Subjective Beliefs and Ex-Ante Trade*

Luca Rigotti
Duke University
Fuqua School of Business

Chris Shannon
U.C. Berkeley
Department of Economics

Tomasz Strzalecki
Northwestern University
Department of Economics

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Abstract

We study a definition of subjective beliefs applicable to preferences that allow for the perception of ambiguity, and provide a characterization of such beliefs in terms of market behavior. Using this definition, we derive necessary and sufficient conditions for the efficiency of ex-ante trade, and show that these conditions follow from the fundamental welfare theorems. When aggregate uncertainty is absent, our results show that full insurance is efficient if and only if agents share some common subjective beliefs. Our results hold for a general class of convex preferences, which contains many functional forms used in applications involving ambiguity and ambiguity aversion. We show how our results can be articulated in the language of these functional forms, confirming results existing in the literature, generating new results, and providing a useful tool for applications.

Keywords
Common Prior, Absence of Trade, Ambiguity Aversion, General Equilibrium.

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1 Introduction

In a model with risk averse agents who maximize subjective expected utility, betting occurs if and only if agents' priors differ. This link between common priors and speculative trade in the absence of aggregate uncertainty is a fundamental implication of expected utility for risk-sharing in markets. A similar relationship holds when ambiguity is allowed and agents maximize the minimum expected utility over a set of priors, as in the model of Gilboa and Schmeidler (1989). In this case, purely speculative trade occurs when agents hold no priors in common; full insurance is Pareto optimal if and only if agents have at least one prior in common, as Billot, Chateauneuf, Gilboa, and Tallon (2000) show. This note develops a more general connection between subjective beliefs and speculative trade applicable to a broad class of convex preferences, which encompasses as special cases not only the previous results for expected utility and maxmin expected utility, but all the models central in studies of ambiguity in markets, including the convex Choquet model of Schmeidler (1989), the smooth second-order prior models of Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006), the second-order expected utility model of Ergin and Gul (2004), the confidence preferences model of Chateauneuf and Faro (2006), the multiplier model of Hansen and Sargent (2001), and the variational preferences model of Maccheroni, Marinacci, and Rustichini (2006).

By casting our results in the general setting of convex preferences, we are able to focus on several simple underlying principles. We identify a notion of subjective beliefs based on market behavior, and show how it is related to various notions of belief that arise from different axiomatic treatments. We highlight the close connection between the fundamental welfare theorems of general equilibrium and results that link common beliefs and risk-sharing. Finally, by establishing these links for general convex preferences, we provide a framework for studying ambiguity in markets while allowing for heterogeneity in the way ambiguity is expressed through preferences. The generality of this approach identifies the forces underlying betting without being restricted to any one particular representation, and in so doing unifies our thinking about models of ambiguity aversion in economic settings.

The note is organized as follows. Section 2 studies subjective beliefs and behavioral characterizations, with illustrations for various familiar representations. Section 3 studies trade between agents with convex preferences. Appendix A develops an extension of these results to infinite state spaces, while Appendix B collects some proofs omitted in the text.

2 Beliefs and Convex Preferences

2.1 Convex Preferences

Let S be a finite set of states of the world. The set of consequences is $\mathbb{R}_+$, which we interpret as monetary payoffs. The set of acts is $\mathcal{F} = \mathbb{R}_+^S$ with the natural topology. Acts are denoted by f, g, h, while $f(s)$ denotes the monetary payoff from act f when state s obtains. For any $x \in \mathbb{R}_+$ we abuse notation by writing $x \in \mathcal{F}$, which stands for the constant act with payoff x in each state of the world.

Let $\succsim$ be a binary relation on $\mathcal{F}$. We say that $\succsim$ is a *convex preference relation* if it satisfies the following axioms:

Axiom 1 (Preference)**.** *$\succsim$ is complete and transitive.*

Axiom 2 (Continuity)**.** *For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} \mid g \succsim f\}$ and $\{g \in \mathcal{F} \mid f \succsim g\}$ are closed.*

Axiom 3 (Monotonicity)**.** *For all $f, g \in \mathcal{F}$, if $f(s) > g(s)$ for all $s \in S$, then $f \succ g$.*

Axiom 4 (Convexity)**.** *For all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} \mid g \succsim f\}$ is convex.*

These axioms are standard, and well-known results imply that a convex preference relation $\succsim$ is represented by a continuous, increasing and quasi-concave function $V : \mathcal{F} \to \mathbb{R}$.[1] Convex preferences include as special cases many common models of risk aversion and ambiguity aversion. In many of these special cases, one element of the representation identifies a notion of beliefs. In what follows, we adopt the notion of subjective probability suggested in Yaari (1969) to define *subjective beliefs* for general convex preferences. We then study characterizations of this concept in terms of market behavior, and illustrate particular special cases including maxmin expected utility, Choquet expected utility, and variational preferences.

2.2 Supporting Hyperplanes and Beliefs

The decision-theoretic approach of de Finetti, Ramsey, and Savage identifies a decision maker's subjective probability with the odds at which he is willing to make small bets. In this spirit, Yaari (1969) identifies subjective probability with a hyperplane that supports the upper contour set.[2] If this set has kinks, for example because of non-differentiabilities often associated with

[1] Axiom 4 captures convexity in monetary payoffs. For Choquet expected utility agents, who evaluate an act according to the Choquet integral of its utility with respect to a non-additive measure (capacity), the relation between payoff-convexity and uncertainty aversion has been studied by Chateauneuf and Tallon (2002). Dekel (1989) studies the relation between payoff-convexity and risk aversion.

[2] In the finance literature this is commonly called a risk-neutral probability, or risk-adjusted probability.

ambiguity, there may be multiple supporting hyperplanes at some acts. To encompass such preferences, we consider the set of *all* (normalized) supporting hyperplanes.[3]

Definition 1 (Subjective Beliefs). *The set of subjective beliefs at an act f is*

$$\boldsymbol{\pi}(f) := \{p \in \Delta S \mid p \cdot g \geq p \cdot f \textit{ for all } g \succsim f\}$$

Given the interpretation of the elements of $\boldsymbol{\pi}(f)$ as beliefs, we will write $E_p g$ instead of $p \cdot g$. For any convex preference relation, $\boldsymbol{\pi}(f)$ is nonempty, compact and convex, and is equivalent to the set of (normalized) supports to the upper contour set of $\succsim$ at f. In the next section we explore behavioral implications of this definition, including willingness or unwillingness to trade, and their market consequences.

2.3 Market Behavior and Beliefs

We begin with a motivating example, set in the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989). The agent's preferences are represented using a compact, convex set of priors $P \subseteq \Delta S$ and a utility index $u : \mathbb{R}_+ \to \mathbb{R}$ that is concave and differentiable. The utility of an act f is given by the minimum expected utility over the set of priors P:

$$V(f) := \min_{p \in P} \sum_{s \in S} p_s u(f(s)) = \min_{p \in P} E_p u(f)$$

where we abuse notation by writing $u(f)$ for $\big(u(f(1)), \ldots, u(f(S))\big)$.

Imagine that the agent is initially endowed with a constant act x. First, consider an act f such that $E_p f = x$ for some $p \in P$, as depicted in the left panel of Figure 1 (the shaded area collects all such acts). One can see that the agent will have zero demand for f. Second, consider an act f such that $E_p f > x$ for all $p \in P$, as depicted in the right panel of Figure 1. One can see that there exists $\varepsilon > 0$ sufficiently small such that $\varepsilon f + (1 - \varepsilon) x \succ x$.

[3] Alternatively, Chambers and Quiggin (2002) define beliefs using superdifferentials of the *benefit function*. Their definition turns out to be equivalent to ours.

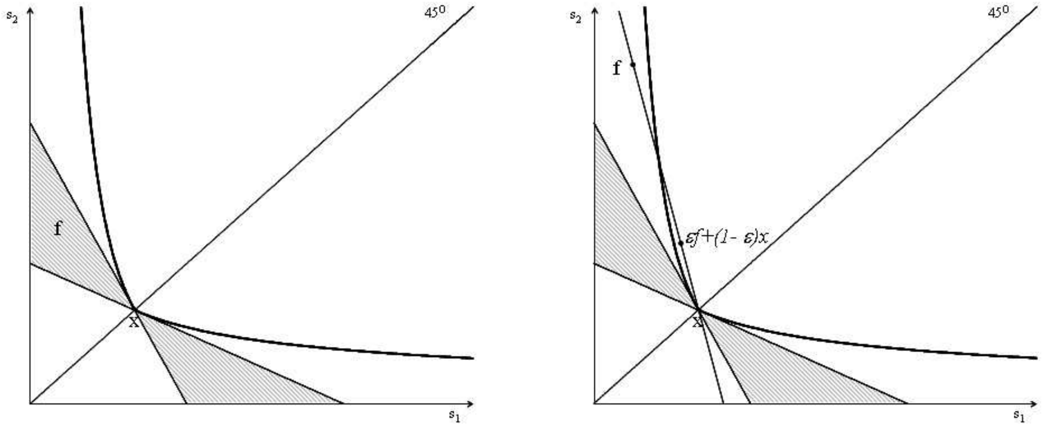

Figure 1. Behavioral properties of beliefs in the MEU model.

In the MEU model, the set P captures two important aspects of market behavior (both evident in Figure 1). First, agents are unwilling to trade from a constant bundle to a random one if the two have the same expected value for *some* prior in the set P. In particular, the set P is the largest set of beliefs revealed by this unwillingness to trade based on zero expected net returns. Second, agents are willing to trade from a constant bundle to (a possibly small fraction of) a random one whenever the random act has greater expected value according to *every* prior in the set P. In particular, the set P is the smallest set of beliefs revealing this willingness to trade based on positive expected net returns.

We introduce two notions of beliefs revealed by market behavior that attempt to capture these properties for general convex preferences. The first notion collects all beliefs that reveal an unwillingness to trade from a given act f.

Definition 2 (Unwillingness-to-trade Revealed Beliefs). *The set of beliefs revealed by unwillingness to trade at f is*

$$\boldsymbol{\pi}^u(f) := \{p \in \Delta S \mid f \succsim g \textit{ for all } g \textit{ such that } E_p g = E_p f\}.$$

This set gathers all beliefs for which the agent is unwilling to trade assets with zero expected net returns. It can also be interpreted as the set of Arrow-Debreu prices for which the agent endowed with f will have zero net demand. For a convex preference, it is straightforward to see that this gives a set of beliefs equivalent to that defined by our subjective beliefs in Definition 1.

Our second notion collects beliefs revealed by a willingness to trade from a given act f. To formalize this, let $\mathcal{P}(f)$ denote the collection of all compact, convex sets $P \subseteq \Delta S$ such that if $E_p g > E_p f$ for all $p \in P$ then $\varepsilon g + (1-\varepsilon) f \succ f$ for sufficiently small ε.[4] We define the willingness-to-trade revealed beliefs as the smallest such set.[5]

Definition 3 (Willingness-to-trade Revealed Beliefs)**.** *The set of beliefs revealed by willingness to trade at an act f is*

$$\boldsymbol{\pi}^w(f) := \bigcap \{P \subseteq \Delta S \mid P \in \mathcal{P}(f)\}.$$

The following proposition establishes the equivalence between the different notions of belief presented in this section, and therefore gives behavioral content to Definition 1. Subjective beliefs are related to observable market behavior in terms of willingness or unwillingness to make small bets or trade small amounts of assets.

Proposition 1. *If $\succsim$ is a convex preference relation, then $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^u(f) = \boldsymbol{\pi}^w(f)$ for every strictly positive act f.*

2.4 Special cases

In this section we explore the relationships between our notion of subjective belief and those arising in several common models of ambiguity. For the benchmark case of classical subjective expected utility, as observed by Yaari (1969), our subjective beliefs coincide with the local trade-offs or risk-neutral probabilities that play a central role in many applications of risk. If we restrict attention to constant acts, then subjective beliefs will coincide with the unique prior of the subjective expected utility representation. This property generalizes beyond SEU. The subjective beliefs we calculate at a constant act, at which risk and ambiguity are absent, coincide with the beliefs identified axiomatically in particular representations.

Maxmin Expected Utility Preferences

We begin with MEU preferences, represented by a particular set of priors P and utility index u.[6] These preferences also include the convex case of Choquet expected utility, for which P has additional structure as the core of a convex capacity.

To derive a simple characterization of the set $\boldsymbol{\pi}(f)$ for MEU preferences, let $U : \mathbb{R}^S_+ \to \mathbb{R}^S$ be the function $U(f) := (u(f(1)), \ldots, u(f(S)))$ giving ex-post utilities in each state. For any

[4] Notice that $\mathcal{P}(f)$ is always nonempty, because $\Delta S \in \mathcal{P}(f)$ by Axiom 3.

[5] The proof of Proposition 1 shows that $\mathcal{P}(f)$ is closed under intersection.

[6] The MEU model is a special case of the model of invariant biseparable preferences in Ghirardato and Marinacci (2001). Ghirardato, Maccheroni, and Marinacci (2004) introduce a definition of beliefs for such preferences and propose a differential characterization. For invariant biseparable preferences that are also convex, their differential characterization is equivalent to ours when calculated at constant bundle. The only invariant biseparable preferences that are convex are actually MEU preferences, however, so these are already included in our present discussion.

$f \in \mathbb{R}^S_{++}$, $DU(f)$ is the $S \times S$ diagonal matrix with diagonal given by the vector of ex-post marginal utilities $(u'(f(1)), \ldots, u'(f(S)))$. For each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in P} E_p u(f)$$

be the set of minimizing priors realizing the utility of f. Note that $V(f) = E_p u(f)$ for each $p \in M(f)$. Using a standard envelope theorem, we can express the set $\boldsymbol{\pi}(f)$ as follows.

Proposition 2. *Let $\succsim$ be a MEU preference represented by a set of priors P and a concave, strictly increasing and differentiable utility index u. Then $\succsim$ is a convex preference, and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(f) \text{ for some } p \in M(f) \right\}.$$

In particular, $\boldsymbol{\pi}(x) = P$ for all constant acts x.

Variational Preferences

Introduced and axiomatized by Maccheroni, Marinacci, and Rustichini (2006), variational preferences have the following representation:

$$V(f) = \min_{p \in \Delta S} [E_p u(f) + c^\star(p)]$$

where $c^\star \colon \Delta S \to [0, \infty]$, is a convex, lower semicontinuous function such that $c^\star(p) = 0$ for at least one $p \in \Delta S$. The function $c^\star$ is interpreted as the cost of choosing a prior. As special cases, this model includes MEU preferences, when $c^\star$ is 0 on the set P and ∞ otherwise, the multiplier preferences of Hansen and Sargent (2001), when $c^\star(p) = R(p \,\|\, q)$ is the relative entropy between p and some reference distribution q, and the mean-variance preference of Markovitz and Tobin, when $c^\star(p) = G(p \,\|\, q)$ is the relative Gini concentration index between p and some reference distribution q.

For each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in \Delta S} \{E_p[u(f)] + c^\star(p)\}$$

be the set of minimizing priors realizing the utility of f. Note that $V(f) = E_p u(f) + c^\star(p)$ for each $p \in M(f)$. The set $\boldsymbol{\pi}(f)$ can be characterized as follows.

Proposition 3. *Let $\succsim$ be a variational preference for which u is concave, increasing, and differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(x) \text{ for some } p \in M(f) \right\}.$$

In particular, $\boldsymbol{\pi}(x) = \{p \in \Delta S \mid c^\star(p) = 0\}$ for all constant acts x.

The set of subjective beliefs at a constant act x, $\boldsymbol{\pi}(x)$, is equal to the set of probabilities for which $c^\star$, the cost of choosing a prior, is zero. An interesting implication of this result is that at a constant act, the subjective beliefs of an agent with Hansen and Sargent (2001) multiplier preferences are equal to the singleton $\{q\}$ consisting of the reference probability, since $R(p \,\|\, q) = 0$ if and only if $p = q$.[7] A similar result holds for mean-variance preferences.

Confidence Preferences

Chateauneuf and Faro (2006) introduced and axiomatized a class of preferences in which ambiguity is measured by a confidence function $\varphi : \Delta S \to [0,1]$. The value of $\varphi(p)$ describes the decision maker's confidence in the probabilistic model p; in particular $\varphi(p) = 1$ means that the decision maker has full confidence in p. By assumption, the set of such measures is nonempty; moreover, the function φ is assumed to be upper semi continuous and quasiconcave. Preferences in this model are represented by:

$$V(f) = \min_{p \in L_\alpha} \frac{1}{\varphi(p)} E_p u(f),$$

where $L_\alpha = \{q \in \Delta S \,|\, \varphi(q) \geq \alpha\}$ is a set of measures with confidence above α.

As before, for each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in L_\alpha} \left\{ \frac{1}{\varphi(p)} E_p u(f) \right\}$$

be the set of minimizing priors realizing the utility of f. Note that $V(f) = \frac{1}{\varphi(p)} E_p u(f)$ for each $p \in M(f)$. By standard envelope theorems, $\boldsymbol{\pi}(f)$ can be characterized in this case as follows.

Proposition 4. *Let $\succsim$ be a confidence preference for which u is concave, increasing, and continuously differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(x) \text{ for some } p \in M(f) \right\}.$$

In particular, $\boldsymbol{\pi}(x) = \{p \in \Delta S \mid \varphi(p) = 1\}$ for all constant acts x.

Smooth Model

The *smooth model* of ambiguity developed in Klibanoff, Marinacci, and Mukerji (2005) allows preferences to display non-neutral attitudes towards ambiguity, but avoids kinks in the indifference curves.[8] This model has a representation of the form

$$V(f) = E_\mu \phi(E_p u(f))$$

[7] This result also follows from an alternate representation $V(f) = -E_q \exp\left(-\theta^{-1} \cdot u(f)\right)$ of those preferences. Strzalecki (2007) obtains an axiomatization of multiplier preferences along these lines.

[8] For similar models, see Segal (1990), Nau (2006) and Ergin and Gul (2004).

where μ is interpreted as a probability distribution on the set of probabilities that the decision maker considers plausible, $\phi : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. When the indexes ϕ and u are concave, increasing, and differentiable, this utility represents a convex preference relation, and the set of subjective beliefs is a singleton consisting of a weighted mixture of all probabilities in the support of the measure μ.

Proposition 5. *Let $\succsim$ be a smooth model preference for which u and ϕ are concave, increasing, and differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \frac{1}{\|E_\mu[\phi'(E_p u(f)) pDU(f)]\|} E_\mu[\phi'(E_p u(f)) pDU(f)].$$

In particular, $\boldsymbol{\pi}(x) = \{E_\mu p\}$ for all constant x.

Ergin-Gul Model

Ergin and Gul (2004) introduce a model in which the state space takes the product form $S = S_a \times S_b$. This model permits different decision attitudes toward events in S_a and S_b, thereby inducing Ellsberg-type behavior. Consider a product measure $p = p_a \otimes p_b$ on S; for any $f \in \mathbb{R}^S$ let $E_a f$ be the vector of conditional expectations of f computed for all elements of S_b (thus $E_a f \in \mathbb{R}^{S_b}$) and for any $g \in \mathbb{R}^{S_b}$ let $E_b g$ denote the expectation of g according to p_b. The preferences are represented by

$$V(f) = E_b \phi(E_a u(f)).$$

In order to express subjective beliefs, let $U(f)$ and $DU(f)$ be defined as before, with the convention that the states in S are ordered lexicographically first by a, then by b. Analogously, for each f define the vector $\Phi(E_a u(f)) \in \mathbb{R}^{S_b}$ and the diagonal matrix $D\Phi(E_a u(f))$.

Proposition 6. *Let $\succsim$ be an Ergin-Gul preference for which u and ϕ are concave, increasing, and differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \frac{1}{\|pDU(f)[I_a \otimes D\Phi(E_a u(f))]\|} pDU(f)[I_a \otimes D\Phi(E_a u(f))],$$

where I_a is the identity matrix of order S_a and $\otimes$ is the tensor product. In particular, $\boldsymbol{\pi}(x) = \{p\}$ for all constant x.

Remark 1: Our notion of beliefs may not agree with the beliefs identified by some representations, in part because we have focused on beliefs revealed by market behavior rather than those identified axiomatically. An illustrative case in point is rank-dependent expected utility (RDEU) of Quiggin (1982) and Yaari (1987) in which probability distributions are distorted by a transformation function. When the probability transformation function is concave, this model reduces to Choquet expected utility with a convex capacity, a special case of MEU. By using the MEU representation, beliefs would be identified with a set of priors P, in general not a singleton. As we showed above, this set P coincides with the set $\boldsymbol{\pi}(x)$, the subjective beliefs given by any constant act x. However, RDEU preferences are also probabilistically sophisticated in the sense of Machina and Schmeidler (1992), with respect to some measure p^*.[9] Using the alternative representation arising from probabilistic sophistication, beliefs would instead be identified with this unique measure p^* rather than with the set P. Although $p^* \in P$, these different representations nonetheless lead to different ways of identifying subjective beliefs, each justified by differing behavioral axioms.[10] This indeterminacy could lead to different ways of attributing market behavior to beliefs. For example, Segal and Spivak (1990) attribute unwillingness to trade to probabilistic first-order risk aversion, while Dow and Werlang (1992) instead attribute unwillingness to trade to non-probabilistic ambiguity aversion.

3 Ex-Ante Trade

In this section, we use subjective beliefs to characterize efficient allocations. As our main result, we show that in the absence of aggregate uncertainty, efficiency is equivalent to full insurance under a "common priors" condition. While we maintain the assumption of a finite state space for simplicity, all of these results extend directly to the case of an infinite state space with appropriate modifications; for details see Appendix A.

We study a standard two-period exchange economy with one consumption good in which uncertainty at date 1 is described by the set S. There are m agents in the economy, indexed by i. Each agent's consumption set is the set of acts $\mathcal{F}$. The aggregate endowment is $e \in \mathbb{R}^S_{++}$. An allocation $f = (f_1, \ldots, f_m) \in \mathcal{F}^m$ is *feasible* if $\sum_{i=1}^m f_i = e$. An allocation f is *interior* if $f_i(s) > 0$ for all s and for all i. An allocation f is a *full insurance* allocation if f_i is constant across states for all i; any other allocation will be interpreted as betting. An allocation f is *Pareto optimal* if there is no feasible allocation g such that $g_i \succsim_i f_i$ for all i and $g_j \succ_j f_j$ for some j.

Proposition 7. *Suppose $\succsim_i$ is a convex preference relation for each i. An interior allocation $(f_1, \ldots, f_m)$ is Pareto optimal if and only if $\bigcap_i \boldsymbol{\pi}_i(f_i) \neq \emptyset$.*

Proof: First, suppose $(f_1, \ldots, f_m)$ is an interior Pareto optimal allocation. By the second welfare theorem, there exists $p \in \mathbb{R}^S$, $p \neq 0$, supporting this allocation, that is, such that

[9] For more on probabilistic sophistication, RDEU and MEU, see Grant and Kajii (2005).

[10] A similar issue arises in the differing definitions of ambiguity found in the ambiguity aversion literature. One definition of ambiguity, due to Ghirardato and Marinacci (2002), takes the SEU model as a benchmark and attributes all deviations from SEU to non-probabilistic uncertainty aversion. Another definition, due to Epstein (1999), uses the probabilistic sophistication model as a benchmark and hence attributes some deviations from SEU to probabilistic first-order risk aversion rather than non-probabilistic uncertainty aversion.

$p \cdot g \geq p \cdot f_i$ for all $g \succsim_i f_i$ and each i. By monotonicity, $p > 0$, thus after normalizing we may take $p \in \Delta S$. By definition, $p \in \boldsymbol{\pi}_i(f_i)$ for each i, hence $\bigcap_i \boldsymbol{\pi}_i(f_i) \neq \emptyset$. For the other implication, take $p \in \bigcap_i \boldsymbol{\pi}_i(f_i)$. By standard arguments, $(f_1, \ldots, f_m; p)$ is a Walrasian equilibrium in the exchange economy with endowments $(f_1, \ldots, f_m)$. By the first welfare theorem, $(f_1, \ldots, f_m)$ is Pareto optimal. ■

This result provides a helpful tool to study mutual insurance and contracting between agents, regardless of the presence of aggregate uncertainty. The following example illustrates. Consider an exchange economy with two agents. The first agent has MEU preferences with set of priors P_1 and linear utility index, while the second agent has SEU preferences with prior p_2, also with a linear utility index. Assume p_2 belongs to the relative interior of P_1 (and hence that P_1 has a nonempty relative interior).[11] Thus this is an economy in which one agent is risk and ambiguity neutral, while the other is risk neutral but strictly ambiguity averse; moreover, the second agent is more ambiguity averse than the first, using the definition of Ghirardato, Maccheroni, and Marinacci (2004). In this case, an interior allocation is Pareto optimal if and only if it fully insures the ambiguity averse agent. Proposition 7 implies an interior allocation f can be Pareto optimal if and only if $p_2 \in \boldsymbol{\pi}_1(f_1)$. If f_1 does not involve full insurance for agent 1, then $\boldsymbol{\pi}_1(f_1)$ will be a subset of the extreme points of P_1, and in particular, will not contain p_2. Alternatively, at any constant bundle x_1, $\boldsymbol{\pi}_1(x_1) = P_1 \ni p_2 = \boldsymbol{\pi}_2(e - x_1)$, so any such allocation is Pareto optimal. This result can be easily extended to the case in which agent 1 is also ambiguity averse, with MEU preferences given by the same utility index and a set P_2, provided P_2 is contained in the relative interior of P_1. Similarly, risk aversion can be introduced, although for given beliefs the result will fail for sufficiently high risk aversion.

Our main results seek to characterize desire for insurance and willingness to bet as a function of shared beliefs alone. To isolate the effects of beliefs, we first rule out aggregate uncertainty by taking the aggregate endowment e to be constant across sates. In addition, we must rule out pure indifference to betting, as might occur in an SEU setting with risk neutral agents. The following two axioms guarantee that such indifference to betting is absent.

Axiom 5 (Strong Monotonicity)**.** *For all $f \neq g$, if $f \geq g$, then $f \succ g$.*

Axiom 6 (Strict Convexity)**.** *For all $f \neq g$ and $\alpha \in (0,1)$, if $f \succsim g$, then $\alpha f + (1-\alpha) g \succ g$.*

Finally, we focus on preferences for which local trade-offs in the absence of uncertainty are independent of the (constant) level of consumption. These preferences are characterized by the fact that the directions of local improvement, starting from a constant bundle at which uncertainty is absent, are independent of the particular constant.

Axiom 7 (Translation Invariance at Certainty)**.** *For all $g \in \mathbb{R}^S$ and all constant bundles $x, x' > 0$, if $x + \lambda g \succsim x$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $x' + \lambda' g \succsim x'$.*

[11] By relative interior, here we mean relative to ΔS.

This axiom will be satisfied by all of the main classes of preferences we have considered. A simple example violating this axiom is the SEU model with state-dependent utility; in this case, the slopes of indifference curves can change along the 45^o line. In fact, in the class of SEU preferences Axiom 7 is equivalent to a state-independent and differentiable utility function. We show below that for a convex preference relation, translation invariance at certainty suffices to ensure that subjective beliefs are instead constant across constant bundles.

Proposition 8. *Let $\succsim$ be a convex preference relation satisfying Axiom 7. Then $\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x')$ for all constant acts $x, x' > 0$.*

By this result, we can write $\boldsymbol{\pi}$ in place of $\boldsymbol{\pi}(x)$ when translation invariance at certainty is satisfied; we maintain this notational simplification below.

Our main result follows. For any collection of convex preferences satisfying translation invariance at certainty, the sets $\boldsymbol{\pi}_i$ of subjective beliefs contain all of the information needed to predict the presence or absence of purely speculative trade. Regardless of other features of the representation of preferences, the existence of a common subjective belief, understood to mean $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$, characterizes the efficiency of full insurance. Moreover, these results can be understood as straightforward consequences of the basic welfare theorems.

Proposition 9. *If $\succsim_i$ satisfies Axioms 1-7 for each i, then the following statements are equivalent:*

(i) There exists an interior full insurance Pareto optimal allocation.

(ii) Any Pareto optimal allocation is a full insurance allocation.

(iii) Every full insurance allocation is Pareto optimal.

(iv) $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

Proof: We show the sequence of inclusions:

(i) ⇒ (iv): Suppose that $x = (x_1, \ldots, x_m)$ is an interior full insurance allocation that is Pareto optimal. By the second welfare theorem, there exists $p \neq 0$ such that p supports the allocation x, that is, such that for each i, $p \cdot f \geq p \cdot x_i$ for all $f \succsim_i x_i$. By monotonicity, $p > 0$, so after normalizing we can take $p \in \Delta S$. By definition $p \in \boldsymbol{\pi}_i$ for all i, hence $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

(iv) ⇒ (ii): Let $p \in \bigcap_i \boldsymbol{\pi}_i$ and suppose f is a Pareto optimal allocation such that f_j is not constant for some j. Define $x_i := E_p f_i$ for each i. By strict monotonicity, $p \gg 0$. Thus $x_i \geq 0$ for all i, and $x_i = 0 \iff f_i = 0$. Since $p \in \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i(x_i) = \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i^u(x_i)$, $x_i \succsim f_i$ for all i, and by strict convexity, $x_j \succ_j f_j$. Then the allocation $x = (x_1, \ldots, x_m)$ is feasible, and Pareto dominates f, which is a contradiction.

(ii) $\Rightarrow$ *(iii)*: Suppose that x is a full insurance allocation that is not Pareto optimal. Then there is a Pareto optimal allocation f that Pareto dominates x. By *(ii)*, f must be a full insurance allocation, which is a contradiction.

(iii) $\Rightarrow$ *(i)*: The allocation $(\frac{1}{m}e, \ldots, \frac{1}{m}e)$ is an interior full insurance allocation. By *(iii)* it is Pareto optimal. ■

Remark 2: Billot, Chateauneuf, Gilboa, and Tallon (2000) derive a version of this result for the particular case of maxmin preferences using an ingenious separation argument.[12] In this case, the common prior condition *(iv)* becomes the intuitive condition $\cap_i P_i \neq \emptyset$.[13] Billot, Chateauneuf, Gilboa, and Tallon (2000) also consider the case of an infinite state space. In the appendix, we show that our result can be similarly extended to an infinite state space, although the argument is somewhat more delicate.

We view a main contribution of our result (and its extension to the infinite state space case) not as establishing the link between efficiency and notions of common priors per se, but in illustrating that these results are a simple consequence of the welfare theorems linking Pareto optimality to the existence of linear functionals providing a common support to agents' preferred sets, coupled with the particular form these supports take for various classes of preferences.

Proposition 9 can be articulated in the language of specific functional forms discussed in Section 2.4. For SEU preferences, condition *(iv)* becomes the standard common prior assumption, whereas for MEU preferences we recover the result of Billot, Chateauneuf, Gilboa, and Tallon (2000). For the smooth model of Klibanoff, Marinacci, and Mukerji (2005) condition *(iv)* means that the *expected* measures have to coincide, while for variational preferences of Maccheroni, Marinacci, and Rustichini (2006) the sets of measures with zero cost have to intersect. Interestingly, it follows that for Hansen and Sargent (2001) multiplier preferences condition *(iv)* means that the reference measures coincide.

Finally, we note that extending Propositions 7 and 9 to allow for incomplete preferences is fairly straightforward, after appropriately modifying axioms 1 and 2. [14]

Appendix A: Infinite State Space

Now we imagine that the state space S may be infinite, and let Σ be a σ-algebra of measurable subsets of S. Let $B(S, \Sigma)$ be the space of all real-valued, bounded, and measurable functions

[12] In Billot, Chateauneuf, Gilboa, and Tallon (2000) there is an imprecision in the proof that *(ii)* $\Rightarrow$ *(iii)*, which implicitly uses condition *(iv)*.

[13] See Kajii and Ui (2006) for related results regarding purely speculative trade and no-trade theorems.

[14] A similar observation is made by Rigotti and Shannon (2005), while a recent paper by Mandler (2006) studies Pareto optima for general incomplete preferences.

on S, endowed with the sup norm topology. Let $ba(S, \Sigma)$ be the space of bounded, finitely additive measures on (S, Σ), endowed with the weak* topology, and let ΔS be the subset of finitely additive probabilities. As in the finite case, we let $\mathcal{F}$ denote the set of acts, which is now $B(S, \Sigma)_+$. We continue to use $x \in \mathbb{R}_+$ interchangeably for the constant act delivering x in each state s. For an act f, a constant $x \in \mathbb{R}_+$ and an event $E \subset S$, let xEf denote the act such that

$$(xEf)(s) = \begin{cases} x & \text{if } s \in E \\ f(s) & \text{if } s \notin E \end{cases}$$

The goal of this section is to establish an analogue of our main result regarding the connection between the efficiency of full insurance and the existence of shared beliefs, Proposition 9, for infinite state spaces. Our work in section 3 renders this analogue fairly straightforward by highlighting the close link between these results and the fundamental welfare theorems, appropriate versions of which hold in infinite-dimensional settings as well.

Because topological issues are often subtle in infinite-dimensional spaces due to the multiplicity of non-equivalent topologies, we begin by emphasizing the meaning of our basic continuity axiom in this setting.

Axiom 2 (Continuity)**.** *For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} \mid g \succsim f\}$ and $\{g \in \mathcal{F} \mid f \succsim g\}$ are closed in the sup-norm topology.*

To accommodate an infinite state space, we will need several additional axioms that serve to restrict agents' beliefs, first by ensuring that beliefs are countably additive, and that beliefs are all mutually absolutely continuous both for a given agent and between different agents. To that end, consider the following:

Axiom 8 (Countable Additivity)**.** *For each f, each $p \in \boldsymbol{\pi}(f)$ is countably additive.*

Axiom 9 (Mutual Absolute Continuity)**.** *If $xEf \sim f$ for some event E and some acts x, f with $x > \sup f$, then $yEg \sim g$ for every y and every act g.*

Proposition 10. *Let $\succsim$ be monotone, continuous, convex, and satisfy mutual absolute continuity. If f, g are acts such that $\inf f, \inf g > 0$, then $\boldsymbol{\pi}(f)$ and $\boldsymbol{\pi}(g)$ contain only measures that are mutually absolutely continuous.*

Proof: Suppose, by way of contradiction, that acts f, g with $\inf f, \inf g > 0$, an event E, and measures $p \in \boldsymbol{\pi}(f), \bar{p} \in \boldsymbol{\pi}(g)$ such that $p(E) = 0$ while $\bar{p}(E) > 0$. Choose $x > \sup f$. By monotonicity, $x \succ f$ and $xEf \succsim f$. Since $p(E) = 0$,

$$p \cdot (xEf) = p \cdot f$$

Together with $p \in \boldsymbol{\pi}(f)$ this implies $xEf \sim f$. Choose y such that $y < \inf g$. By mutual continuity, $yEg \sim g$. Since $\bar{p}(E) > 0$,

$$\bar{p} \cdot (yEg) < \bar{p} \cdot g$$

But $\bar{p} \in \boldsymbol{\pi}(g)$, which yields a contradiction. ■

The same argument will show that if mutual continuity holds across agents, then all beliefs of all agents are mutually absolutely continuous. We say that a collection $\{\succsim_i : i = 1, \dots, m\}$ of preference orders on $\mathcal{F}$ *satisfies mutual absolute continuity* if whenever $xEf \sim_i f$ for some agent i, some event E, and some $x > \sup f$, then $yEg \sim_j g$ for every agent j, every y, and every act g.

Proposition 11. *Let $\succsim_i$ be monotone, continuous, and convex for each i, and let $\{\succsim_i : i = 1, \dots, m\}$ satisfy mutual absolute continuity. Then for every i, j and any acts f, g such that $\inf f, \inf g > 0$, $\boldsymbol{\pi}_i(f)$ and $\boldsymbol{\pi}_j(g)$ contain only measures that are mutually absolutely continuous.*

Mutual absolute continuity is a strong assumption, and is close to the desired conclusion of mutual absolute continuity of agents' beliefs. Without more structure on preferences, it does not seem possible to weaken, however. Without the additional structure available in various representations, nothing needs to tie together beliefs at different acts. This gives us very little to work with for general convex preferences. In contrast, in particular special cases, much weaker conditions would suffice to deliver the same conclusion. For example, Epstein and Marinacci (2007) show that a version of the modularity condition of Kreps (1979) is equivalent to mutual absolute continuity of priors in the MEU model.

For a complete analogue of our main result regarding the connection between common priors and the absence of betting, we must ensure that individually rational Pareto optimal allocations exist given any initial endowment allocation. This is needed to show that *(ii)* $\Rightarrow$ *(iii)* in Proposition 9 without the additional assumption of a common prior, that is, to show that if every Pareto optimal allocation must involve full insurance, then all full insurance allocations are in fact Pareto optimal. Since no two full insurance allocations can be Pareto ranked, this conclusion will follow immediately from the existence of individually rational Pareto optimal allocations. Instead Billot, Chateauneuf, Gilboa, and Tallon (2000) use the existence of a common prior, condition *(iv)*, to argue that any Pareto improvement must itself be Pareto dominated by the full insurance allocation with consumption equal to the expected values, computed with respect to some common prior. In the finite state space case, it is straightforward to give an alternative argument that does not make use of the common prior condition. If a full insurance allocation is not Pareto optimal, then there must exist a Pareto optimal allocation that dominates it, as a consequence of the existence of individually rational Pareto optimal allocations. When all Pareto optimal allocations involve full insurance, this leads to a contradiction that establishes the desired implication.

With an infinite state space, the existence of individually rational Pareto optimal allocations is more delicate. Typically, this existence is derived from continuity of preferences in some topology in which order intervals, and hence sets of feasible allocations, are compact. In our setting, such topological assumptions are problematic, as order intervals in $B(S,\Sigma)$ fail to be compact in topologies sufficiently strong to make continuity a reasonable and not overly restrictive assumption. Instead we give a more subtle argument that makes use of countable additivity and mutual continuity to give an equivalent formulation of the problem recast in $L_\infty(S,\Sigma,\mu)$ for an appropriately chosen measure μ.

More precisely, suppose that $\{\succsim_i : i = 1,\dots,m\}$ satisfy mutual absolute continuity. Choose a measure $\mu \in \boldsymbol{\pi}_1(x)$ for some constant x. We can extend each $\succsim_i$ to $L_\infty(S,\Sigma,\mu)_+$ in the natural way, first by embedding $B(S,\Sigma)_+$ in $L_\infty(S,\Sigma,\mu)_+$ via the identification of an act f with its equivalence class $[f] \in L_\infty(S,\Sigma,\mu)_+$, and then by noticing that a preference order satisfying our basic axioms will be indifferent over any acts $f, f' \in B(S,\Sigma)_+$ such that $f' \in [f]$. This allows us to extend each preference order $\succsim_i$ to $L_\infty(S,\Sigma,\mu)_+$ in the natural way, by defining $[f] \succsim_i [g] \iff f \succsim_i g$ for any $f, g \in B(S,\Sigma)_+$. Similarly, given a utility representation V_i of $\succsim_i$ on $B(S,\Sigma)_+$, define $V_i : L_\infty(S,\Sigma,\mu)_+ \to \mathbb{R}$ by $V_i([f]) = V_i(f)$ for each $f \in B(S,\Sigma)_+$.

With this recasting of the problem, the existence of individually rational Pareto optimal allocations follows from an additional type of continuity.

Axiom 10 (Countable Continuity)**.** *There exists $\bar{x}$ and $\mu \in \boldsymbol{\pi}(\bar{x})$ such that for all $g, f, x \in \mathcal{F}$, if $\{f^\alpha\}$ is a net in $\mathcal{F}$ with $f^\alpha \succsim x$ and $f^\alpha \leq g$ for all α, and $q \cdot f^\alpha \to q \cdot f$ for all $q \in ca(S,\Sigma)$ such that $q \ll \mu$, then $f \succsim x$.*

Proposition 12. *Let $\succsim_i$ be monotone, continuous, countably continuous, countably additive, and convex for each i, and let $\{\succsim_i : i = 1,\dots,m\}$ satisfy mutual absolute continuity. For any initial endowment allocation $(e_1,\dots,e_m)$, individually rational Pareto optimal allocations exist.*

Proof: Fix a constant act $x > 0$ and choose a measure $\mu \in \boldsymbol{\pi}_1(x)$. If f and g are μ-equivalent, so $\mu(\{s : f(s) \neq g(s)\}) = 0$, then $f \sim_i g$ for each i. To see this, fix μ-equivalent acts f and g, and an agent i. Without loss of generality suppose $g \succsim_i f$. First suppose that $\inf f, \inf g > 0$. In this case, every $p \in \boldsymbol{\pi}_i(f)$ is absolutely continuous with respect to μ, so

$$p \cdot g = p \cdot f \quad \forall p \in \boldsymbol{\pi}_i(f)$$

Thus $f \succsim_i g$, and we conclude $g \sim_i f$ as desired. For the general case, consider the sequence of constant acts $\{x^n\}$ with $x^n = \frac{1}{n}$ for each n: $\inf x^n > 0$ for each n while $x^n \to 0$ in the sup-norm topology. For each n, the acts $f + x^n$ and $g + x^n$ are μ-equivalent, and $\inf(f + x^n), \inf(g + x^n) > 0$. By the previous argument, $f + x^n \sim_i g + x^n$ for each n, and by continuity $f \sim_i g$ as desired.

For each i, extend V_i to $L_\infty(S,\Sigma,\mu)_+$ using this observation, by defining $V_i([f]) := V_i(f)$ for each $f \in B(S,\Sigma)_+$.

Fix an initial endowment allocation $(e_1, \ldots, e_m)$, and set $e := \sum_i e_i$. By the Banach-Alaoglu Theorem, the order interval $[0, e]$ is weak*-compact in $L_\infty(S, \Sigma, \mu)_+$, and by mutual absolute continuity and countable continuity, V_i is weak*-upper semi-continuous on $[0, e]$.

From this it follows by standard arguments that for every initial endowment allocation $(e_1, \ldots, e_m)$, an individually rational Pareto optimal allocation exists; for completeness we reproduce an argument from Boyd (1995); see also Theorem 1.5.3 in Aliprantis, Brown, and Burkinshaw (1989).

Define a preorder on the compact set of feasible allocations

$$\mathcal{A} := \{f \in [L_\infty(S, \Sigma, \mu)_+]^m : \sum_i f_i = e\}$$

as follows. Given feasible allocations $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_m)$, define $f \succsim g$ if $f_i \succsim_i g_i$ for each i. Set

$$\mathcal{B}(g) := \{f \in \mathcal{A} : f \succsim g\}$$

and

$$\mathcal{S} := \mathcal{B}((e_1, \ldots, e_m)) = \{f \in \mathcal{A} : f \succsim (e_1, \ldots, e_m)\}$$

Let $\mathcal{R}$ be a chain in $\mathcal{S}$. For any finite subset $\bar{\mathcal{R}}$ of $\mathcal{R}$, $\cap_{g \in \bar{\mathcal{R}}} \mathcal{B}(g) = \mathcal{B}(\max\ \bar{\mathcal{R}})$ is nonempty, by transitivity. Thus $\{\mathcal{B}(g) : g \in \mathcal{R}\}$ has the finite intersection property. Each $\mathcal{B}(g)$ is weak*-closed, hence, by compactness of $\mathcal{A}$, $\cap_{g \in \mathcal{R}} \mathcal{B}(g) \neq \emptyset$, and any element of $\cap_{g \in \mathcal{R}} \mathcal{B}(g)$ provides an upper bound for $\mathcal{R}$. By Zorn's lemma for preordered sets (see, e.g., Megginson (1998), p. 6), $\mathcal{S}$ has a maximal element, which is then an individually rational Pareto optimal allocation. ■

With this in place, we turn to the infinite version of Proposition 9. The proof is analogous, making use of an infinite-dimensional version of the second welfare theorem and our previous result establishing the existence of individually rational Pareto optimal allocations in our model. As in the finite case, the aggregate endowment e is constant, with $e > 0$, hence $\inf e > 0$. We say that $f = (f_1, \ldots, f_m) \in \mathcal{F}^m$ is a norm-interior allocation if $\inf f_i > 0$ for $i = 1, 2, \ldots m$.

Proposition 13. *Let $\{\succsim_i : i = 1, \ldots, m\}$ satisfy Axioms 1-10 and mutual absolute continuity. Then the following statements are equivalent:*

(i) There exists a norm-interior full insurance Pareto optimal allocation.

(ii) Any Pareto optimal allocation is a full insurance allocation.

(iii) Every full insurance allocation is Pareto optimal.

(iv) $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

Proof: As in the proof of Proposition 9, we show the sequence of inclusions:

(i) $\Rightarrow$ *(iv)*: Suppose that $x = (x_1, \ldots, x_m)$ is a norm-interior full insurance allocation that is Pareto optimal. Each x_i is contained in the norm interior of $B(S, \Sigma)_+$, hence by the second welfare theorem, there exists $p \in ba(S, \Sigma)$ with $p \neq 0$ such that p supports the allocation x, that is, such that for each i, $p \cdot f \geq p \cdot x_i$ for all $f \succsim_i x_i$. By monotonicity, $p > 0$, so after normalizing we can take $p \in \Delta S$. By definition $p \in \boldsymbol{\pi}_i$ for all i, hence $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

(iv) $\Rightarrow$ *(ii)*: Let $p \in \bigcap_i \boldsymbol{\pi}_i$ and suppose f is a Pareto optimal allocation such that f_j is not constant for some j. Define $x_i := E_p f_i$ for each i. By strict monotonicity, p is strictly positive, that is, $p \cdot g > 0$ for any act $g > 0$. Together with countable additivity, this yields $x_i \geq 0$ for all i, and $x_i = 0 \iff f_i = 0$. Since $p \in \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i(x_i) = \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i^u(x_i)$, $x_i \succsim f_i$ for all i, and by strict convexity, $x_j \succ_j f_j$. Then the allocation $x = (x_1, \ldots, x_m)$ is feasible, and Pareto dominates f, which is a contradiction.

(ii) $\Rightarrow$ *(iii)*: Suppose that x is a full insurance allocation that is not Pareto optimal. Using Proposition 12, there must be a Pareto optimal allocation f that Pareto dominates x. By *(ii)*, f must be a full insurance allocation, which is a contradiction.

(iii) $\Rightarrow$ *(i)*: The allocation $(\frac{1}{m}e, \ldots, \frac{1}{m}e)$ is a norm-interior full insurance allocation. By *(iii)* it is Pareto optimal. ■

We close with an example illustrating how the additional axioms arising in the infinite state space case might naturally be satisfied. We consider the version of the MEU model studied by Billot, Chateauneuf, Gilboa, and Tallon (2000). They consider an MEU model in which each agent i has a weak*-closed, convex set of priors $P_i \subset ba(S, \Sigma)$ consisting only of countably additive measures, and a utility index $u_i : \mathbb{R}_+ \to \mathbb{R}$ that is strictly increasing, strictly concave, and differentiable. In addition, they assume that all measures in P_i and P_j are mutually absolutely continuous for all i and j. It straightforward to verify that $P_i = \boldsymbol{\pi}_i$ for each i, as in the finite state case, and that the model satisfies countable additivity. To verify mutual absolute continuity, suppose that $x > \sup f$ but $xEf \sim_i f$ for some event E and some agent i. Using Theorems 3 and 5 of Epstein and Marinacci (2007), there must exist $p \in P_i$ such that $p(E) = 0$. Because all measures in P_i and P_j for any other j are assumed to be mutually absolutely continuous, it must be the case that $p(E) = 0$ for any $p \in P_j$ for any agent j, which guarantees that $yEg \sim_j g$ for all j and any other acts y, g.

To see that continuity and countable continuity are also satisfied, first take $\{f^n\}, f$ in $\mathcal{F}$ with $\|f^n - f\| \to 0$. Then

$$\begin{aligned}
|V_i(f^n) - V_i(f)| &= |\min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f^n)) - \min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f))| \\
&\leq \max\{|E_{p^{n*}}(u_i(f^n) - u_i(f))|, |E_p(u_i(f^n) - u_i(f))|\}
\end{aligned}$$

where $p^{n*} \in M(f^n)$ and $p^* \in M(f)$.[15] Since $\|u_i(f^n) - u_i(f)\| \to 0$, $|V_i(f^n) - V_i(f)| \to 0$, and the desired conclusion follows.

Next, to see that countable continuity is also satisfied, fix $\mu \in \boldsymbol{\pi}_1$ and an agent i. Take $g, f, x \in \mathcal{F}$ and a net $\{f^\alpha\}$ in $\mathcal{F}$ with $f^\alpha \succsim_i x$ and $f^\alpha \leq g$ for all α. Notice that it suffices to show that the set $\{f \in L_\infty(S, \Sigma, \mu)_+ : f \succsim_i x, f \in [0, g]\}$ is $\sigma(L_\infty(S, \Sigma, \mu), L_1(S, \Sigma, \mu))$-closed, with $\succsim_i$ and acts recast in $L_\infty(S, \Sigma, \mu)$ as in Proposition 12. Using convexity, this is equivalent to showing that this set is closed in the Mackey topology $\tau := \tau(L_\infty(S, \Sigma, \mu), L_1(S, \Sigma, \mu))$. Thus suppose $f^\alpha \xrightarrow{\tau} f$. By way of contradiction, suppose that $x \succ_i f$, thus $V_i(x) = E_{p^*}(u_i(x)) > E_{p^*}(u_i(f))$, where as above $p^* \in M(f)$. Then for every α,

$$E_{p^*}(u_i(f^\alpha)) \geq V_i(f^\alpha) \geq E_{p^*}(u_i(x)) > E_{p^*}(u_i(f))$$

while

$$\begin{aligned} 0 < E_{p^*}(u_i(x)) - E_{p^*}(u_i(f)) \leq E_{p^*}(u_i(f^\alpha)) - E_{p^*}(u_i(f)) &= E_{p^*}(u_i(f^\alpha) - u_i(f)) \\ &= |E_{p^*}(u_i(f^\alpha) - u_i(f))| \\ &\leq E_{p^*}(|u_i(f^\alpha) - u_i(f)|) \\ &\leq E_{p^*}(K|f^\alpha - f|) \end{aligned}$$

for some $K > 0$, where the last inequality follows from the assumption that u_i is strictly concave, strictly increasing, and differentiable, hence Lipschitz continuous. Since τ is locally solid, $|f^\alpha - f| \xrightarrow{\tau} 0$, from which it follows that $|f^\alpha - f| \xrightarrow{w^*} 0$ as well. Since $p^* \ll \mu$ and p^* is countably additive, by appealing to the Radon-Nikodym Theorem, $E_{p^*}(K|f^\alpha - f|) \to 0$. As this yields a contradiction, $f \succsim_i x$ as desired.

Appendix B: Proofs

We will use the fact that $\{g | g \succ f\} = \operatorname{int}\{g | g \succsim f\}$ and $\{g | g \succsim f\} = \operatorname{cl}\{g | g \succ f\}$. Let $\langle f, g \rangle$ denote the inner product of f and g and ∂I be the superdifferential of a concave function I.

Proof of Proposition 1. Using continuity, monotonicity, and convexity, standard arguments yield the equivalence of $\boldsymbol{\pi}(f)$ and $\boldsymbol{\pi}^u(f)$ for any strictly positive act f.

To show that $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^w(f)$ as well, we first observe that by definition, the set $\boldsymbol{\pi}(f)$ is the set of normals to the convex upper contour set $B(f) := \{g \in \mathbb{R}^S : g \succsim f\}$ at f, normalized to lie in ΔS. Let $T_{B(f)}(f)$ denote the tangent cone to $B(f)$ at f, which is given by:

$$T_{B(f)}(f) = \{g \in \mathbb{R}^S : f + \lambda g \succsim f \text{ for some } \lambda > 0\}$$

[15] As in the finite state space case, $M(f) := \arg\min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f))$.

From standard convex analysis results, $\boldsymbol{\pi}(f)$ is also the set of normals to $T_{B(f)}(f)$, again normalized to lie in ΔS. Thus

$$\boldsymbol{\pi}(f) = \{p \in \Delta S : p \cdot g \geq 0 \text{ for all } g \in T_{B(f)}(f)\}$$

and $g \in T_{B(f)}(f) \iff p \cdot g \geq 0$ for all $p \in \boldsymbol{\pi}(f)$. Then

$$\begin{aligned} & g' \in T_{B(f)}(f) + \{f\} = \{h \in \mathbb{R}^S : (1-\varepsilon)f + \varepsilon h \succsim f \text{ for some } \varepsilon > 0\} \\ \iff & p \cdot g' \geq p \cdot f \text{ for all } p \in \boldsymbol{\pi}(f) \end{aligned}$$

Thus $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^w(f)$. ∎

For many of the results in the section on special cases, we make use of the following lemma.

Lemma 1. *Assume that $\succsim$ satisfies Axioms 1-4 and the representation V of $\succsim$ is concave. Then $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^\partial(f) := \{\frac{q}{\|q\|} | q \in \partial V(f)\}$.*

Proof: First, we show that $\boldsymbol{\pi}^\partial(f) \subseteq \boldsymbol{\pi}(f)$. Let $p = \frac{q}{\|q\|}$ for some $q \in \partial V(f)$. Let $V(g) \geq V(f)$. We have $0 \leq V(g) - V(f) \leq \langle q, g - f\rangle$, hence $\langle q, f\rangle \leq \langle q, g\rangle$, so $E_p g \geq E_p f$. Second, we show that $\boldsymbol{\pi}^\partial(f) \in \mathcal{P}(f)$, thus $\boldsymbol{\pi}^w(f) \subseteq \boldsymbol{\pi}^\partial(f)$. Let g be such that $E_p g > E_p f$ for all $p \in \boldsymbol{\pi}^\partial(f)$. We need to find $\varepsilon > 0$ with $V(\varepsilon g + (1-\varepsilon)f) > V(f)$. The one-sided directional derivatives $V'(f;h)$ exist for all $h \in \mathbb{R}^S$, and $V'(f;h) = \min\{\langle l, h\rangle | l \in \partial V(f)\}$.[16] Hence, for some $q \in \partial V(f)$:

$$\begin{aligned} V(\varepsilon g + (1-\varepsilon)f) &= V(f + \varepsilon(g - f)) \\ &= V(f) + \varepsilon V'(f; g - f) + o(\varepsilon) \\ &= V(f) + \varepsilon \min\{\langle l, g - f\rangle |\, l \in \partial V(f)\} + o(\varepsilon) \\ &= V(f) + \varepsilon\langle q, g - f\rangle + o(\varepsilon) \\ &= V(f) + \varepsilon[\langle q, g - f\rangle + o(1)]. \end{aligned}$$

Because $q = \|q\|p$ for some $p \in \boldsymbol{\pi}^\partial(f)$, $\langle q, g - f\rangle = \|q\| E_p(g - f) > 0$. Therefore, there exists a $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$, $\varepsilon[E_p(g - f) + o(1)] > 0$, hence $V(\varepsilon g + (1-\varepsilon)f) > V(f)$. ∎

Proof of Proposition 3. It follows from the proof of Theorem 3 in Maccheroni, Marinacci, and Rustichini (2006) that $I(\xi) = \min_{p \in \Delta S}(E_p \xi + c^\star(p))$ is concave. This, together with concavity of u, yields the concavity of V. Continuity and monotonicity follow from the fact that I is monotonic and sup-norm Lipschitz continuous. By Theorem 18 of Maccheroni, Marinacci, and Rustichini (2006),

$$\partial V(f) = \{q \in \mathbb{R}^S : q = pDU(f) \text{ for some } p \in M(f)\}.$$

[16]Theorem 23.4 of Rockafellar (1970) implies that $V'(f;h) = \inf\{\langle l, h\rangle | l \in \partial V(f)\}$ for all h. Because V is a proper concave function, $\partial V(f)$ is a compact set, hence the infimum is achieved.

The result follows from Lemma 1. ■

Proof of Proposition 2. This follows from Proposition 3 by noting that MEU is the special case of variational preferences for which

$$c^\star(p) = \begin{cases} 0 & \text{if } p \in P \\ \infty & \text{if } p \notin P. \end{cases}$$ ■

Proof of Proposition 4. It follows from Lemma 8 in Chateauneuf and Faro (2006) that $I(\xi) = \min_{p \in L_\alpha} \frac{1}{\varphi(p)} E_p \xi$ is concave. This, together with concavity of u, yields the concavity of V. Continuity and monotonicity follow from the fact that I is monotonic and sup-norm Lipschitz continuous (see Lemma 6 in Chateauneuf and Faro 2006). By Clarke (1983) (2.8, Cor. 2),

$$\partial V(f) = \{q \in \mathbb{R}^S : q = pDU(f) \text{ for some } p \in M(f)\}.$$

The result follows from Lemma 1. ■

Proof of Proposition 5. Continuity, monotonicity and convexity are routine. When u and ϕ are concave and differentiable, it is straightforward to see that V is also concave and differentiable, and that $\partial V(f) = \{DV(f)\} = \{E_\mu[D\phi(E_p u(f))pDU(f)]\}$. ■

Proof of Proposition 6. Continuity, monotonicity and convexity are routine. When u and ϕ are concave and differentiable, it is straightforward to see that V is also concave and differentiable. A direct calculation of directional derivatives reveals that $\partial V(f) = \{DV(f)\} = \{pDU(f)[I_a \otimes D\Phi(E_a u(f))]\}$. ■

Proof of Proposition 8. Fix constant acts $x, x' > 0$, and let $B(x) := \{f \in \mathbb{R}^S_+ : f \succsim x\}$ denote the upper contour set of $\succsim$ at x. As in the proof of Proposition 1, let $T_{B(x)}(x)$ denote the tangent cone to $B(x)$ at x:

$$T_{B(x)}(x) = \{g \in \mathbb{R}^S : x + \lambda g \succsim x \text{ for some } \lambda > 0\}$$

Again as in the proof of Proposition 1, $\boldsymbol{\pi}(x)$ is the normal cone to $T_{B(x)}(x)$, analogously for $\boldsymbol{\pi}(x')$. By translation invariance at certainty, $T_{B(x)}(x) = T_{B(x')}(x')$, from which we conclude that $\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x')$. ■

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