Unimprovable Allocations in Economies with Incomplete Information

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Abstract

The paper considers an exchange economy with incomplete information in which agents can retrade goods until all gains from trade are exhausted. Unimprovable allocations are defined to be those allocations from which agents would not wish to deviate either by retrading goods or by revealing further information. The concept of unimprovability is then used to analyze a lemons market and an adverse selection insurance market in which agents can renegotiate after information has been revealed. Finally, unimprovability is compared to different concepts of efficiency and to the concept of durability. *Journal of Economic Literature* Classification Number: D82.

1   Introduction

In an exchange economy with complete information, the concept of Pareto efficiency describes all allocations for which gains from trade are exhausted. Thus, if agents obtain through trading an allocation which is Pareto efficient, agents cannot improve through further trading. What concept describes the absence of improving trades for an economy with incomplete information?

Consider an exchange economy with differentially informed agents. In the case of incomplete information, the process of obtaining an allocation and of information revelation cannot be separated. Thus, intuitively, each trade can be thought to occur in two steps. First, information is exchanged. Then agents execute trades which are contingent on the revealed information. In contrast to the case of complete information, the outcome of a trade is therefore not only characterized by the allocation of consumption goods but also by the amount of information which has been revealed. We define such an outcome as unimprovable, if agents would not want to deviate from it either by retrading or by revealing additional information.

Implicit in our concept of unimprovability is that agents cannot be forced to adhere to a prespecified outcome. In contrast, standard notions such as ex-ante, interim, or ex-post incentive efficiency assume that agents cannot retrade. This results in the following differences. In ex-ante, interim, and ex-post incentive efficient allocations one can assume without loss of generality that agents reveal all information. In contrast, complete information revelation does not necessarily arise in our solution concept. In particular, agents' ability to retrade in the presence of complete information would result in allocations which are Pareto efficient with respect to complete information, and such allocations can in general not be obtained because of incentive problems.

There is also another important difference between trade under complete and trade under incomplete information which is incorporated in our solution concept. With complete information, each trade can be seen independent of other trades. That is, in order to show that an allocation can be improved upon, it is sufficient to show that there exists an allocation which improves upon the status quo. It is irrelevant whether or not the new trades and new allocations themselves can be improved upon any further. With incomplete information trades cannot be considered independent of each other as they affect agents' decisions about revealing information.

For example, consider an adverse selection problem in an insurance mar-

ket where the insurer offers a contract which separates the insured agents by their types (low and high risk). Typically, separation of types is possible if the low-risk agents receive only partial insurance. Now consider a second trade which offers complete insurance to the low-risk types given that all information has been revealed. If the first trade is followed by the second one, then a high-risk agent will always pretend to be a low-risk agent. In contrast, if only the first decision rule is considered, separation of types is possible.

In the context of collective choice problems, Green and Laffont [14] and Forges [12] discuss the effects of recontracting. Apart from considering a differential information economy, the concept in this paper differs, by making agents' decisions on information revelation and retrading contingent on their expected further retrades.[1] In order to do this, we specify agents' expected final allocations and revealed information given a particular status quo (characterized by an allocation and the amount of information revealed through previous trades). This expectation function must be consistent, that is, it should not be possible for agents to improve upon any expected final allocations. We define an allocation together with the publicly revealed informed to be unimprovable if it can be obtained starting from agents' initial endowments.

In this paper after introducing the concept of unimprovable allocations, we provide a general existence result. Next, we characterize unimprovable allocations for a lemons market and for an insurance problem with adverse selection. We then compare our solution concept to interim (incentive) efficiency and durability as defined in Holmström and Myerson [17]. Finally, in the concluding remarks we discuss the relationship to the core with differential information of Yannelis [30] and the core of Allen [3] and Vohra [28].

Our solution concept combines cooperative and non-cooperative features. Specifically, information revelation follows a non-cooperative game. In contrast, the choices on alternative trades and information revelation are made cooperatively by all agents. Thus, there are some relationships between the concepts introduced in this paper and those which can be found in the literature on cooperative game theory and its application to economies with differential information.

First, our expectation functions are related to the standards of behavior

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[1]Forges [11] also considers differential information economies, but in contrast to this paper agents do not take future retrades into account.

defined in Greenberg [15]. A standard of behavior defines for each position of a game—in our model a position corresponds to an allocation and to the amount of publicly revealed information—a final position a coalition of agents will obtain. In the language of Greenberg, our definition of unimprovability (Definition 2) then corresponds to a particular consistency requirement for standards of behavior. Similar modeling is used in Chwe [8] to show that in the resulting solution concept agents are forward looking in contrast to the stable set.

In the literature on cooperative solution concepts with incomplete information authors either impose restrictions on how information is shared by coalitions of agents (see Allen [2], Berliant [6], Hahn and Yannelis [16], Koutsougeras [22], Koutsougeras and Yannelis [23], Krasa and Yannelis [24], Wilson [29], Yannelis [30]) or they impose incentive compatibility restrictions on the allocations a coalition of agents can obtain (see Allen [3], Boyd, Prescott and Smith [7], Ichiishi and Sertel [20], Kahn and Mookherjee [21], Lacker and Weinberg [25], Ichiishi and Idzik [19], Vohra [28]) This paper differs from the two approaches by characterizing the outcome of trading not only by an allocation of consumption goods but also by the amount of information which is publicly revealed. Thus, similar to the first group of papers arbitrary new trades which are solely based on publicly revealed information are admissible and not subject to any further incentive constraints. In contrast to the first group of papers but similar to the second group, trades which reveal additional information are subject to standard incentive compatibility restrictions. However, incentive compatibility is defined with respect to the information which has already been revealed. Moreover, unlike the first group of papers, the amount of information revealed by agents is endogenous in our concept.[2]

Finally, there is also a relationship between unimprovability and the concept of renegotiation proofness used in the literature on contracts with incomplete information (see Dewatripont and Maskin [10] for a survey of this literature). In principal-agent problems with incomplete information, contracts which are ex-ante (incentive) efficient can typically be improved upon once agents' information is revealed.[3] This is the case because any new contract

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[2]In the first literature, measurability restrictions on allocations or net-trades (for example, measurability with respect to private information) are assumed exogenously. Thus, measurability is used instead of incentive compatibility to study the effects of incomplete information.

[3]In most papers in the literature renegotiation occurs after agents have entered the

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only needs to be incentive compatible with respect to the information which has already been publicly revealed. Hence the incentive constraint becomes less binding as time goes on. In the principal-agent literature, renegotiation is typically modeled as a non-cooperative game where one party makes a take-it-or-leave-it offer.[4] A contract is renegotiation proof if no such renegotiation offer is made. Given that the motivation for our paper is to describe the result of trading at the interim (i.e., when agents are already differentially informed), unimprovability can be used as a renegotiation proofness constraint. That is, if a contract is specified ex-ante subject to the constraint that the allocation and publicly revealed information at the interim is unimprovable, then agents will not find it beneficial to renegotiate the contract at the interim. There are some obvious differences to the standard version of renegotiation proofness. First, our concept is based on cooperative and on non-cooperative behavior. Secondly, in our model agents takes into account that any renegotiated contract itself can be renegotiated further.[5] Thirdly, unimprovability can be used as a concept of renegotiation proofness in contracting problems with more than two agents or more general information structures, where analyzing a purely non-cooperative renegotiation game might not be tractable.

2   The Model

We now introduce the model of an exchange economy with differential information (c.f., Radner [26]). There is a finite number of agents described by $I = \{1, \ldots, n\}$. Each agent i's consumption set is given by X^i.

It should be noted that as in Gale [13] the consumption sets can be rather general. For example, X^i can be itself a set of contracts agents can enter in. Agents will then be able to retrade contracts in X^i as long as other agents are willing to accept the trades, but they are not able to subsequently alter the form of a contract $x^i \in X^i$ which they obtain as the result of trading. Therefore our model does not solely describe economies where agents are unable to commit abstaining from retrades. Rather, through an appropriate definition

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contract. An alternative approach where contract renegotiation occurs before contracts are entered but after information is revealed is considered in Asheim and Nilssen [4].

[4]For a non-cooperative renegotiation game with multiple rounds see for example Beaudry and Poitevin [5].

[5]That is, they are farsighted as in Chwe [8].

of the consumption sets, economies where agents are able to commit not to change certain types of contracts can be accommodated.[6]

There is uncertainty over the state of the economy. This is described by the probability space $(\Omega, \mathcal{A}, \mu)$. In order to simplify the exposition, we will assume throughout the paper that Ω is finite and that $\mu(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Agents are differentially informed about the state. In particular, each agent i's information is given by $\mathcal{F}^i$, a partition of Ω. That is, each agent i knows the element of her partition which contains the true state of nature. If an agent i receives additional information $\mathcal{G}$ then her information is given by $\mathcal{F}^i \vee \mathcal{G}$ which is the partition generated by $\mathcal{F}^i$ and $\mathcal{G}$.[7]

Throughout the paper we assume that all information revelation is public. That is, all agents refine their own information partition $\mathcal{F}^i$ with the same partition $\mathcal{G}$, which describes the publicly revealed information. The extension of the analysis of this paper to economies where exchange of information can also be restricted to coalitions $S \subset I$ of agents (i.e., where private side deals are possible) is left for future research.

Agent i's utility function is given by $u^i \colon \Omega \times X^i \to I\!\!R$. The agent's endowment is given by $e^i \colon \Omega \to X^i$. Agents know their endowment, i.e., e^i is $\mathcal{F}_i$-measurable.

In summary, an *economy with differential information* is given by

$$\{X^i, u^i, e^i, \Omega, \mathcal{F}^i, \mu \mid i \in I\},$$

where
(i) X^i is agent i's consumption set;
(ii) $u^i \colon \Omega \times X^i \to I\!\!R$ is agent i's von Neumann Morgenstern utility function;
(iii) $e^i \colon \Omega \to X^i$ is agent i's endowment, which is $\mathcal{F}_i$-measurable;
(iv) Ω is the set of states of nature;
(v) $\mathcal{F}^i$ is agent i's private information, a partition of Ω;
(vi) μ is the agents' common prior ex ante, a probability on Ω.

3   Unimprovability

In order to find a concept of unimprovability for incomplete information, the following questions must be addressed.

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[6] See for example, Section 5.2, where X^i will be a set of insurance contracts.
[7] Formally, this is the coarsest partition of Ω which contains both $\mathcal{F}^i$ and $\mathcal{G}$.

(A1) What is the set of allocations that can be obtained given agents' information?

(A2) What does it mean that an alternative allocation and further information revelation improves upon the status quo? Moreover, how can such an allocation be characterized?

(A3) How do agents evaluate alternative allocations that can be improved upon further?

Questions (A1)–(A3) summarize the main difficulties which arise when one analyzes trade under incomplete information.

In Section 3.1, we address (A1), i.e., we characterize incentive compatible decision rules. The differences to the incentive compatibility notion which is used for example in the definition of ex-ante (incentive) efficiency are that agents do not necessarily reveal all their information, that agents update their utility with respect to publicly revealed information rather than using ex-ante utility, and that information revelation may be sequential. (A2) is addressed in Section 3.2. Finally, (A3) is addressed in Section 3.3 and the definition of unimprovability is provided.

3.1 Incentive Compatibility

As mentioned above, in economies with differential information trade and information revelation cannot be treated independent of each other. Thus, we imagine that agents first reveal some of their private information. Then trades are executed based on this information. Further information revelation and trades can be thought of in exactly the same way.

A status quo in our economy will be described by a feasible allocation x^i, $i \in I$ and by the amount of information $\mathcal{G}$ which has been publicly revealed through trading.[8] We now describe the conditions under which it is possible to obtain an allocation $\hat{x}^i$, $i \in I$ and information[9] $\hat{\mathcal{G}} \geq \mathcal{G}$ given that the status quo is given by x^i, $i \in I$ and $\mathcal{G}$.

In this paper we assume that trades are publicly observable. Thus, the net trades $\hat{x}^i - x^i$ must be $\hat{\mathcal{G}}$-measurable for all agents $i \in I$. Truthful revelation of information then means that each agent reports an event in Ω which contains

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[8]For example, x^i, $i \in I$ and $\mathcal{G}$ trivial, corresponds to the situation at which trading starts. Recall that in addition to $\mathcal{G}$, agents also have their private information $\mathcal{F}^i$.

[9]For two information partitions $\mathcal{F}$, $\mathcal{G}$ we write $\mathcal{F} \geq \mathcal{G}$ if $\mathcal{F}$ is weakly finer than $\mathcal{G}$. Similar, $\mathcal{F} > \mathcal{G}$ means that $\mathcal{F} \geq \mathcal{G}$ but $\mathcal{F} \neq \mathcal{G}$.

the true state of nature. For example, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Then reporting $\{\omega_1, \omega_2\}$ whenever ω_1 or ω_2 has occurred, and reporting $\{\omega_3\}$ whenever ω_3 has occurred would be a truthful report. If the agent has complete information then her report would be truthful but not fully revealing.

Is it possible to reveal arbitrary information partitions $\hat{\mathcal{G}}$ through the simultaneous reports which are usually considered in information revelation games? The answer is no. Consider the following example.

Example 1. Assume there are two agents $i = 1, 2$ and four states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The information sets are given by $\mathcal{F}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, and $\mathcal{F}^2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$. Now assume that no information has been publicly revealed yet, i.e., $\mathcal{G}$ is trivial. The agents want to reveal $\hat{\mathcal{G}} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}\}$.

$\hat{\mathcal{G}}$ can clearly not be revealed if only one agent announces information. In particular, if only one of the agents, say agent 1, reveals information then $\mathcal{F}^1$ will be publicly revealed. Thus both agents must announce information. However, if both agents announce information simultaneously then full information revelation will be the result. Consequently, announcements must be sequential in order to obtain $\hat{\mathcal{G}}$.

Information $\hat{\mathcal{G}}$ can be revealed as follows. First agent 1 announces $\mathcal{F}^1$, i.e., depending on the state either $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$ is reported. As a result, agent 2 will have full information. Thus, agent 2 can report $\hat{\mathcal{G}}$. Specifically, the agent reports either $\{\omega_1\}$, or $\{\omega_2\}$ or $\{\omega_3, \omega_4\}$, depending on the state.

Sequential information revelation will naturally occur in our model. For example, assume that starting from their endowment agents reveal information $\mathcal{G}$ and obtain an allocation x^i, $i \in I$. Further information revelation then leads to $\hat{\mathcal{G}} \geq \mathcal{G}$ and retrading to $\hat{x}^i$, $i \in I$. As Example 1 indicates such sequential information revelation can lead to information sets $\hat{\mathcal{G}}$ which cannot be obtained through a single round of information revelation.

We now specify the structure of the information revelation game. The sequential information revelation game can go over $t = 1, \ldots, T$ rounds. In each round t, agents report elements in a partition M_t^i of Ω. These partitions become (weakly) finer i.e., $M_t^i \geq M_{t-1}^i$ for every $i \in I$ and for every $t \in T$. Finally, after T rounds information $\hat{\mathcal{G}}$ is completely revealed, i.e., $\hat{\mathcal{G}} = \bigvee_{i \in I} M_T^i$. We will say that the allocation $\hat{x}^i$, $i \in I$ is incentive compatible given x^i, $i \in I$, $\mathcal{G}$ if announcing truthfully is optimal in each round, where optimal means that the announcement strategies are a perfect Bayesian Nash

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equilibrium.[10] Below we provide a formal description of the game and of the strategies.

For given x^i, $i \in I$, $\mathcal{G}$, a sequential information revelation game is given by $\{M_t^i, \Sigma_t^i, \mathcal{F}_i \vee \mathcal{G}, \Omega, \mu, z^i, \pi^i \mid i \in I\}$, where

(1) M_t^i is agent i's action space at t, a partition of Ω with $M_t^i \leq M_{t+1}^i$ and $\bigvee_{i \in I} M_T^i = \hat{\mathcal{G}}$;

(2) Σ_t^i is the set of all pure strategies $\sigma_t^i \colon \mathcal{F}^i \vee \mathcal{G} \times \prod_{k<t,j\in I} M_k^j \to M_t^i$;

(3) $\mathcal{F}_i \vee \mathcal{G}$ is agent i's information, a partition of Ω, and μ is agents' prior on Ω;

(4) $z^i \colon \Omega \times \prod_{k\leq T,j\in I} M_k^j \to \mathbb{R}^l$ is agent i's net trade, where $z^i(\cdot, m)$ is $\mathcal{G}$-measurable;

(5) $\pi^i \colon \Omega \times \prod_{k\leq T,j\in I} M_k^j \to \mathbb{R}$, defined by $\pi^i(\omega, m) = u^i(\omega, x^i(\omega) + z^i(\omega, m))$ is agent i's payoff function.[11]

Definition 1. *$\hat{x}^i$, $i \in I$, $\hat{\mathcal{G}}$ is **incentive compatible** given x^i, $i \in I$, $\mathcal{G}$ if and only if there exists a sequential information revelation game with a perfect Bayesian Nash equilibrium in which agents report truthfully[12] and where $x^i(\omega) + z^i(\omega, \sigma(\omega)) = \hat{x}^i(\omega)$, for the equilibrium strategies σ.*

Remark. Since agents use pure strategies in the above game, the assumption of truthful revelation of information is without loss of generality.

3.2   Trade and Information Revelation

We now address the question what it means for an alternative allocation $\hat{x}^i$, $i \in I$, $\hat{\mathcal{G}}$ to improve upon the status quo (i.e, upon the current allocation x^i, $i \in I$, $\mathcal{G}$.). In order to obtain the alternative allocation, agents must first reveal information $\hat{\mathcal{G}}$ (Of course, if $\hat{\mathcal{G}} = \mathcal{G}$ no further information revelation is necessary). Each agent i then evaluates $\hat{x}^i$, $i \in I$ with respect to expected utility conditioned on information $\mathcal{F}^i \vee \hat{\mathcal{G}}$. Thus, if

$$E\big(u^i(\omega, \hat{x}^i) \mid \mathcal{F}^i \vee \hat{\mathcal{G}})\big)(\bar{\omega}) \geq E\big(u^i(\omega, x^i) \mid \mathcal{F}^i \vee \hat{\mathcal{G}})\big)(\bar{\omega}), \tag{1}$$

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[10] In a perfect Bayesian Nash equilibrium agents strategies are a Nash equilibrium in all states and time periods given that beliefs are derived using Bayes' rule whenever possible.

[11] In order for $x^i + z^i \in X^i$, we must assume as in Hurwicz, Maskin and Postlewaite [18] that agents cannot over report their endowments. For simplicity of exposition we abstract from this issue.

[12] As mentioned above, a truthful report means that each agent i reports the element of the partition M_t^i that contains the true state of nature.

for all $i \in I$ and for all $\bar{\omega} \in \Omega$, all agents are weakly better off by trading from the status quo to the new allocation. If the strict inequality holds for an agent i with positive probability then agent i is strictly better off.

However, what happens if $\hat{x}^i$, $\hat{\mathcal{G}}$ makes the agents only better off in some but not all states, i.e., if inequality (1) holds only for some but not for all states? Then if the set of states where (1) holds is not common knowledge information, additional information is revealed. This new information must be taken into account when evaluating the alternative allocation. In order to illustrate this point, consider the following example.[13]

Example 2. Assume there are two agents $i = 1, 2$. There are two consumption goods per state and two states of nature $\Omega = \{0, 1\}$ which occur with the same probability. Both agents' utility functions are given by $u(\omega, x_1, x_2) = \omega x_1 + x_2$. Assume that agent 1 has full information, i.e., the agent knows the true state of the economy. On the other hand, agent 2 has no information. Thus, we have specified a "lemons problem" in which good 1 is either worthless or a perfect substitute for good 2. The value of good 1 is known by agent 1 but not by agent 2. Assume the current (state-independent) allocation is $x^1 = (4, 0)$, $x^2 = (0, 4)$ and no public information is revealed. The agents consider whether or not they should switch to $y^1 = (1, 1)$, $y^2 = (3, 3)$.

Recall that agent 1 knows the true state. Thus, if $\omega = 0$ agent 1 will agree to change since $u(\omega, x^1) < u(\omega, y^1)$. In contrast, if $\omega = 1$ then $u(\omega, x^1) > u(\omega, y^1)$ and agent 1 will choose to stay with the original allocation. On the other hand, agent 2 does not know the state and therefore puts equal probabilities on states 1 and 2. As a consequence, agent 2 is better off under the alternative allocation. Does this mean that the alternative allocation will be adopted in state $\omega = 0$?

The alternative allocation would be adopted if agent 2 behaves naively and does not take into account that agent 1 will only switch to the alternative allocation in state 0. However, in state 0, agent 2 is better off under the original allocation. Thus, if agent 2 is rational, the alternative allocation would not be adopted and hence does not block the original allocation.

How do agents evaluate alternative allocations which are adopted in some

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[13]See also Ichiishi and Sertel [20] for an example in which switching to an alternative decision rule reveals information. However, in their model this information revelation does not matter.

but not all states? Assume that an alternative allocation $\hat{x}^i$, $i \in I$, $\hat{\mathcal{G}}$ is adopted in states $\omega \in A$. In all states $\omega \notin A$, agents stay with the original allocation x^i. Agents' consumption is then given by

$$y^i(\omega) = \begin{cases} \hat{x}^i(\omega) & \text{if } \omega \in A; \\ x^i(\omega) & \text{otherwise.} \end{cases} \tag{2}$$

Clearly, the net trades $y^i - x^i$ are $\mathcal{H} = \hat{\mathcal{G}} \vee \{A, A^c\}$-measurable. Thus, the problem of deciding whether $\hat{x}^i$, $i \in I$, $\hat{\mathcal{G}}$ improves in some states but not all states upon the status quo, can be replaced by the decision whether or not to obtain y^i, $i \in I$ and to reveal information $\mathcal{H}$. Therefore $\hat{x}^i$, $i \in I$ will only be adopted if there exists an A which occurs with positive probability such that y, defined by equation (2), fulfills condition (1). However, this means that using (2) we can restrict ourselves without loss of generality to allocations which are weakly better in all states.[14]

Our solution concept allows agents to trade until no further improvements can be made. However, it can easily be the case that in a sequence of trades each trade improves upon the previous allocation but that the final allocation is not necessarily better in all states for all agents than the original allocation. That is, consider allocations x_k^i, $i \in I$, $\mathcal{G}_k$, $k = 1, 2, 3$, where each allocation improves upon the previous one, i.e., $E(u^i(\omega, x_k^i) \mid \mathcal{F}_i \vee \mathcal{G}_k) \geq E(u^i(\omega, x_{k-1}^i) \mid \mathcal{F}_i \vee \mathcal{G}_k)$. Then in general, it need not be the case that $E(u^i(\omega, x_3^i) \mid \mathcal{F}_i \vee \mathcal{G}_3) \geq E(u^i(\omega, x_1^i) \mid \mathcal{F}_i \vee \mathcal{G}_3)$.[15] Thus, in order to get a notion of "improving" which is consistent with the our solution concept we say that y^i, $i \in I$, $\mathcal{H}$ improves upon the status quo x^i, $i \in I$, $\mathcal{G}$ if y^i, $i \in I$, $\mathcal{H}$ can be obtained by executing two trades, where the first trade does not reveal any new information. Both trades (weakly) increase all agents' expected utilities.[16] We now provide the formal definition.

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[14]Consider how this argument applies for Example 2. For any given A we use (2) to define the corresponding allocation. In order to ensure that (1) holds for agent 1 we must choose $A = \{0\}$. However, given this choice of A, condition (1) does not hold for agent 2. Thus, the alternative allocation will not be adopted as we have already indicated in Example 2.

[15]Note that the law of iterated expectations only implies that $E(u^i(\omega, x_3^i) \mid \mathcal{F}_i \vee \mathcal{G}_1) \geq E(u^i(\omega, x_1^i) \mid \mathcal{F}_i \vee \mathcal{G}_1)$.

[16]Note that if each agent i's utility function is $\mathcal{F}_i$-measurable, then each agent i knows the utility $u^i(\omega, x^i)$. Thus, y^i, $i \in I$, $\mathcal{G}$ improves upon the status quo if and only if $u^i(\omega, y^i) \geq u^i(\omega, x^i)$ for all $\omega \in \Omega$, $i \in I$. However, knowing the utility function rules out many interesting cases of adverse selection.

Definition 2. y^i, $i \in I$, $\mathcal{H}$ **improves upon** x^i, $i \in I$, $\mathcal{G}$ if and only if $\mathcal{H} \geq \mathcal{G}$ and there exist a feasible allocation $\hat{x}^i$, $i \in I$ where $\hat{x}^i - x^i$ is $\mathcal{G} - measurable$ for all $i \in I$ such that $E(u^i(\omega, y^i) \mid \mathcal{F}_i \vee \mathcal{H})(\bar{\omega}) \geq E(u^i(\omega, \hat{x}^i) \mid \mathcal{F}_i \vee \mathcal{H})(\bar{\omega})$, and $E(u^i(\omega, \hat{x}^i) \mid \mathcal{F}_i \vee \mathcal{G})(\bar{\omega}) \geq E(u^i(\omega, x^i) \mid \mathcal{F}_i \vee \mathcal{G})(\bar{\omega})$, for all $i \in I$, $\bar{\omega} \in \Omega$.

y^i, $i \in I$, $\mathcal{H}$ **strictly improves upon** x^i, $i \in I$, $\mathcal{G}$ if and only if one of the above inequalities is strict.

3.3 Unimprovable Allocations

Let x^i, $i \in I$ be an allocation and let $\mathcal{G}$ denote the information which has been revealed. Assume that agents decide whether or not to obtain allocation $\hat{x}^i$, $i \in I$ through trading and to reveal information $\hat{\mathcal{G}} > \mathcal{G}$ in the process. However, unless $\hat{x}^i$, $i \in I$ is itself an allocation which cannot be improved upon, agents will expect further trades to occur. This will have an impact on incentive compatibility. For example, assume agents expect that after $\hat{x}^i$, $i \in I$, $\hat{\mathcal{G}}$ further trades will finally lead to y^i, $i \in I$, $\mathcal{H}$. Then agents' decision whether or not to report $\hat{\mathcal{G}}$ truthfully will depend on y^i, $i \in I$, rather than on $\hat{x}^i$, $i \in I$.

Thus, in order to have a consistent notion of incentive compatibility, agents must have for each allocation x^i, $i \in I$, $\mathcal{G}$ a correct expectation of the final allocation they will obtain trough further trades.

Thus, let $C = \{((x^1, \ldots, x^n), \mathcal{G}) \mid \sum_{i \in I} x^i = \sum_{i \in I} e^i$, and $\mathcal{G}$ is a partition of $\Omega\}$. Then the expected final allocation and information is given by the function $\psi: C \to C$. Of course, we need a consistency condition on ψ which ensures that all allocations $\psi(x, \mathcal{G})$ are in fact final, i.e., that agents cannot improve upon them any further.

It should be noted that ψ is related to the "standards of behavior" defined in Greenberg [15]. A standard of behavior describes what happens if agents start at a particular "position" in a game. A position in our model corresponds to an allocation and the amount of publicly revealed information.

Definition 3. ψ is **consistent** if and only if the following conditions hold.
(C1) $(\tilde{x}, \tilde{\mathcal{G}}) = \psi(x, \mathcal{G})$ is incentive compatible given $(x, \mathcal{G})$; $\tilde{\mathcal{G}} \geq \mathcal{G}$; and $\tilde{x}^i - x^i$ is $\tilde{\mathcal{G}}$-measurable for every $i \in I$ and for all $(x, \mathcal{G}) \in C$.
(C2) $\psi(x, \mathcal{G})$ improves upon $(x, \mathcal{G})$, for all $(x, \mathcal{G}) \in C$.
(C3) $\psi(\psi(x, \mathcal{G})) = \psi(x, \mathcal{G})$ for all $(x, \mathcal{G}) \in C$.
(C4) Let $(\tilde{x}, \tilde{\mathcal{G}}) = \psi(x, \mathcal{G})$, where $(x, \mathcal{G}) \in C$ is arbitrary. Then the following does not hold:

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There exists $(y, \mathcal{H}) \in C$ with $\mathcal{H} \geq \tilde{\mathcal{G}}$ such that $(\tilde{y}, \tilde{\mathcal{H}}) = \psi(y, \mathcal{H})$ is incentive compatible given $(\tilde{x}, \tilde{\mathcal{G}})$ and strictly improves upon $(\tilde{x}, \tilde{\mathcal{G}})$.

(C1) indicates that information $\tilde{\mathcal{G}}$ can be revealed in an incentive compatible way. Condition (C2) specifies that agents will not trade unless they expect to improve as a consequence. (C3) is the requirement that no further trade occurs once a final allocation and the final amount of information is obtained. (C4) specifies that agents do not expect to be able to improve upon a final allocation either by reallocating goods or by revealing more information.

Remark. In (C4) we only excluded the possibility that the grand coalition can improve upon a final outcome $(\tilde{x}, \tilde{\mathcal{G}}) = \psi(x, \mathcal{G})$. Would it be nevertheless possible for some coalition $S \subset I$ to improve themselves through further trading and information revelation?

The answer is no. In particular, assume by way of contradiction that there exists a coalition S which can strictly improve upon $(\tilde{x}, \tilde{\mathcal{G}})$. Note that this improvements are posterior, i.e., coalition S can obtain allocations y_i, $i \in S$ with $\sum_{i \in S} y_i = \sum_{i \in S} \tilde{x}_i$ rather than those for which $\sum_{i \in S} y_i = \sum_{i \in S} e_i$. Let $y_i = \tilde{x}_i$ for all $i \notin S$. Let $\mathcal{H}$ be the information necessary to obtain allocation y. Then agents will expect $(\tilde{y}, \tilde{\mathcal{H}}) = \psi(y, \mathcal{H})$ to occur. Now (C2) implies that $(\tilde{y}, \tilde{\mathcal{H}})$ improves upon $(y, \mathcal{H})$ for all $s \in S$. However, since $y_i = \tilde{x}_i$ for all $i \notin S$, it follows that $(\tilde{y}, \tilde{\mathcal{H}})$ strictly improves upon $(\tilde{x}, \tilde{\mathcal{G}})$ for all agents $i \in I$. This violates $(C4)$, a contradiction.

Definition 4. *x^i, $i \in I$, $\mathcal{G}$ is **unimprovable** if and only if there exists a consistent ψ such that $\psi(e^1, \ldots, e^n, \mathcal{H}) = ((x^1, \ldots, x^n), \mathcal{G})$, where $\mathcal{H} = \{\{\Omega\}\}$ is the trivial information set.*

Note that when trading starts each agent i has only her private information $\mathcal{F}^i$ and public information is trivial. This is the reason why $\mathcal{H} = \{\{\Omega\}\}$ is chosen in Definition 3 to describe those unimprovable allocations agents can obtain given their initial endowment of information.

4 Existence of Unimprovable Allocations

Theorem 1. *Let $\mathcal{E}$ be a differential information economy. Assume that Ω is finite, that each agent's utility function $u^i(\omega, \cdot)$ is continuous for all fixed*

$\omega \in \Omega$, and that X^i is a closed subset of $\mathbb{R}_+^l$.[17] *Then unimprovable allocations exist.*

Corollary 1 below is useful for finding unimprovable allocations. Corollary 1 indicates that one can always restrict attention to $(x, \mathcal{G}) \in C$ that cannot be strictly improved upon by some other allocation $(y, \mathcal{G}) \in C$. Thus, let $\hat{C} = \{(x, \mathcal{G}) \in C \mid$ there does not exist $(y, \mathcal{G}) \in C$ which strictly improves upon $(x, \mathcal{G})\}$. Moreover, it is sufficient to find ψ on a subset of $\hat{C}$. In particular, let $\hat{C}_{(x,\mathcal{G})} = \{(\tilde{x}, \tilde{\mathcal{G}}) \in \hat{C} \mid E(u^i(\omega, \tilde{x}^i) \mid \mathcal{F}_i \vee \mathcal{G}) \geq E(u^i(\omega, x^i) \mid \mathcal{F}_i \vee \mathcal{G})$, where at least one inequality is strict, and $\tilde{\mathcal{G}} \geq \mathcal{G}\}$.

For example, if $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{G} = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, and if $(x, \mathcal{G}) \in \hat{C}$, then $\hat{C}_{(x,\mathcal{G})}$ consists of all allocations that are complete information Pareto efficient and dominate $(x, \mathcal{G})$.[18] Moreover, if $(x, \mathcal{G})$ is complete information Pareto efficient then $\hat{C}_{(x,\mathcal{G})} = \emptyset$. Thus, Corollary 1 immediately implies Corollary 2.

Corollary 1. *Let $(x, \mathcal{G}) \in \hat{C}$ improve upon and be incentive compatible with respect to $(e, \{\{\Omega\}\})$. Then $(x, \mathcal{G})$ is unimprovable if and only if $(x, \mathcal{G}) \in \hat{C}$ and there exists $\psi \colon \hat{C}_{(x,\mathcal{G})} \to \hat{C}_{(x,\mathcal{G})}$ that fulfills (C1)–(C4) on $\hat{C}_{(x,\mathcal{G})}$ such that no element of $\psi(\hat{C}_{(x,\mathcal{G})})$ strictly improves upon and is incentive compatible with respect to $(x, \mathcal{G})$.*

Corollary 2. *All complete information Pareto efficient, individually rational, and incentive compatible allocations are unimprovable.*

The proof of Theorem 1 follows from a backward induction argument over the amount of public information $\mathcal{G}$. One starts with complete information. Allocations must then be complete information Pareto efficient. One then constructs inductively a consistent ψ for coarser information sets. The backward induction argument also implies that in an application, ψ does not need to be constructed on all of C, as Corollaries 1 indicates. Moreover, Corollary 1 implies that unimprovable allocations cannot be dominated by any other allocation which can be obtained by using the publicly available information.

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[17]Note that under the assumptions of Yannelis [30], X^i can also be a subset of an infinite dimensional space.

[18]This is the case since no allocation $(x', \mathcal{G})$ can dominate $(x, \mathcal{G})$ if $(x, \mathcal{G}) \in \hat{C}$.

5 Examples of Unimprovable Allocations

5.1 A Lemons Market

Assume there are two agents referred to as the buyer B and the seller S. There are two goods: a car x, whose quality is only known to the seller, and money m. Assume that the agents' utility functions are given by

$$u_B(\omega, x, m) = \begin{cases} 6x + m & \text{if } \omega = \omega_1; \\ 3x + m & \text{if } \omega = \omega_2; \end{cases} \qquad u_S(\omega, x, m) = \begin{cases} 5x + m & \text{if } \omega = \omega_1; \\ 2x + m & \text{if } \omega = \omega_2; \end{cases}$$

where $\omega \in \{\omega_1, \omega_2\}$ describes the quality of the car. Each of the two states occurs with probability 0.5. $\mathcal{F}_S = \{\{\omega_1\}, \{\omega_2\}\}$ and $\mathcal{F}_B = \{\{\omega_1, \omega_2\}\}$. For simplicity assume that x is indivisible. The seller is endowed with one car. The buyer is endowed only with money.

Note that this economy corresponds to a classic lemons market, in which trade would be Pareto efficient in both states. However, since the state is private information, the seller has always the incentive of claiming that the car is of good quality. Akerlof [1] argues that in such a framework only lemons will be traded.

What allocations in this model are unimprovable? Given that there are just two states, either $\mathcal{G} = \{\{\omega_1\}, \{\omega_2\}\}$ or $\mathcal{G} = \{\{\omega_1, \omega_2\}\}$. First, assume that there exists an unimprovable allocation for which $\mathcal{G} = \{\{\omega_1\}, \{\omega_2\}\}$. Then trade would have to occur in both states. In particular, the buyer would have to pay $5 \leq m \leq 6$ units of money if $\omega = \omega_1$, and $2 \leq m \leq 3$ if $\omega = \omega_2$. The resulting allocation, however, is not incentive compatible, as the seller will always have the incentive to misreport the quality of the car.

Thus, $\mathcal{G} = \{\{\omega_1, \omega_2\}\}$ in all unimprovable allocations (since unimprovable allocations exist by Theorem 1). Then because net trades are $\mathcal{G}$-measurable, they must be state independent. Thus, the buyer's expected utility is $4.5x + m$. However, this means that the buyer is willing to pay at most 4.5 units of money for the car. Consequently, the seller will not be willing to trade if $\omega = \omega_1$. Thus, no trade will occur, i.e., neither lemons nor good quality cars will be sold.

What explains the difference between this result and that of the competitive model used by Akerlof? In the competitive model a price is realized at which only owners of lemons sell their car. However, this means that after trade has occurred, sellers who did not trade are known to have a high quality car. Such an outcome can be improved upon, that is, trade of high quality

cars should now take place. Thus, sellers of lemons have the incentive not to offer their car, expecting that a buyer will believe the car to be of good quality. Our solution concepts takes this effect into account and it is the reason why markets break down completely.

We now show that it is possible for some trade to occur, if there are more than two types of cars. In particular, assume that there are three states $\Omega = \{\omega_1, \omega_2, \omega_3\}$ which occur with the same probability. The agents' utility functions for $\omega = \omega_3$ are given by

$$u_B(\omega, x, m) = 2x + m, \qquad u_S(\omega, x, m) = x + m.$$

In states $\omega = \omega_1, \omega_2$ the utilities are the same as above.

We show that there exists an unimprovable allocation in which only cars of the lowest quality are traded. Thus, let $\mathcal{G} = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, i.e., the seller either announces the car to be of the lowest quality, or to be of medium or high quality. If $\omega = \omega_1$, the buyer pays $1 \leq m \leq 2$ units of money for the car. If $\omega \neq \omega_1$ no trade takes place. It is now easy to see that the resulting allocation is unimprovable. First, if $\omega = \omega_1$, no further improving trade exist. Moreover, if $\omega \in \{\omega_2, \omega_3\}$ then according to the buyer's updated prior each state occurs with probability 0.5. Thus, as we have shown above, only no trade is unimprovable. Using Corollary 1, we can therefore conclude that we found an unimprovable allocation in which only cars of the lowest quality are traded.

Our concept applied to the lemons problem therefore has two predictions. First, as in Akerlof [1] only low quality cars are traded. However, unlike Akerlof's result we need more than two "qualities" in order to prevent full information revelation ex-post.

5.2 Insurance with Adverse Selection

We now apply our general model and the solution concept to insurance markets with adverse selection. That is, we will consider a risk neutral agent who can insure some risk averse agents against an accident (a low endowment realization). In this application, the agents' consumption set will be the set of all insurance contracts.[19] In this environment, unimprovability can be

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[19]Moral hazard can also be introduced in this model. In this case, the consumption set would consist of all insurance contracts in which the insured agent chooses an incentive compatible effort level.

used as a renegotiation proofness constraint. In particular, as in Asheim and Nilssen [4] once a contract is agreed upon and before it is known whether an accident has occurred, agents can renegotiate. Moreover, agents have correct expectations on how new contracts will be further renegotiated upon.

First, we show that if there are two risk types, only pooling contracts are unimprovable. Pooling also includes the case where none of the agents is insured. At then end of this section, we argue that if there are more than two types, some separation is possible. Specifically, the agents with the highest risk will be insured, and the lower risk agents will remain uninsured. The result has some similarity with those in Asheim and Nilssen [4]. However, in their model some of the high risk agents together with all low risk agents remain uninsured.

Consider a standard model of an economy with adverse selection. That is, there is a risk averse agent (agent 1) who wants to receive insurance from a risk neutral agent (agent 2) against a low endowment realization. However, agent 1 has private information about the probability of a loss (a low realization).

Specifically, assume that in the high endowment state the agent receives a units of income, whereas she receives 0 units in the low state. The agent's utility from x units of income is $v(x)$, where v is continuous, monotone, and strictly concave. If the consumer is of the "good" type, then the probability of the high realization is q_g. In contrast, if the agent is of the bad type then this probability is q_b, where $q_b < q_g$. The risk neutral agent does not know whether the insured agent is of the good or the bad type. Assume that agent 1 is of the good or the bad type with probabilities p_g and p_b, respectively.

Below we show how this economy can be written in the language of our general model. We then show that only pooling contracts, i.e., insurance contracts which are not type dependent, are unimprovable. The intuition behind the result is the following.

In order to induce agents to self select by their types, a contract must provide partial insurance to low risk agents. However, this outcome can be improved upon. In particular, because agents self select, full revelation of information is obtained. Unimprovable allocations would then be complete information Pareto efficient and consequently entail complete insurance. However, any contract which involves complete insurance cannot be used to separate types. This rules out separating contracts. Therefore, unimprovable allocations must involve pooling.

The only relevant information in the model is the type g or b of the agent. Thus, choose $\Omega = \{g, b\}$ where g and b occur with probabilities p_g and p_b, respectively. Note that we will have either full information revelation $\mathcal{G} = \{\{g\}, \{b\}\}$, or no information revelation, $\mathcal{G} = \{\{g, b\}\}$. Full revelation corresponds to type separation whereas no information revelation corresponds to pooling.

Furthermore, let x_h and x_l denote the consumption in the high and in the low state, respectively. Thus, $X^i = I\!\!R_+^2$.

Agent 1's utility function is given by $u^1(\omega, x_1, x_2) = q_\omega v(x_1) + (1 - q_\omega) v(x_2)$, and agent 2's utility function is $u^2(\omega, x_1, x_2) = q_\omega x_1 + (1 - q_\omega) x_2$ where $\omega = g, b$. Finally, the agents' endowments are $e^1 \equiv (a, 0)$ and $e^2 \equiv (b, b)$, respectively, independent of the state. The information sets are given by $\mathcal{F}^1 = \{\{g\}, \{b\}\}$ and $\mathcal{F}^2 = \{\{g, b\}\}$.

We want to show that pooling contracts are always optimal. In a pooling contract, the net-trade between the two agents is independent of the state $\omega = g, b$. Thus, since the endowments are state independent, agents' consumption in both states is the same, i.e., $x^i(g) = x^i(b)$, $i = 1, 2$. In contrast, in a separating contracts the net trades are state dependent and consequently, $x^1(g) \neq x^i(b)$.

Now assume by way of contradiction that there exists an unimprovable allocation in which agents are separated by their types. Separation of types implies that there is full information revelation, i.e., $\mathcal{G} = \{\{g\}, \{b\}\}$. Then Corollary 2 implies that $x^i(\omega)$, $i \in I$ must be Pareto efficient in each state ω. Clearly, this implies that agent 1 is fully insured in both states, i.e., $x_{1\omega}^1 = x_{2\omega}^1$ for $\omega = g, b$. Moreover, in order for the allocation to be incentive compatible, agent 1's consumption in states g and b must be the same, i.e, $x_g^1 = x_b^1 = (\bar{x}, \bar{x})$. Thus, we have a pooling contract, a contradiction.

Now consider an insurance problem with three risk types, ω_i, $i = 1, 2, 3$, where $i = 1$ is the lowest and $i = 3$ the highest risk type. Thus, $q_{\omega_1} > q_{\omega_2} > q_{\omega_3}$. Assume that ω_i occurs with probability p_i. Moreover, assume that the probabilities and the utility functions are chosen such that if it is know that $\omega \in \{\omega_1, \omega_2\}$ then the only pooling contract is not to be insured. Thus, $\mathcal{G} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ together with full insurance if $\omega = \omega_1$ and no insurance for $\omega \in \{\omega_2, \omega_3\}$ is unimprovable. The argument is similar to that for the lemons market with three states. In other words, the highest risk types are completely insured, whereas both lower risk types are uninsured.

6 Relationship to Efficiency Concepts

6.1 Ex-Ante and Interim Incentive Efficiency

We now discuss the relationship of unimprovability to efficiency concepts which can be found in the literature. First, consider the definition of interim incentive efficiency of Holmström and Myerson [17].[20] According to their definition, a feasible, incentive compatible allocation x^i, $i \in i$ is interim (incentive) efficient if and only if there does not exist another feasible, incentive compatible allocation y^i, $i \in I$ which dominates x^i, $i \in I$ at the interim, i.e., which makes all agents weakly (and some strictly) better off in all states with respect to interim expected utility. Formally,

$$E\big(u^i(\omega, y^i) \mid \mathcal{F}^i\big)(\bar{\omega}) \geq E\big(u^i(\omega, x^i) \mid \mathcal{F}^i\big)(\bar{\omega}), \tag{3}$$

for all $i \in I$ and for all $\bar{\omega} \in \Omega$, where the strict inequality holds for at least one agent on a set of positive probability. Now note that if we replace in (3) interim by ex-ante expected utility then we get the definition of ex-ante incentive efficiency.

However, as pointed out in Holmström and Myerson [17, Section 6] interim incentive efficiency is not a positive solution concept. Specifically, they provide an example in which agents can improve upon a decision rule which is interim incentive efficient. We now discuss this example as it shows immediately why interim incentive efficient allocations are not necessarily unimprovable. The example also has the property that the interim incentive efficient allocation is ex-ante incentive efficient. Thus, the analysis in this section also shows that ex-ante incentive efficient allocations need not be unimprovable.

Instead of an exchange economy, Holmström and Myerson consider a collective choice problem. Two agents $i = 1, 2$ must make a joint decision among three possible choices which are denoted by A, B, C.[21] Thus, the model differs from the exchange economy with differential information considered in this paper. However, one can easily map the decision problem above into our general problem. In particular, consider an exchange economy with two agents. Let e^i, $i = 1, 2$ denote the agents' endowments. Now assume there

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[20] For other concepts of interim efficiency see Hahn and Yannelis [16].

[21] If agents disagree, they receive an autarky level of utility. If this utility is sufficiently low, agreement can always be ensured.

are solely three feasible allocations, x_A, x_B, x_C which make both agents better off than under autarky. All other allocations give agents the same utility $\bar{u}$ as under autarky, independent of the state ω. Thus, in addition to $\bar{u}$ the following three utilities can be obtained: $u_{\omega,A}$, $u_{\omega,B}$, $u_{\omega,C}$. We choose $\bar{u}$ such that $\bar{u} < u_{\omega,j}$, for all $\omega \in \Omega$ and $j = A, B, C$. Then in any ex-ante and interim incentive efficient, and in unimprovable allocations only x_A, x_B, and x_C will occur. Thus, these allocations correspond exactly to the three joint actions in the decision problem. As a consequence, our definition of unimprovability can be applied to such problems.[22]

We now continue with the description of the example in Holmström and Myerson [17]. The example, indicates that there exists an interim incentive efficient allocations which can be improved upon. Each agent i can be of two types $T^i = \{ia, ib\}$. The type is private information. Thus, $\Omega = T^1 \times T^2$. Each of the four states in Ω occur with the same probability. The information set are given by $\mathcal{F}^1 = \{\{(1a, 2a), (1a, 2b)\}, \{(1b, 2a), (1b, 2b)\}\}$; $\mathcal{F}^2 = \{\{(1a, 2a), (1b, 2a)\}, \{(1a, 2b), (1b, 2b)\}\}$. Each agent's utility only depends on her own type. The utilities for the three decision are listed below.

	u_{1a}	u_{1b}	u_{2a}	u_{2b}
A	2	0	2	2
B	1	4	1	1
C	0	9	0	−8

In this example Holmström and Myerson point out that choosing A in state $(1a, 2a)$, choosing B in states $(1a, 2b)$, $(1b, 2b)$, and choosing C in $(1b, 2a)$ is ex-ante and interim incentive efficient. Now assume that agent 1 is of type 1a. Since agent 2 is always best off with choice A, one would imagine that agent 1 would propose choosing A instead of B, thus deviating from the interim incentive efficient decision rule. The interim incentive efficient decision rule specified above is therefore not one which we would expect agents to use.

Is the above decision rule unimprovable? The argument in the previous paragraph immediately shows that the answer is no. In particular, the above decision rule requires that each agent reports all her information. Thus, $\mathcal{G}$ corresponds to full information. Now consider the alternative decision rule where A instead of B is chosen in state $(1a, 2b)$. This decision rule is again

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[22]Holmström and Myerson [17] also considers stochastic decision rules. Stochastic decision rules can be incorporated in our model in a similar way.

measurable with respect to $\mathcal{G}$. Moreover, it dominates the original one and it is incentive compatible with respect to $\mathcal{G}$, because $\mathcal{G}$ already corresponds to complete information.[23]

Thus, conditions (C2) and (C4) of Definition 3 imply that the decision rule can be improved upon.

What is the structure of all unimprovable decision rules? In order to determine them one can proceed by backward induction on $\mathcal{G}$. Thus, we first assume that $\mathcal{G}$ corresponds to full information. The only incentive compatible decision rules which cannot be dominated by another $\mathcal{G}$-measurable decision rule involves choosing A in states $(1a, 2a)$ and $(1a, 2b)$. In the remaining states the same action must be chosen. Secondly, assume that $\mathcal{G}' < \mathcal{G}$ is the information revealed in an unimprovable decision rule. Then the decision rule must also involve choosing A in states $(1a, 2a)$ and $(1a, 2b)$. Otherwise, agents could reveal all information and modify their decision rule to one where A is chosen in both of the two states.

Thus, all unimprovable decision rules involve choosing A in states $(1a, 2a)$, $(1a, 2b)$, and choosing, because of incentive compatibility, the same action $j = A, B, C$ in the remaining states. Thus, if $j = A$ the information $\mathcal{G}$ revealed can be arbitrary. If $j \neq A$, then we can have $\mathcal{G} = \{\{(1a, 2a), (1a, 2b)\}, \{(1b, 2a), (1b, 2b)\},$ or $\mathcal{G}$ corresponding to complete information.

6.2 Durability

The concept of Durability was introduced in Holmström and Myerson [17] to describe decision rules from which agents would not deviate at the interim. They define a decision rule γ as durable if and only if the following holds: Let γ' be an arbitrary decision rule. Then it must be the case that in a voting game where agents decide whether or not to switch from γ to any alternative decision rule γ', staying with γ is a perfect equilibrium (in the sense of Selten [27]). This formulation is modified in Crawford [9], where agents first choose a mechanism at the interim and then follow the chosen mechanism.

There are a number of differences between our concept and durability. Specifically, durable decision rules allow agents to fully reveal all information.

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[23]Note that this alternative decision rule is not incentive compatible with respect to agents' initially endowed information. This is one of the main reasons for the differences between the concepts of interim incentive efficiency and unimprovability.

Moreover, once a decision rule is agreed upon, agents must adhere to it, i.e., further retrading is not allowed. Recall that in our model agents are free to trade as long as they desire to do so. Also, because of coordination problems in the voting game, agents will not always adopt decision rules γ' which dominate γ. Consider the following example discussed in Holmström and Myerson [17, Section 9].

As in the previous example, there are two agents $i = 1, 2$. Each agent i can be of one of two types $t^i = ia, ib$. Thus, we have again four states in Ω. In contrast to the previous decision problem there are now only two choices, A and B. Agents' utilities now also depend on the type of the other agent and are given by

$$u^1(\omega, A) = u^2(\omega, A) = 2 \quad \text{for all } \omega \in \Omega;$$
$$u^1(\omega, B) = u^2(\omega, B) = \begin{cases} 3 & \text{if } t \in \{(1a, 2a), (1b, 2b)\}; \\ 0 & \text{if } t \in \{(1a, 2b), (1b, 2a)\}. \end{cases}$$

In other words, if agents' types match then they receive the highest payoff from choosing B. If the types do not match, then agents are better off choosing A. In fact, we have just described the only interim incentive efficient decision rule. However, Holmström and Myerson point out that choosing A independent of types is durable.

What decision rules in this example are unimprovable? If $\mathcal{G}$ corresponds to full information then agents must choose B if their types match and A otherwise. Any other decision rule is dominated. Now assume that only information $\mathcal{G}'$ less than full information is revealed. Then revealing all information and choosing A or B depending on whether or not types match dominates any other decision rule and it is incentive compatible. Thus, there exists exactly one unimprovable decision rule and it coincides with the interim incentive efficient one. The durable decision rule where agents choose A independent of their types can be improved upon.

7   Concluding Remarks and Extensions

We started this paper with the following question: What allocations describe absence of improving trades for exchange economies with incomplete information?

We have shown that one can find a consistent solution concept (unimprovability) which exists under weak conditions, has interesting properties,

and is relatively easy to compute in standard examples of economies with incomplete information. In order to get a consistent solution concept it turns out to be essential that agents are forward looking when deciding about what information to reveal. This is a feature which should be important in defining any cooperative solution concepts (e.g., the core) for economies with adverse selection.

For economies with two agents, the concept in this paper already has characteristics of the core and can as a consequence be compared to existing concepts of a core with differential information (e.g., Yannelis [30], Allen [4] and Vohra [28]).

In the private core of Yannelis [30] net trades are required to be measurable with respect to each agent's private information $\mathcal{F}_i$. In the case of two agents, feasibility implies that net trades are therefore measurable with respect to common knowledge information $\mathcal{F}_1 \wedge \mathcal{F}_2$. In contrast, in our solution concept measurability of net trades with respect to some information set $\mathcal{G}$ is not assumed but rather a consequence of the solution concept (proved in Theorem 1). Moreover, $\mathcal{G}$ is endogenously determined in our solution concept.

In Vohra [28] and Allen [3], a coalition S can obtain all feasible, (individually) Bayesian incentive compatible allocations. Blocking can be defined with respect to agents' ex-ante or interim expected utility. Information revealed through trades, however, cannot be used for blocking by coalitions of agents. Thus, this concept can be useful to describe trade ex-ante, when agents can sign binding agreements before they become informed, but it can be problematic in environments where agents trade (or can change past agreements) at a time when they are already differentially informed. It should be noted that adverse selection problems arise precisely in the latter case.

Finally, we illustrate the differences to our solution concept by means of two examples.

First, consider the economy with adverse selection and two states in Section 5.2. Here $\mathcal{G} = \mathcal{F}_1 \wedge \mathcal{F}_2$, and hence our solution coincides with the private core. In contrast, in the core of Allen [3] and Vohra [28] agents would be able to obtain state contingent trades (i.e., separating contracts). The reason for this result is that by assumption the information revealed by type separation cannot be used for further blocking.

Now consider as another example an economy with two agents, two goods, and two states. Agent 1's utility function is $u^1(\omega, x_1, x_2) = x_1 x_2$. Agent 1's endowment in the two states is $(4, 0)$ and $(0, 4)$. On the other hand, the two

goods are perfect substitutes for agent 2, i.e., $u^2(\omega, x_1, x_2) = x_1 + x_2$. The agent has no information, and an endowment of $(2,2)$ in both states. It is easy to see that in all unimprovable allocations $\mathcal{G}$ corresponds to complete information, and agents will obtain complete information Pareto efficient allocations. This seems to be a reasonable prediction as agent 1 has no advantage from misreporting his endowment. Moreover, the state can also be credibly signaled if agent 1 shows the endowment to agent 2. Vohra and Allen's core provides the same result as our concept. In contrast, in the private core, net trades would have to be measurable with respect to $\mathcal{F}_1 \wedge \mathcal{F}_2$ and hence autarky will be the result.

8   Appendix

Proof of Theorem 1. Let B denote the set of all partitions of Ω. We endow B with the natural order. That is, let $\mathcal{F}$, $\mathcal{G} \in B$ then $\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{G}$ is a finer partition than $\mathcal{F}$. We now construct a consistent belief function ψ by means of backward induction on B.

Let $\mathcal{G}$ be a partition which corresponds to complete information, i.e., $\mathcal{G} = \{\{\omega\} \mid \omega \in \Omega\}$. First, if x^i, $i \in I$ is Pareto efficient with respect to complete information[24] then define $\psi(x, \mathcal{G}) = (x, \mathcal{G})$. Now assume that x^i, $i \in I$ is not Pareto efficient. Then because of continuity of the agents' utility functions, there exists a complete information Pareto efficient allocation y^i, $i \in I$ with $u^i(\omega, y^i) \geq u^i(\omega, x^i)$ for all agents $i \in I$ and for all $\omega \in \Omega$. Then define $\psi(x, \mathcal{G}) = (y, \mathcal{G})$. Clearly, (C1) and (C2) hold. (C3) holds since $\psi(x, \mathcal{G}) = (y, \mathcal{G})$ is Pareto efficient, and because by construction $\psi(y, \mathcal{G}) = (y, \mathcal{G})$ for all Pareto efficient $(y, \mathcal{G})$. Finally, for $\mathcal{G}$ corresponding to complete information, (C4) is equivalent to complete information Pareto efficiency.

Now assume by way of induction that we have constructed the belief function ψ for all $(x, \mathcal{H}) \in C$ with $\mathcal{H} > \bar{\mathcal{H}}$. The objective is to define $\psi(x, \bar{\mathcal{H}})$ for all $(x, \bar{\mathcal{H}})$. We classify three different cases for all $(x, \bar{\mathcal{H}}) \in C$, and define ψ for each of them.

Case 1. There exists $(y, \mathcal{H}) = \psi(\tilde{x}, \tilde{\mathcal{H}})$, $\tilde{\mathcal{H}} > \bar{\mathcal{H}}$ such that $(y, \mathcal{H})$ improves upon and is incentive compatible with respect to $(x, \bar{\mathcal{H}})$.

We define $\psi(x, \bar{\mathcal{H}}) = (y, \mathcal{H})$.

Case 2. The condition of case 1 does not hold and there does not exist $(y, \bar{\mathcal{H}}) \in C$ that strictly improves upon $(x, \bar{\mathcal{H}})$.

We define $\psi(x, \bar{\mathcal{H}}) = (x, \bar{\mathcal{H}})$.

Case 3. The conditions of case 1 and 2 do not hold.

Choose $(y, \bar{\mathcal{H}})$ that fulfills the conditions of Case 2 and improves upon $(x, \bar{\mathcal{H}})$. Then let $\psi(x, \bar{\mathcal{H}}) = (y, \bar{\mathcal{H}})$.

It now remains to prove that ψ is well defined and fulfills (C1)–(C4). We first show that ψ is well defined. For cases 1 and 2 this is obvious. Thus, assume that $(x, \bar{\mathcal{H}})$ fulfills the conditions of Case 3. We first show that

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[24]That is, there does not exist a feasible allocation y^i, $i \in I$ such that $u^i(\omega, y^i) \geq u^i(\omega, x^i)$, for all $i \in I$, $\omega \in \Omega$ with the strict inequality holding for at least on agent i and for at least one state ω.

there exists $(y, \bar{\mathcal{H}})$ which dominate $(x, \bar{\mathcal{H}})$ and which cannot be dominated by another allocation $(\tilde{y}, \bar{\mathcal{H}})$.

Define modified consumption sets $L_{X^i} = \{y : \Omega \rightarrow X^i \mid y - x^i \text{ is } \bar{\mathcal{H}}\text{-}$ measurable$\}$ as in Yannelis [30]. Then L_{X_i} is closed and nonempty (since $x^i \in L_{X_i}$). Thus, the set of feasible allocations y^i, $i \in I$ with $y^i \in L_{X_i}$ is compact. Then the following maximization problem has a solution for all $\lambda_i > 0$.

$$\max_y \sum_{i \in I} \lambda_i E(u^1(\omega, y^1(\omega)),$$

subject to

1. $y_i \in L_{X_i}$,
2. $\sum_{i \in I} y_i = \sum_{i \in I} x_i$;
3. $E(u^i(\omega, y^i) \mid \mathcal{F}_i \vee \bar{\mathcal{H}}) \geq E(u^i(\omega, x^i) \mid \mathcal{F}_i \vee \bar{\mathcal{H}})$.

Clearly, the solution cannot be dominated by another allocation $\tilde{y}$ where $\tilde{y}^i - y^i$ is $\bar{\mathcal{H}}$-measurable.

In order to show that $(y, \bar{\mathcal{H}})$ fulfills the conditions of case 2, it is sufficient to prove that $\psi(y, \bar{\mathcal{H}}) = (y, \bar{\mathcal{H}})$. Assume by way of contradiction that $(\tilde{y}, \tilde{\mathcal{H}}) = \psi(y, \bar{\mathcal{H}}) \neq (y, \bar{\mathcal{H}})$. Then by construction $\tilde{\mathcal{H}} > \bar{\mathcal{H}}$. Moreover, $(\tilde{y}, \tilde{\mathcal{H}})$ improves upon $(y, \bar{\mathcal{H}})$. Thus, $(\tilde{y}, \tilde{\mathcal{H}})$ improves upon $(x, \bar{\mathcal{H}})$.

We next show that $(\tilde{y}, \tilde{\mathcal{H}})$ is IC with respect to $(x, \bar{\mathcal{H}})$. Since $(\tilde{y}, \tilde{\mathcal{H}})$ is incentive compatible with respect to $(y, \bar{\mathcal{H}})$, there exists an information revelation game $\{M_t^i, \Sigma_t^i, \mathcal{F}_i \vee \mathcal{G}, \Omega, \mu, z^i, \pi^i\}$ such that truth telling is an equilibrium and such that $x^i(\omega) + z^i(\omega, \sigma(\omega)) = \hat{x}^i(\omega)$. We now define a new game, starting from $(x, \bar{\mathcal{H}})$. The message spaces, strategy spaces and information sets are the same as above. Define $\hat{z}^i(\omega, m) = z^i(\omega, m) + y^i(\omega) - x^i(\omega)$, for all (ω, m). Then $\hat{\pi} = \pi$. Thus, since truth telling is an equilibrium of the original game, it is also an equilibrium of the new game. Moreover, the resulting allocation for the new game is again $(\tilde{y}, \tilde{\mathcal{H}})$. Thus, $(\tilde{y}, \tilde{\mathcal{H}})$ is incentive compatible with respect to $(x, \bar{\mathcal{H}})$. Hence, $(x, \bar{\mathcal{H}})$ fulfills the conditions of Case 1, a contradiction. Thus, $\psi(y, \bar{\mathcal{H}}) = (y, \bar{\mathcal{H}})$ and hence $(y, \bar{\mathcal{H}})$ fulfills the conditions of Case 2. Thus, ψ is well defined.

It now remains to prove conditions (C1)–(C4). For case 1 the conditions hold by construction and since ψ fulfill the conditions for all $\mathcal{H} > \bar{\mathcal{H}}$ by the induction argument. Similarly, (C1)–(C3) hold for Case 2. We now show (C4). First note that there does not exist $(y, \mathcal{H}) = \psi(\tilde{x}, \tilde{\mathcal{H}})$, $\tilde{\mathcal{H}} \geq \bar{\mathcal{H}}$ that strictly improves upon $(x, \bar{\mathcal{H}})$. In particular, if $\tilde{\mathcal{H}} > \bar{\mathcal{H}}$ then the condition of Case 1 would be fulfilled. Similarly, if $\tilde{\mathcal{H}} = \bar{\mathcal{H}}$ then the above argument used

to show that ψ is well defined implies that $(y, \mathcal{H})$ is incentive compatible and improves upon $(x, \bar{\mathcal{H}})$. Again, $\mathcal{H} > \bar{\mathcal{H}}$ would imply that $(x, \mathcal{G})$ fulfills the conditions of case 1. Similarly, $\mathcal{H} = \bar{\mathcal{H}}$ means that $(x, \mathcal{G})$ fulfills the conditions of case 3. These contradictions imply that (C4) holds.

Now assume that $(x, \bar{\mathcal{H}})$ fulfills the conditions of Case 3. Then since $\psi(x, \bar{\mathcal{H}}) = (y, \bar{\mathcal{H}})$ (i.e., no additional information is revealed), $(y, \bar{\mathcal{H}})$ is IC. Moreover, by construction (C2) and (C3) holds. Finally, note that $(y, \bar{\mathcal{H}}) \notin \psi(C)$. Thus, (C4) is trivially fulfilled.

Thus, we have shown that a consistent belief function ψ exist. Finally, we only need to define $(x, \mathcal{G}) = \psi((e^1, \ldots, e^n), \mathcal{H})$, where $\mathcal{H}$ is the trivial information set. Then $(x, \mathcal{G})$ is unimprovable. This concludes the proof of the Theorem.

Proof of Corollary 1. Let $(x, \mathcal{G})$ be unimprovable. We first show that $(x, \mathcal{G}) \in \hat{C}$.

Assume by way of contradiction that there exists a $(y, \mathcal{G})$ that strictly improves upon $(x, \mathcal{G})$. Then as shown in the proof of Theorem 1, $\psi(y, \mathcal{G})$ strictly improves upon and is incentive compatible with respect to $(x, \mathcal{H})$, a contradiction.

The properties of ψ on $\hat{C}_{(x, \mathcal{G})}$ follow since ψ is consistent on C.

In order to prove the other implication, we must extend ψ to C. This can be done by the induction argument of Theorem 1. However one must ensure that when defining ψ on $C \setminus \hat{C}_{(x, \mathcal{G})}$, we do not violate (C4) on $\hat{C}_{x, \mathcal{G}}$. Thus, it is sufficient to prove the following:

Let $(y, \bar{\mathcal{H}}) \notin \hat{C}_{(x, \mathcal{G})}$. Let $(y', \mathcal{H}') \in C$, $\mathcal{H}' \geq \mathcal{H}$ such that $\psi(y', \mathcal{H}')$ strictly improves upon $(y, \bar{\mathcal{H}})$. Then $\psi(y', \mathcal{H}')$ does not strictly improve upon or is not incentive compatible with respect to any $(\tilde{x}, \tilde{\mathcal{G}}) \in \psi(\hat{C}_{(x, \mathcal{G})})$.

Assume by way of contradiction that $(y'', \mathcal{H}'') = \psi(y', \mathcal{H}')$ strictly improves upon and is incentive compatible for some $(\tilde{x}, \tilde{\mathcal{G}}) \in \psi(C_{(x, \mathcal{G})})$. Then $E(u^i(\omega, y^{i''}) \mid \mathcal{F}_i \vee \tilde{\mathcal{G}}) \geq E(u^i(\omega, \tilde{x}^i) \mid \mathcal{F}_i \vee \tilde{\mathcal{G}})$, where at least one inequality is strict. The definition of $\hat{C}_{(x, \mathcal{G})}$ and the law of iterated expectations immediately imply that $(y'', \mathcal{H}'') \in C_{(x, \mathcal{G})}$. By the step-wise construction of Theorem 1 and because (C3) holds on $\hat{C}_{(x, \mathcal{G})}$, it follows that $\psi(y'', \mathcal{H}'') = (y'', \mathcal{H}'')$. Moreover, by assumption $(y'', \mathcal{H}'')$ strictly improves upon $(\tilde{x}, \tilde{\mathcal{G}})$. Because both $(y'', \mathcal{H}'')$ and $(\tilde{x}, \tilde{\mathcal{G}})$ are in $\hat{C}_{(x, \mathcal{G})}$, this is a contradiction to the assumption that ψ fulfills (C4) on $\hat{C}_{(x, \mathcal{G})}$.

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