the distribution of $\alpha$ is not extremely concentrated around 0.5. Moreover, the positive effect dominates very soon, and is considerably larger in size. The intuition for this is as follows: Both the positive and the negative effect arise only if an additional voter is pivotal. However, the negative effect stems from an externality affecting other voters, and in a pivot situation, the number of negatively-affected voters is at most one larger than the number of positively affected ones; the expected net per capita effect is therefore very small. In contrast, the positive effect for higher values of $\alpha$ is due to the increase of the winning probability of the majority-preferred candidate caused by increased voter participation, which affects many non-voters.

Figure 1 also indicates that the region of values of $\alpha$ where subsidies are harmful decreases with $N$, which suggest that this region vanishes for the limit case that $N$ goes to infinity. We now prove this result formally.

**Proposition 4** Let $A_N$ be the set of all $\alpha$ such that there exists a subsidy $s > 0$ that strictly increases surplus relative to voluntary voting in an economy with $N$ agents. Then $A_N \to [0, 1]$ as $N \to \infty$ (i.e., the set of limit points of sequences $\alpha_N$ with $\alpha_N \in A_N$ for all $N$ is $[0, 1]$.)

In the proof we first show that, for any $\alpha$, the probability that the wrong candidate gets elected under voluntary voting is strictly greater than 0, even as $N \to \infty$, else no citizen would have an incentive to vote. To understand the intuition, first, note that the expected number of voters must remain finite, as otherwise, no individual voter’s pivot probability would justify incurring strictly positive voting costs of at least $c$. Hence, both cost cutoffs that characterize the equilibrium must converge to $c$. Therefore, in the limit, the probability to be pivotal must be the same for an A- and a B-supporter. We show in the proof that this implies that the winning probability for A and B must both be 1/2. This outcome is highly inefficient, because if $\alpha > 1/2$, then the number of A-supporters exceeds that of B supporters with probability close to 1. Thus, voluntary voting leads to a substantial mistake probability in equilibrium. Subsidies increase turnout, which lowers the mistake probability.

We prove Proposition 4 by showing that, starting from $s = 0$, a small increase in $s$ is beneficial for given $\alpha \neq 0.5$. We now turn to the question which subsidy is actually optimal. For any finite $N$, this question is very hard to answer analytically. However, for $N \to \infty$, the following proposition shows that the optimal subsidy converges to $c$. 

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Proposition 5 Let $s_N$ be the optimal subsidy of an economy with $N$ agents. Then $s_N \to c$ as $N \to \infty$. Moreover, subsidies strictly improve upon voluntary voting.

To gain intuition, consider a subsidy of $s = c + \varepsilon$. Such a subsidy makes it a dominant strategy for everyone with cost below $c + \varepsilon$ to vote, where $\varepsilon$ is positive but small. Essentially, such a subsidy allows drawing a sample that is relatively small compared to the total population to avoid significant voting costs, but it is still sufficiently large to yield the correct decision with probability 1.

4 Robustness

Our results show that subsidizing participation is typically optimal in a costly voting model. In this section, we show that this result extends to other models of endogenous participation in elections. The empirical evidence on the costly voting model is mixed. On the one hand, the costly voting model predicts that the minority supporters are more active in equilibrium which makes up for their smaller numbers (underdog effect). The model also predicts higher participation rates in close elections (competition effect). These comparative static results are strongly supported by empirical studies. On the other hand, the simple costly voting model predicts that participation rates go to zero for large elections, which is evidently not the case. It is therefore important to analyze whether voting subsidies may still play an important role in models that can account for positive equilibrium participation rates in large elections. We first consider a model in which some voters enjoy voting and show that this gives rise to a new effect that may make subsidies beneficial. We then sketch how to extend that model to allow for endogenous participation decisions by leaders that can motivate some voters with positive voting costs to participate.

Consider first a very simple model with an infinite number of citizens. Proportion $\delta$ of the population has zero or negative voting costs (group D – dutiful voters), while proportion $1 - \delta$ has positive voting costs distributed on $[c, \bar{c}]$ as in our base model (group C – costly voters). Furthermore, let $\alpha_D$ ($\alpha_C$) be the probability a group D (group C) individual prefers candidate A. From an ex-ante point of view, $g(\alpha_C, \alpha_D)$ is the density from which $\alpha_C$ and $\alpha_D$ are drawn.

Clearly, if there are no subsidies, only citizens who enjoy voting (i.e., who belong to group D) vote. Therefore, candidate A wins the election if and only if $\alpha_D > 1/2$. From a social point of view, candidate A should win if and only if $\alpha \equiv \delta \alpha_D + (1 - \delta) \alpha_C > 1/2$. Hence, even if a significant group of individuals have negative voting costs, voluntary participation guarantees that the correct candidate wins only for a very small class of distributions that have the property that $\alpha_D > 1/2$ ($< 1/2$) implies that $\alpha > 1/2$ ($\alpha < 1/2$) with probability 1. More generally, the equilibrium per-capita utility under voluntary voting is

$$\int_0^1 \int_{[c, \bar{c}]} \left[ \delta \alpha_D + (1 - \delta) \alpha_C \right] g(\alpha_C, \alpha_D) d\alpha_D d\alpha_C + \int_0^{1/2} \left[ 1 - \delta \alpha_D - (1 - \delta) \alpha_C \right] g(\alpha_C, \alpha_D) d\alpha_D + \delta E_B,$$

where $E_B$ is the average benefit that voters in group D receive from voting. Clearly, paying a subsidy of only $\zeta$ does not significantly increase participation and hence is unlikely to change the outcome of the election, but paying a subsidy of $\tau$ (equivalently, compulsory voting) yields a per-capita utility of

$$\int_0^1 \int_0^{1/2} \max \left[ \delta \alpha_D + (1 - \delta) \alpha_C, 1 - \delta \alpha_D - (1 - \delta) \alpha_C \right] g(\alpha_C, \alpha_D) d\alpha_D d\alpha_C + \delta E_B - (1 - \delta) E_C,$$

where $E_C$ is the average cost that voters incur from voting. Clearly, paying a subsidy of only $\zeta$ does not significantly increase participation and hence is unlikely to change the outcome of the election, but paying a subsidy of $\tau$ (equivalently, compulsory voting) yields a per-capita utility of
where $EC$ is the expected per-capita voting cost among voters with positive voting costs. Since the integral terms in (3) are generically larger than the corresponding terms in (2), compulsory voting leads to a higher welfare, provided that the average voting cost $EC$ of the costly voters is sufficiently small.

Note that the model so far can “explain” positive participation rates by assuming that some people enjoy voting, but it cannot explain the underdog effect and the competition effect, which says that participation rates (in finite electorates) are the higher, the closer is the size of the two preference groups. Both effects are well documented, both for large scale elections (see, e.g., Blais [2], Nalebuff and Shachar [19]) and in laboratory experiments (see Levine and Palfrey [14]). Alternative endogenous participation models also generate these effects. For example, Feddersen and Sandroni [9] develop a model of ethical voters who receive a moral benefit that possibly offsets their voting costs, if they feel that they “should” vote. The ethical rule that specifies the voting cost threshold below which an individual voter “should” vote is derived endogenously, taking into account both cost considerations and the preference group’s chance of winning. While no voter is individually pivotal with positive probability in the model of Feddersen and Sandroni, a positive percentage of ethical voters participates in the election. Moreover, there is a positive probability that the minority candidate wins, and ethical minority voters are more likely to participate in equilibrium.

Nalebuff and Shachar [19] and Herrera and Martinelli [12] develop models with a number of “leaders” who favor either candidate A or candidate B and can decide whether to take a costly action that mobilizes a (random) number of voters to vote for their respective preferred candidate. Equilibria in these models also feature positive participation rates, a positive mistake probability and the underdog effect.

We now sketch an extension of the above model that corrects this shortcoming of the costly voting model. Suppose there are $L_A$ “A-leaders” and $L_B$ “B-leaders,” who each have influence over a set of measure $\ell$ of costly voters who support A and B, respectively. Clearly, $L_A\ell \leq \alpha_C$ and $L_B\ell \leq 1 - \alpha_C$. A leader can “mobilize” his followers by imposing (through moral pressure etc.) a cost of not voting. Without loss of generality, we can assume that all his followers vote if a leader mobilizes (otherwise, we can just decrease $\ell$ accordingly). A leader’s objective function is a weighted average between utility from the election outcome and the average voting costs of his followers. Also assume that the weight on voting costs is sufficiently small, so that a leader who knows that he can change the election outcome by mobilizing would strictly prefer to do so. Candidate A wins the election if and only if $\delta\alpha_B + (1 - \delta)\ell M_A > \delta(1 - \alpha_D) + (1 - \delta)\ell M_B$, where $M_A$ and $M_B$ are the number of A– and B–leaders who mobilize, respectively. Rearranging this condition, we get

$$M_A - M_B > \frac{\delta}{1 - \delta} \frac{1 - 2\alpha_D}{\ell} \equiv K. \quad (4)$$

If the expression $K$ on the right hand side of (4) is positive, then it is the critical number by which the number of mobilizing A–leaders must exceed the number of mobilizing B-leaders in order for Candidate A to win.

A leader’s strategic calculations are very similar to the calculations of a costly voter in our basic model. Provided that $\delta$ is not too large (so that costly voters can, in principle, swing the election to the candidate who is not preferred by the majority of the dutiful voters), both candidates have a strictly positive chance of

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6 An alternative possibility that leads to qualitatively the same results is that leaders do not care about their followers’ voting costs, but face some mobilization costs.
winning in equilibrium. If, instead, $\delta$ is large (how large depends also on the value of $\alpha_D$), then only dutiful voters participate in the election and the majority among them is decisive. This model therefore can explain the fact that in presidential elections in Utah and Massachusetts, the Republican or Democratic candidate, respectively, consistently wins: The advantage of one party over the other among dutiful voters is just so large that leaders do not try to mobilize costly voters and consequently, the larger preference group wins always. In contrast, the simple costly voting model (with all voters having voting costs) would predict that Democratic and Republican presidential candidates should each have a significant probability of winning Utah or Massachusetts.

The effect of leaders on welfare relative to a model in which only dutiful voters vote can be ambiguous. On the one hand, if the majority group among dutiful voters is not equal to the majority preference group in the total population, then the presence of leaders makes it at least possible that the majority-preferred candidate wins. On the other hand, if the majority group among dutiful voters is equal to the majority preference group in the total population, then the presence of leaders introduces the possibility that the wrong candidate wins.

As in the basic model and the voluntary voting model above, it is quite conceivable that voting subsidies are desirable from a social point of view. With voting subsidies, more costly voters choose to vote, regardless of the actions of leaders. If the voting subsidy is sufficiently large to induce most citizens to vote, then the voting outcome is correct decision from a social perspective. If the average voting cost of costly voters is sufficiently small relative to the importance of the electoral decision, then subsidies are welfare enhancing in this framework.

5 Conclusion

A central theme in many texts by political scientists dealing with participation in elections (as well as in newspaper editorials) is that electoral turnout is “too low”. In this paper, we provide a theoretical framework for this idea. We show that costly voting induces suboptimal equilibrium participation and frequently leads to wrong choices. We prove that in such a world, providing incentives for citizens to vote increases the quality of electoral decisions and social welfare.

Our setup is the simplest model in which questions of costly voting can be studied: Citizens know which candidate they prefer, they only have to decide whether or not to vote, and the voting costs are drawn from the same distribution for both $A$ and $B$ supporters. Extending our model to allow for incomplete information about candidates and differential voting costs for $A$ and $B$ supporters should reinforce our qualitative finding that voting is similar to providing a public good.

Following Börgers [3], we assume that each voter’s benefit from the election of his preferred candidate is normalized to one, while costs are random draws for individual voters. More generally, one could assume that each voter’s benefit is also random. However, if both costs and benefits are drawn from the same distribution for $A$ and $B$-supporters, the analysis for voluntary and compulsory voting is largely unchanged.
6 Appendix

Proof of Proposition 1. Let $c_A$ be given. We construct $c_B$ such that an individual who prefers $B$ is indifferent between voting and not voting if $c_A = c_B$.

Consider a supporter of $B$. The probability that $a$ of the remaining $N - 1$ individuals support candidate $A$ and that $k$ of these $A$ supporters participate in the election, is given by

$$\text{Prob}\{\#A-supporters = a, \#A-voters = k\} = \binom{N-1}{a} a^a (1-a)^{N-1-a} \binom{a}{k} F(c_A)^k (1 - F(c_A))^{a-k}. \quad (5)$$

If there are $a$ supporters of $A$, of whom $k$ participate in the election then our $B$ supporter’s expected benefit of voting including subsidy $s$ but excluding voting costs is

$$\text{Benefit}(a, k) = 0.5 \left[ \binom{N-a-1}{k-1} F(c_B)^{k-1} (1 - F(c_B))^{N-a-k} \right.$$

$$\left. + \binom{N-a-1}{k} F(c_B)^{k} (1 - F(c_B))^{N-a-k} \right] + s. \quad (6)$$

It follows immediately that $\text{Prob}\{\#A = a, \#A-voters = k\}$ and $\text{Benefit}(a, k)$ are continuous in $c_A$. The expected benefit from voting for a $B$-supporter with voting costs $c_B$ is

$$\text{EB}_B(c_A, c_B) = \frac{N-1}{a} \sum_{k=0}^{N-1} \text{Prob}\{\#A = a, \#A-voters = k\} \text{Benefit}(a, k), \quad (7)$$

which is continuous in $c_A$ and $c_B$. Similarly, an $A$-supporter’s gross benefit, $\text{EB}_A(c_A, c_B)$, is continuous. We now define the function $T: [\underline{c}, \overline{c}]^2 \rightarrow [\underline{c}, \overline{c}]^2$ by

$$T(c_A, c_B) = \left( \max\{\min[\text{EB}_A(c_A, c_B), \overline{c}], \underline{c}\}, \max\{\min[\text{EB}_B(c_A, c_B), \overline{c}], \underline{c}\} \right).$$

Clearly, $T$ is continuous. Brouwer’s fixed point theorem therefore implies that there exist $c_A^*, c_B^*$ with $T(c_A^*, c_B^*) = (c_A^*, c_B^*)$. Consider $c_A^*$. If $\underline{c} < c_A^* < \overline{c}$ then the gross benefit of an $A$ supporter with costs $c_A^*$ who participates in the election is exactly $c_A^*$. As a consequence an $A$ supporter with cost $c_A^*$ is indifferent between voting and not voting, and every $A$ supporter with a lower cost will strictly prefer to vote. Now let $c_A^* = \underline{c}$. Then $\text{EB}_A(c_A, c_B) \leq \underline{c}$. Thus, no $A$ supporter will will participate, because the probability that $c = \underline{c}$ is 0. Finally, $c_A^* = \overline{c}$ implies that all $A$ supporters participate in the voting. Therefore, $c_A^*$ is the cost cutoff for $A$ supporters. Similarly, it follows that $c_B^*$ is the equilibrium cutoff for $B$ supporters. \[\]

Proof of Proposition 2. First, suppose that $N$ is odd. Consider one particular agent, and suppose that all remaining agents vote. Then the pivot probability for that agent is given by $\binom{N-1}{(N-1)/2} \alpha^{(N-1)/2} (1 - \alpha)^{(N-1)/2}$.

If the agent is pivotal, then his benefit is 0.5. Thus, $c_A = \overline{c}$ if and only if

$$\overline{c} \leq 0.5 \binom{N-1}{(N-1)/2} \alpha^{(N-1)/2} (1 - \alpha)^{(N-1)/2}. \quad (8)$$

Choose $\hat{\alpha} > 0.5$ such that (8) holds with equality. Then $\delta = \hat{\alpha} - 0.5$. If $\alpha < \hat{\alpha}$ there is already full participation and subsidies are irrelevant. It remains to show that welfare is decreasing if subsidies are introduced for any $\alpha > \hat{\alpha}$ that is close to $\hat{\alpha}$. 10
We first show that \( c_A \) and \( c_B \) are continuous in \( \alpha \) around \( \hat{\alpha} \). We use the implicit function theorem, as generalized by Clarke [5] and [6] to non-smooth functions. Because \((1 - F(c_A)) = (1 - F(c_B)) = 0 \) at \( \hat{\alpha} \), we can ignore terms with \((1 - F(c_A))^k, (1 - F(c_B))^k \) for \( k \geq 2 \), because their derivatives with respect to \( c_A \) or \( c_B \) are zero. Omitting these terms, (7) implies that the benefit of a \( B \) supporter from voting is given by

\[
0.5 \left[ \left( \frac{N-1}{N+1} \right)^{\frac{N}{2}} \left( 1 - \alpha \right)^{\frac{N}{2}} F(c_A) \left( \frac{N-1}{N+1} \right)^{\frac{N}{2}} \left( 1 - \alpha \right)^{\frac{N}{2}} F(c_B) \right] + \left( \frac{N-1}{N+1} \right)^{\frac{N}{2}} \left( 1 - \alpha \right)^{\frac{N}{2}} \frac{N+1}{2} \left( 1 - F(c_A) \right) F(c_B) - c_B. \tag{9}
\]

If we take the derivatives with respect to \( c_A \) and \( c_B \) from the right, \( F(c_A) \) and \( F(c_B) \) do not change, and the matrix of derivatives has full rank. Taking the derivative with respect to \( c_A \) and \( c_B \) from the left, and using the fact that \( F(c_A) = F(c_B) = 1 \), it follows that

\[
\frac{\partial}{\partial c_A} : 0.5 f(\bar{c}) \left( \frac{N-1}{N+1} \right)^{\frac{N}{2}} \left( 1 - \alpha \right)^{\frac{N}{2}} \left( 1 - 2\alpha \right), \quad \frac{\partial}{\partial c_B} : -1. \tag{10}
\]

By symmetry, the derivatives for \( A \) supporters are

\[
\frac{\partial}{\partial c_A} : -1, \quad \frac{\partial}{\partial c_B} : -0.5 f(\bar{c}) \left( \frac{N-1}{N+1} \right)^{\frac{N}{2}} \left( 1 - \alpha \right)^{\frac{N}{2}} \left( 1 - 2\alpha \right). \tag{11}
\]

Thus, the determinant is strictly positive. Similarly, it follows that the matrix of derivatives has full rank for any combination of left and right derivatives, which implies that \( c_A \) and \( c_B \) are continuous functions of \( \alpha \).

Consider \( \alpha \) that is marginally greater than \( \hat{\alpha} \). If all agents participate, then a particular citizen is pivotal if the number of \( A \)- and \( B \)-voters is the same among the remaining voters. If our particular citizen votes for candidate \( A \), then the total benefit of all other \( A \)-voters increases by \((N-1)/4\), while the benefit to \( B \) voters is reduced by exactly the same amount. A subsidy would turn our citizen from a non-voter to a voter if and only if the private benefit of the person equals his voting costs, i.e., if the person’s net-benefit from voting is zero. Thus, if all agents participate, there is no effect on surplus.

Now suppose that one of the remaining agents does not vote. Then a particular \( A \) supporter is pivotal if and only if there are \((N - 3)/2 \) \( A \)-voters and \((N - 1)/2 \) \( B \)-voters among the remaining agents. Thus, the net-change in benefit to these voters is \(-0.5\) if our \( A \) supporter participates. As above, if a subsidy would induce our \( A \)-supporter to vote, then the private benefit is zero. Thus, the subsidy would lower surplus.

Note that we can ignore the effects when two or more agents do not participate, because this event is on the order of \((1 - F(c_A))^2\) and \((1 - F(c_B))^2\), while the above effect is of the order \((1 - F(c_A))\) and \((1 - F(c_B))\), and \( F(c_A), F(c_B) \) are close to 1.

The proof for even \( N \) is similar. With an even number of agents and full participation in the election, a particular person person, say an \( A \) supporter, is pivotal if there are \((N - 3)/2 \) \( A \)-supporters and \((N - 1)/2 \) \( B \) supporters, which implies a negative externality. Again, all higher order terms can be ignored.

**Proof of Proposition 3.** Let \( \alpha = 1 \). Then an \( A \)-supporter is indifferent between voting and not voting if

\[
[1 - F(c_A)]^{N-1} \frac{1}{2} = c_A - s, \tag{12}
\]
It follows immediately that $\zeta < c_A < \bar{c}$ for small $s$. Applying the implicit function theorem yields

$$\frac{dc_A}{ds} = \frac{2}{2 + (N - 1)f(c_A)[1 - F(c_A)]^{N-3}} > 0.$$  \hspace{1cm} (13)

Substituting $a = N, k_B = 0$, and $\alpha = 1$ in (1) yields

$$W = \frac{N}{2} (1 - F(c_A))^N + \sum_{k_A=1}^{N} (N - k_A E[c|c \leq c_A]) {N \choose k_A} F(c_A)^{k_A}(1 - F(c_A))^{N-k_A}$$

$$= N \left[ 1 - \frac{1}{2} (1 - F(c_A))^N - \int_{\zeta}^{c_A} cf(c) dc \right].$$  \hspace{1cm} (14)

Taking the derivative of (14) with respect to $s$ at $s = 0$ and using (12) yields

$$\frac{\partial W}{\partial s} = f(c_A)N(N - 1)c_A \frac{dc_A}{ds} > 0,$$  \hspace{1cm} (15)

which proves that subsidies increase welfare. Finally, using (12), (13) and the fact that $c_A \to \zeta$ immediately implies that derivative of per-capita surplus $\frac{\partial W}{\partial s/N}$ converges to 1 as $N \to \infty$.

Because expected social welfare $W$ is a continuous function of $(\alpha, c_A, c_B)$, it remains to prove that $(c_A, c_B)$ is continuous in $\alpha$ near $\alpha = 1$. Since we can ignore all terms $(1 - F(c_B))$, (7) implies

$$\frac{1}{2} (1 - F(c_A))^{N-1} - c_A = 0$$  \hspace{1cm} (16)

$$\frac{1}{2} \left\{ (1 - F(c_A))^{N-1} + (N - 1)F(c_A)(1 - F(c_A))^{N-2} \right\} - c_B = 0.$$  \hspace{1cm} (17)

Differentiating (16) with respect to $c_A$ and $c_B$ from the left yields $-0.5(N - 1)f(c_A) [(1 - F(c_A))]^{N-2} - 1$ and 0, respectively. The derivatives from the right are $-1$ and 0. Similarly, the derivatives of (17) from the left are $-0.5(N - 1)(N - 2)f(c_A) [(1 - F(c_A))]^{N-3} - 1$ and $-1$. The derivatives from the right are $-1, 1$. Thus, all possible combinations of right and left derivatives yield a matrix that has full rank. Clarke [5] and [6] therefore implies that $c_A$ and $c_B$ are continuous in $\alpha$.

**Lemma 1** Suppose that $\zeta > 0$. Then the expected number of $A$ and $B$ voters, $\bar{v}_A(N)$ and $\bar{v}_B(N)$, are bounded away from $\infty$, i.e., there exists an $M$ such that $\bar{v}_A(N), \bar{v}_B(N) \leq M$ for all $N \in \mathbb{N}$.

**Proof of Lemma 1.** The strategy of the proof is to show that if the expected number of voters goes to infinity as $N \to \infty$ then the pivot probabilities go to zero. This provides a contradiction because the voting costs $c$ are always strictly positive, i.e., $c \geq \zeta > 0$.

Suppose by way of contradiction that the expected number of $A$ and $B$ voters are both infinite (the case where only the expected number of one type of voter is finite is similar and omitted). Then because the expected number of $A$ and $B$ voters are given by $\bar{v}_A(N) = F(c_A(N))\alpha N$ and $\bar{v}_B(N) = F(c_B(N))(1 - \alpha)N$, respectively, we have

$$\lim_{N \to \infty} F(c_A(N))N = \infty, \text{ and } \lim_{N \to \infty} F(c_B(N))N = \infty.$$  \hspace{1cm} (18)

Let $N_A(N)$ be the realized number of $A$-supporters out of $N$ citizens. Let $N_B(N) = N - N_A(N)$. 

Claim 1. The expected number of $A$ and $B$ voters goes to infinity for almost all realizations $N_A$ and $N_B$, i.e.,
$$\lim_{N \to \infty} N_A(N)F(c_A(N)) = \lim_{N \to \infty} N_B(N)F(c_B(N)) = \infty.$$ 

Let $X_i$ be a random variable such that $X_i = 1$ if person $i$ is an $A$ supporter and $X_i = 0$ if agent $i$ is a $B$ supporter. Because the $X_i$ are i.i.d., the central limit theorem implies that
$$\lim_{N \to \infty} P\left(\left|\frac{\sum_{i=1}^{N} (X_i - \alpha)}{\sqrt{N\alpha(1-\alpha)}} \right| \leq \lambda\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{x^2}{2}} dx. \quad (19)$$

Thus, for every $\varepsilon > 0$ there exists a $\lambda > 0$ such that
$$\lim_{N \to \infty} P\left(\left|N\alpha - \lambda \sqrt{N\alpha(1-\alpha)}\right| \leq \sum_{i=1}^{N} X_i \leq N\alpha + \lambda \sqrt{N\alpha(1-\alpha)}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} e^{-\frac{x^2}{2}} dx > 1 - \varepsilon. \quad (20)$$

Hence, with probability arbitrarily close to 1,
$$N_A(N) \in [N\alpha - \lambda \sqrt{N\alpha(1-\alpha)}, N\alpha + \lambda \sqrt{N\alpha(1-\alpha)}], \quad (21)$$
so that $N_A(N)F(c_A(N))$ and $N_B(N)F(c_B(N)) \to \infty$, proving claim 1.

Let $Y^A_i$ be the random variable which assumes the value 1 if the $i^{th}$ $A$ supporter votes and 0, otherwise. Similarly, define $Z^B_i$ for $B$ supporters. The probability that a particular agent is pivotal is less or equal to $P\left(\{|\sum_{i=1}^{N_A} Y^A_i - \sum_{i=1}^{N_B} Z^B_i| \in \{-1, 0, 1\}\}\right)$. To determine this upper bound for the pivot probability, we next show that the limit distribution is normal. Care must be taken in applying the central limit theorem, because $Y^A_i$ and $Z^B_i$ converge to zero a.e., as $N_A$ and $N_B \to \infty$.

Claim 2. Suppose that $N_A, N_B \to \infty$ such that $N_A F(c_A(N)), N_B F(c_B(N)) \to \infty$. Then
$$\lim_{N_A, N_B \to \infty} P\left(\left|\frac{\sum_{i=1}^{N_A} Y^A_i - \sum_{i=1}^{N_B} Z^B_i}{\sqrt{N_A \text{var}[Y^A_i] + N_B \text{var}[Z^B_i]}} \leq \lambda\right| \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{x^2}{2}} dx,$$
where the convergence is uniform in $\lambda$.

Let $C_N = \sum_{i=1}^{N_A} E[|Y^A_i - F(c_A(N))|^{2+\delta}] + \sum_{i=1}^{N_B} E[|Z^B_i - F(c_B(N))|^{2+\delta}]$, for some $\delta > 0$. According to Theorem 4.4 in Doob [8] it is sufficient to check that
$$\lim_{N_A, N_B \to \infty} \left(\frac{C_N}{N_A \text{var}[Y^A_i] + N_B \text{var}[Z^B_i]}\right)^{1+\delta/2} = 0. \quad (22)$$

Recall that $Y^A_i$ and $Z^B_i$ assume the value 1 with probabilities $F(c_A(N))$ and $F(c_B(N))$, respectively; and

\footnote{In fact, the distinction between this result and that in Lemma 2 is as follows. Here we show that if the expected number of voters were to go to infinity, then the limit distribution would be normal (which, as shown below, leads to a contradiction). In contrast the Poisson limit distribution in Lemma 2 is compatible with strictly positive voting costs.}
the value 0 otherwise. Thus, we get

\[
\lim_{N_A, N_B \to \infty} \left( \frac{C_N}{N_A \var{Y_i^{N_A}} + N_B \var{Z_i^{N_B}}} \right)^{1+\delta/2} \\
\leq \lim_{N_A \to \infty} \sum_{i=1}^{N_A} E[|Y_i^{N_A} - F(c_A(N))|^{2+\delta}] + \lim_{N_B \to \infty} \sum_{i=1}^{N_B} E[|Z_i^{N_B} - F(c_B(N))|^{2+\delta}] \\
= \lim_{N_A \to \infty} \frac{N_A F(c_A(N))(1 - F(c_A(N))) \left[(1 - F(c_A(N)))^{1+\delta} + F(c_A(N))^{1+\delta}\right]}{\left[N_A F(c_A(N))(1 - F(c_A(N)))\right]^{1+\delta/2}} \\
+ \lim_{N_B \to \infty} \left[N_B F(c_B(N))(1 - F(c_B(N)))\right]^{1+\delta/2} \\
= \frac{(1 - F(c_A(N)))^{1+\delta} + F(c_A(N))^{1+\delta}}{\left[N_A F(c_A(N))(1 - F(c_A(N)))\right]^{1+\delta/2}} + \lim_{N_B \to \infty} \left[N_B F(c_B(N))(1 - F(c_B(N)))\right]^{1+\delta/2} = 0,
\]

because \(N_A F(c_A(N))\) and \(N_B F(c_B(N))\) \(\to \infty\) by claim 1. Thus, condition (22) is satisfied, proving claim 2.

In the remainder of the proof we use claim 2 to derive a contradiction. In particular, if an agent is pivotal then \(\sum_{i=1}^{N_A} Y_i^{N_A} - \sum_{i=1}^{N_B} Z_i^{N_B} \in \{-1, 0, 1\}\). However, the normalized sum (i.e., the expression in claim 2) then converges to zero because the standard deviation goes to infinity. Because the limit distribution is continuous, this implies that the probability of being pivotal converges to zero, which is incompatible with strictly positive voting costs. Formally, define

\[
b = \lim_{N \to \infty} \frac{N_A F(c_A(N)) - N_B F(c_B(N))}{\sqrt{N_A \var{Y_i^{N_A}} + N_B \var{Z_i^{N_B}}}}.
\]

Note that we allow for the possibility that \(b\) is negative or positive infinity. Furthermore, we can assume without loss of generality that the sequence converges. Otherwise, we can take a converging subsequence. Let \(\varepsilon > 0\) be arbitrary. Then there exists \(\lambda > 0\) such that

\[
\frac{1}{\sqrt{2\pi}} \int_{b-\lambda}^{b+\lambda} e^{-\frac{x^2}{2}} dx < \varepsilon.
\]

Furthermore \(\sqrt{N_A \var{Y_i^{N_A}} + N_B \var{Z_i^{N_B}}} \leq \sqrt{N_A \var{Y_i^{N_A}}} + \sqrt{N_B \var{Z_i^{N_B}}}\), where each of the summands converges to \(\infty\) because of Claim 1, i.e., because \(N_A F(c_A(N))\) and \(N_B F(c_B(N))\) converge to \(\infty\). This and (23) imply that for sufficiently large \(N\) a necessary condition for being pivotal is that

\[
b - \lambda \leq \frac{\sum_{i=1}^{N_A} Y_i^{N_A} - \sum_{i=1}^{N_B} Z_i^{N_B} - N_A F(c_A(N)) + N_B F(c_B(N))}{\sqrt{N_A \var{Y_i^{N_A}} + N_B \var{Z_i^{N_B}}}} \leq b + \lambda.
\]

Thus, (24), (25), and claim 2 imply that the probability of being pivotal is less than \(\varepsilon\). Because \(\varepsilon\) was chosen arbitrarily, the pivot probability converge to zero, for almost all realizations \(N_A\) and \(N_B\). Taking expectations over all possible realizations of \(N_A\) and \(N_B\) we can therefore conclude that the pivot probability is zero.

Now recall that voting costs \(c \geq \zeta > 0\). Because the pivot probability converges to zero, the payoff to a voter is strictly negative as \(N\) gets large. This is a contradiction, since the citizen would better off not voting.
Lemma 2 Suppose that $\varepsilon > 0$. Then the probability that candidate $A$ wins the election under voluntary voting converges to $\frac{1}{2}$ as $N \to \infty$.

Proof of Lemma 2. Let $N_A(N)$ be a sequence that satisfies (21). Define the random variables $X^A_{N_A(N),i}$ and $X^B_{N_A(N),i}$ as in Lemma 1. We first prove that $\sum_{i=1}^{N_A(N)} X^A_{N_A(N),i}$ converges to a Poisson distribution.

The expected number of $A$ voters given that there are $N_A$ supporters of $A$ is $\bar{v}_A(N|N_A) = N_A F(c_A(N))$. Thus,

$$\left(\frac{N_A(N)}{k}\right) F(c_A(N))^k (1 - F(c_A(N)))^{N_A(N) - k} = \frac{(N_A(N) - 1) \ldots (N_A(N) - k + 1)}{N_A(N)^{k-1}} \frac{\bar{v}_A(N|N_A)^k}{k!} \left(1 - \frac{\bar{v}_A(N|N_A)}{N_A(N)}\right)^{N_A(N) - k}. \tag{26}$$

Note that

$$\lim_{N \to \infty} \frac{\bar{v}_A(N|N_A)}{\bar{v}_A(N)} = \lim_{N \to \infty} \frac{N_A(N) F(c_A(N))}{\alpha NF(c_A(N))} = 1, \tag{27}$$

because $N_A(N)$ satisfies (21).

Lemma 1 implies that there exist a subsequence $\bar{v}_A(N_n)$ of $\bar{v}_A(N)$ such that $\lim_{n \to \infty} \bar{v}_A(N_n) = \bar{v}_A$. With a slight abuse of notation we denote the subsequence again by $\bar{v}_A(N)$. Thus, (26) and (27) imply

$$\lim_{N \to \infty} P\left(\{\#A-voters = k\} | \{\#A-supporters = N_A(N)\}\right) = \frac{\bar{v}_A^k}{k!} e^{-\bar{v}_A},$$

where the convergence is uniform for all sequences $N_A(N)$ that satisfy (21). Thus, (20) implies

$$\lim_{N \to \infty} \left| P\left(\{\#A-voters = k\} | N\right) - \frac{\bar{v}_A^k}{k!} e^{-\bar{v}_A}\right| < \varepsilon.$$

Because $\varepsilon > 0$ was chosen arbitrarily, it follows that

$$\lim_{N \to \infty} P\left(\{\#A-voters = k\} | N\right) = \frac{\bar{v}_A^k}{k!} e^{-\bar{v}_A}, \tag{28}$$

Similarly, $\lim_{N \to \infty} P\left(\{\#B-voters = k\} | N\right) = \frac{\bar{v}_B^k}{k!} e^{-\bar{v}_B}$. An $A$ supporter with cost $c_i$ votes if and only if

$$\frac{1}{2} P\left(\{\#A-voters - \#B-voters \in [0, -1]\} | N\right) \geq c_i, \tag{29}$$

where $P(\{\#A-voters - \#B-voters \in [0, -1]\} | N)$ is the probability that candidate $A$ gets either one less vote or the same number of votes as candidate $B$, given that there are $N$ individuals. Lemma 1 implies that for arbitrary $\varepsilon > 0$, there exists $K$ such that $P(\{\#A-voters \geq K\} | N) < \varepsilon$ and $P(\{\#B-voters \geq K\} | N) < \varepsilon$ for all sufficiently large $N$. Thus, (28) implies

$$\lim_{N \to \infty} P\left(\{\#A-voters - \#B-voters \in [0, -1]\} | N\right) = \sum_{i=0}^{\infty} \frac{\bar{v}_B^i}{i!} e^{-\bar{v}_B} \left(\frac{\bar{v}_A}{i!} e^{-\bar{v}_A} + \frac{\bar{v}_A^{i+1}}{(i+1)!} e^{-\bar{v}_A}\right), \tag{30}$$

By formula 9.6.10 in Abramowitz and Stegun [1] (see also Myerson [15]), we get

$$\lim_{N \to \infty} P\left(\{\#A-voters - \#B-voters \in [0, -1]\} | N\right) = \sqrt{\frac{\bar{v}_A}{\bar{v}_B}} \frac{I_1(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A+\bar{v}_B}} + \frac{I_0(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A+\bar{v}_B}}, \tag{31}$$

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where $I_k$ is a modified Bessel function. Similarly, the pivot probability for a $B$ supporter is

$$
\lim_{N \to \infty} P(#A\text{-voters} - #B\text{-voters} \in \{0, 1\}|N) = \frac{\sqrt{\bar{v}_A} I_1(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A + \bar{v}_B}} + \frac{I_0(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A + \bar{v}_B}}. \tag{32}
$$

As $N \to \infty$ both pivot probabilities in (31) and (32) must converge to $2\zeta$, by (29) and Lemma 1. Thus,

$$
\sqrt{\bar{v}_A \bar{v}_B} I_1(2\sqrt{\bar{v}_A \bar{v}_B}) e^{\bar{v}_A + \bar{v}_B} = \frac{\sqrt{\bar{v}_B} I_1(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A + \bar{v}_B}} + \frac{I_0(2\sqrt{\bar{v}_A \bar{v}_B})}{e^{\bar{v}_A + \bar{v}_B}}. \tag{33}
$$

However, since the Bessel function $I_1$ is never zero, (33) implies that $\bar{v}_A = \bar{v}_B$, i.e., in the limit the number of $A$ and $B$ voters are drawn from the same Poisson distribution. As a consequence, each candidate wins with probability $\frac{1}{2}$, independent of $\alpha$. ■

**Proof of Proposition 4.** Lemma 2 implies that the wrong candidate is elected with probability $0.5$ as $N \to \infty$. Thus, expected per-capita surplus under voluntary voting converges to $0.5$.

Now consider a subsidy $s = \zeta + \epsilon$. Then the number of voters goes to infinity as $N \to \infty$, because at least all citizens with costs $c \leq \zeta + \epsilon$ vote. Thus, the pivot probability goes to zero, which implies that $c_A, c_B \to \zeta + \epsilon$. As a consequence, if $\alpha \neq 0.5$, then the majority candidate wins with probability $1$ as $N \to \infty$. Thus, ex-ante expected per-capita surplus converges to $\max\{\alpha, (1 - \alpha)\} - \int_{\zeta}^{\zeta + \epsilon} c \, dF(c)$. Since $\epsilon$ can be chosen arbitrarily, the expected surplus given the subsidy exceeds the expected surplus from voluntary voting. As a consequence, $[0, 1]$ is the set of limit points of sequences $\alpha_N \in A_N$. ■

**Proof of Proposition 5.** Let $S_{N,s}(\alpha)$ be the ex-ante expected surplus, given $N$ citizens and a subsidy $s$. The proof of Proposition 4 implies that $\lim_{N \to \infty} S_{N,s}(\alpha) = \max\{\alpha, (1 - \alpha)\} - \int_{\zeta}^{\zeta + \epsilon} c \, dF(c)$, for a.e. $\alpha$. Thus, Lebesgue’s dominated convergence theorem implies $\lim_{N \to \infty} \int S_{N,s} g(\alpha) \, d\alpha = \int \max\{\alpha, (1 - \alpha)\} g(\alpha) \, d\alpha - \int_{\zeta}^{\zeta + \epsilon} c \, dF(c)$. If $s \downarrow \zeta$, we can therefore get arbitrarily close to the first best. In contrast, if $s \geq \gamma > \zeta$, then there remains a loss of at least $\int_{\zeta}^{\zeta + \gamma} c \, dF(c)$ in the limit. Similarly, if $s \leq \gamma < \zeta$, then Lemma 2 implies a strictly positive probability that the wrong candidate is elected, when $N$ is large, and expected surplus is again bounded away from the first best. Thus, $s_N \to \zeta$. ■
References


