On existence of rich Fubini extensions*

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Abstract

This note supplements recent work by Sun (2006) on Fubini extensions. In particular, it is shown that sufficiently rich Fubini extensions can be obtained without appealing to constructions from nonstandard analysis.

1 Introduction

In many contexts of economics, a large finite set is idealized by the continuum. The prototype example is Aumann's (1964) model of economies with a continuum of agents. In this spirit, there is also the desire to get, with a continuum of random variables, an "exact version" of the classical law of large numbers. However, there are mathematical difficulties with this idea, as was first noted in the economic literature by Judd (1985) and Feldman and Gilles (1985). Nevertheless, there are positive results in the direction of an exact law of large numbers. The pioneering work is by Green (1994, first draft 1988). Subsequent results are by Al-Najjar (2004), Alós-Ferrer (2002), Anderson (1991), Sun (1998, 2006), and Uhlig (1996). The views of these authors differ, though.

The approach in Sun (2006) is to derive an exact law of large numbers from measurability of a process with respect to a Fubini extension of the product measure corresponding to a parameter probability space and a sample space. Existence of Fubini extensions that are "rich" in the sense that non-trivial processes for which an exact law of large numbers holds indeed exist was shown by Sun using Loeb space constructions. In this note, we show that such Fubini extensions can be obtained without appealing to Loeb spaces.

Let us start with some definitions, taken from Sun (2006), but slightly reformulated here concerning notation.

Definition 1. Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $(X \times Y, \Lambda, \lambda)$ the corresponding product probability space. Let $\overline{\lambda}$ be a probability measure on $X \times Y$, and $\overline{\Lambda}$ its domain. Then $\overline{\lambda}$ is said to be a *Fubini extension* of λ if (a) $\overline{\Lambda} \supset \Lambda$ and (b) for each $H \in \overline{\Lambda}$—denoting by χH the characteristic function of H—the integrals $\iint \chi H(x,y)\,d\nu(y)\,d\mu(x)$ and $\iint \chi H(x,y)\,d\mu(x)\,d\nu(y)$ are well defined and $\iint \chi H(x,y)\,d\nu(y)\,d\mu(x) = \overline{\lambda}(H) = \iint \chi H(x,y)\,d\mu(x)\,d\nu(y)$.

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Note that (a) and (b) in this definition imply that $\overline{\lambda}$ must agree with λ on Λ. Also note that this definition implies that if $f\colon X \times Y \to \mathbb{R}$ is a $\overline{\Lambda}$-measurable function, then for almost all $x \in X$, the x-sections $f(x,\cdot)$ are measurable for the ν-completion of T, and similarly for the y-sections. From this it follows in turn that an analogous statement holds for functions from $X \times Y$ to any Polish space. The definition also implies that the *conclusion* of Fubini's theorem holds for $\overline{\lambda}$-integrable functions from $X \times Y$ to $\mathbb{R}$.

Definition 2. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, Z a Polish space, and $f\colon X\times Y \to Z$ a function such that for almost all $x \in X$, $f(x,\cdot)$ is measurable for the μ-completion of T and the Borel sets of Z. Then the family $\langle f(x,\cdot)\rangle_{x\in X}$ is said to be *essentially pairwise independent* if there is a null set N in X such that for each $x \in X \setminus N$ the functions $f(x,\cdot)$ and $f(x',\cdot)$ are stochastically independent for almost all $x' \in X$.

Let (X, Σ, μ), (Y, T, ν), and Z be as in this latter definition, and let λ be the product measure on $X \times Y$ given by μ and ν. As shown by Sun (2006), if a function $f\colon X \times Y \to Z$ is measurable with respect to the domain $\overline{\Lambda}$ of some Fubini extension $\overline{\lambda}$ of λ, then essentially pairwise independence of the family $\langle f(x,\cdot)\rangle_{x\in X}$ implies that this family satisfies an exact law of large numbers. Of course, the requirements are trivially satisfied for a constant valued function, and a question is whether there exist Fubini extensions such that the criterion in the following definition is satisfied.

Definition 3. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure. Let $\overline{\lambda}$ be a Fubini extension of λ, and $\overline{\Lambda}$ its domain. The Fubini extension $\overline{\lambda}$ is called a *rich Fubini extension* if there is a $\overline{\Lambda}$-measurable function $f\colon X \times Y \to [0,1]$ such that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent and for almost all $x \in X$, the distribution of the function $f(x,\cdot)$ is the equal distribution on $[0,1]$.

Recall that any Borel probability measure τ on a Polish space Z is the distribution of some measurable function g defined on $([0,1], \mathcal{B}, \rho)$, where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$ and ρ is Lebesgue measure. Hence, if f is as in Definition 3, and f' is the composition $g \circ f$, then f' is a $\overline{\Lambda}$-measurable function from $X \times Y$ to Z such that the family $\langle f'(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent, and for almost all $x \in X$, the distribution of $f'(x,\cdot)$ is τ. In particular, by the Fubini property of $\overline{\Lambda}$, the distribution of f' is equal to τ. Thus the word "rich" in Definition 3 is justified.

Results on the existence of rich Fubini extensions were established by Sun (1998, Theorem 6.2) and Sun (2006, Proposition 5.6), using nonstandard analysis, and in particular Loeb space constructions. The purpose of this note is to show that one can get rich Fubini extensions without appealing to Loeb spaces. In view of the results in Sun (2006) on the exact law of large numbers via general Fubini extensions, the results in our note imply, in particular, that one can get exact laws of large numbers without appealing to Loeb spaces and nonstandard analysis.

2 Notation, conventions, and further definitions

If (X, Σ, μ) is any measure space, $\operatorname{cov}\mathcal{N}(\mu)$ denotes the least cardinal of any family of μ-null sets which covers X, provided such a family exists. We let $\operatorname{cov}\mathcal{N}(\mu)$ be undefined if no such family exists. Thus, if κ is a cardinal and it is written, e.g., "$\operatorname{cov}\mathcal{N}(\mu) \le \kappa$," then this is understood to imply that X *can* be covered by a family of μ-null sets.

For a non-empty set I, ν_I denotes the usual measure on $\{0,1\}^I$. In particular, $\nu_{\mathbb{N}}$ denotes the usual measure on $\{0,1\}^{\mathbb{N}}$; $\nu_{\mathbb{N}}^B$ denotes the restriction of $\nu_{\mathbb{N}}$ to the Borel σ-algebra of $\{0,1\}^{\mathbb{N}}$.

If (X, Σ, μ) is any measure space, "measurable" for a mapping $f\colon X \to \{0,1\}^{\mathbb{N}}$ always means measurable with respect to the Borel (= Baire) sets of $\{0,1\}^{\mathbb{N}}$.

For convenience, we will work with the following restatement of Definition 3. (Recall for this that $[0,1]$ with Lebesgue measure and $\{0,1\}^{\mathbb{N}}$ with its usual measure are isomorphic as measure spaces.)

Definition 4. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure. Let $\overline{\lambda}$ be a Fubini extension of λ, and $\overline{\Lambda}$ its domain. The Fubini extension $\overline{\lambda}$ is called a *rich Fubini extension* if there is a $\overline{\Lambda}$-measurable function $f\colon X \times Y \to \{0,1\}^{\mathbb{N}}$ such that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent and for almost all $x \in X$, the distribution of the function $f(x,\cdot)$ is equal to $\nu_{\mathbb{N}}^B$.

Let (X, Σ, μ), (Y, T, ν), and λ be as in this definition. By Sun (2006, Theorem 4.2) (see also Theorem 3 below), there can be no rich Fubini extension of λ if one of the σ-algebras Σ and T, say Σ, has a non-negligible element A such that the trace of Σ on A is essentially countably generated. For this reason we consider probability spaces that satisfy the criterion in the following definition.

Definition 5. Let (X, Σ, μ) be a probability space and $(\mathfrak{A}, \hat{\mu})$ its measure algebra. The measure μ (or the measure space (X, Σ, μ)) is said to be *super-atomless* if each non-zero principal ideal in $\mathfrak{A}$ has uncountable Maharam type.[1]

Examples of super-atomless probability spaces are $\{0,1\}^I$ with its usual measure when I is an uncountable set, the product measure space $[0,1]^I$ where each factor is endowed with Lebesgue measure when I is uncountable, subsets of these spaces with full outer measure when endowed with the subspace measure, atomless Loeb probability spaces. Further, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure (see Podczeck, 2008).[2] We also need the following definition.

[1]We refer to Fremlin (2002) for terminology and facts concerning measure algebras.

[2]A super-atomless probability measure obtained in this way differs in a significant way from atomless Loeb measures. Indeed, let (X, Σ, μ) be any atomless Loeb probability space, Z a Polish space, $f\colon X \to Z$ a measurable mapping, and ν the distribution of f on Z. Then by Keisler and Sun (2002), for ν-almost all $z \in Z$ the inverse image $f^{-1}(\{z\})$ has a cardinality at least as large as that of the continuum. Of course, such an implication does not hold for a super-atomless probability space constructed via an extension of a Borel probability measure on some Polish space.

Definition 6. Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \hat{\mu})$. For an uncountable cardinal κ, the measure μ (or the measure space (X, Σ, μ)) is said to be *κ-super-atomless* if $\kappa = \min\{\kappa' : \kappa'$ is the Maharam type of some non-zero principal ideal in $\mathfrak{A}\}$.[3]

3 Results

Theorem 1. *Given any super-atomless probability space (X, Σ, μ), there is probability space (Y, T, ν) (also super-atomless) such that the product measure corresponding to μ and ν has a rich Fubini extension.*

Note that in Theorem 1, for the given probability space (X, Σ, μ) we can in particular have that $X = [0, 1]$ and μ is any extension of Lebesgue measure on $[0, 1]$ to a super-atomless measure. We also note, writing $\mathfrak{c}$ for the cardinal of the continuum:

Remark 1. In Theorem 1, if $\#(X) \leq \mathfrak{c}$ then the probability space (Y, T, ν) can be chosen so that $\#(Y) = \mathfrak{c}$. (For an argument establishing this, see subsection 4.3.) In particular, (Y, T, ν) can be chosen with $\#(Y) = \mathfrak{c}$ if $X = [0, 1]$ and μ is any extension of Lebesgue measure on $[0, 1]$ to a super-atomless measure.

A concrete version of Theorem 1 is contained in the next result. Actually, this next result is a "Loeb space free" version of Proposition 5.6 in Sun (2006).

Theorem 2. *Let (X, Σ, μ) be any super-atomless probability space. Then there is a probability measure ν on $(\{0, 1\}^{\mathbb{N}})^X$ such that the product probability measure on $X \times (\{0, 1\}^{\mathbb{N}})^X$ corresponding to μ and ν has a rich Fubini extension, say $\overline{\lambda}$ with domain $\overline{\Lambda}$. The measure ν and the Fubini extension $\overline{\lambda}$ can be chosen in such a way that the coordinate projections function $f : X \times (\{0, 1\}^{\mathbb{N}})^X \to \{0, 1\}^{\mathbb{N}}$, given by $f(x, y) = y(x)$, has the following properties: (a) f is $\overline{\Lambda}$-measurable; (b) the family $\langle f(x, \cdot) \rangle_{x \in X}$ is i.i.d. for ν with distribution $\nu_{\mathbb{N}}^{B}$, thus, in particular, essentially pairwise independent for the marginals μ and ν of $\overline{\lambda}$.*

Can it be proved that, given *any two* super-atomless probability spaces, the corresponding product measure has a rich Fubini extension? Unfortunately, the answer is no. Consider $\{0, 1\}^{\omega_1}$ with its usual measure ν_{ω_1}, where ω_1 is the least uncountable cardinal. It cannot be proved in ZFC that $\operatorname{cov} \mathcal{N}(\nu_{\omega_1}) = \omega_1$.[4] On the other hand, $\{0, 1\}^{\omega_1}$ is Maharam-type-homogeneous with Maharam type ω_1. But this implies that if $\operatorname{cov} \mathcal{N}(\nu_{\omega_1}) > \omega_1$, then the product measure corresponding to two copies of $\{0, 1\}^{\omega_1}$ cannot have a rich Fubini extension. In fact, the next theorem provides necessary conditions for rich Fubini extensions to exist.

[3] Recall that the cardinals are well-ordered, so the definition makes sense.

[4] Recall that Martin's axiom implies that $\operatorname{cov} \mathcal{N}(\nu_{\omega_1}) = \mathfrak{c}$ (see Fremlin, 2005, 523Y(f)(ii) and 517O(b) and (d)) and that it is (relatively) consistent with ZFC that Martin's axiom holds and $\omega_1 < \mathfrak{c}$.

Theorem 3. *Let (X, Σ, μ) and (Y, T, ν) be probability spaces. If the product probability measure on $X \times Y$ determined by μ and ν has a rich Fubini extension, then the following hold.*

(a) *Each non-zero principal ideal of the measure algebra of ν has Maharam type $\geq \operatorname{cov} \mathcal{N}(\mu)$.*

(b) *Each non-zero principal ideal of the measure algebra of μ has Maharam type $\geq \operatorname{cov} \mathcal{N}(\nu)$.*

Theorem 3 implies in particular that if (X, Σ, μ) and (Y, T, ν) are probability spaces, then in order for the corresponding product probability measure to have a rich Fubini extension, it is necessary that the measure algebras of both μ and ν do not contain non-zero principal ideals with countable Maharam type; in particular, it is necessary that both probability spaces are atomless.[5]

The following result provides sufficient conditions in order that the product measure corresponding to two given probability spaces have a rich Fubini extension.

Theorem 4. *Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure on $X \times Y$. Suppose that for some uncountable cardinals α and β, μ is α-super-atomless and ν is β-super-atomless. Further suppose that for some cardinal κ, with $\kappa \leq \min\{\alpha, \beta\}$, there is a non-decreasing family $\langle M_\xi \rangle_{\xi<\kappa}$ of null sets in X with $\bigcup_{\xi<\kappa} M_\xi = X$ and a non-decreasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of null sets in Y with $\bigcup_{\xi<\kappa} N_\xi = Y$. Then λ has a rich Fubini extension.*

The hypotheses in Theorem 4 can be satisfied, as shown in the following example.

Example. Let κ be any cardinal with uncountable cofinality, and consider $\{0,1\}^\kappa$ with its usual measure ν_κ. Fix any $\overline{x} \in \{0,1\}^\kappa$ and for each $\xi < \kappa$, let

$$N_\xi = \{x \in \{0,1\}^\kappa : x(\eta) = \overline{x}(\eta) \text{ for all } \eta < \kappa \text{ with } \eta \geq \xi\}.$$

Set $X = \bigcup_{\xi<\kappa} N_\xi$, let μ be the subspace measure on X induced by ν_κ, and Σ the domain of μ. As κ has uncountable cofinality, X intersects every non-empty subset of $\{0,1\}^\kappa$ that is determined by coordinates in some countably subset of κ. Thus X has full outer measure for ν_κ. This implies that μ is a probability measure and that the measure algebra of μ can be identified with that of ν_κ. According to a standard fact, ν_κ is Maharam-type-homogeneous with Maharam type κ, and it follows that μ has the same property. In our terminology, this means μ is κ-super-atomless. Note that for any $\xi < \kappa$, N_ξ is a ν_κ-null set in $\{0,1\}^\kappa$ since all of its elements agree on some infinite subset of κ. Hence for any $\xi < \kappa$, N_ξ is a μ-null set in X. Evidently the family $\langle N_\xi \rangle_{\xi<\kappa}$ is non-decreasing. Thus (X, Σ, μ) provides an example as desired. (If $\kappa \leq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinal of the continuum, the argument can be refined to yield an X with $\#(X) = \mathfrak{c}$; c.f. the proof of Theorem 5.)

[5]We remark that these latter facts were already noted by Sun (2006).

Recall that if (X, Σ, μ) is any complete atomless probability space, there is a mapping $f \colon X \to [0,1]$ which is inverse measure preserving for μ and Lebesgue measure on $[0,1]$. Hence if $[0,1]$ can be covered by a non-deceasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of Lebesgue null sets for some cardinal κ, then any atomless probability space (X, Σ, μ) can be covered by a non-deceasing family $\langle M_\xi \rangle_{\xi<\kappa}$ of μ-null sets (with the same κ). Thus we have the following corollary of Theorem 4.

Corollary 1. *Suppose that $[0,1]$ can be covered by a non-decreasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of Lebesgue null sets. Then given any probability spaces (X, Σ, μ) and (Y, T, ν) such that μ is α-super-atomless with $\alpha \geq \kappa$, and ν is β-super-atomless with $\beta \geq \kappa$, the product measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension.*

If the continuum hypothesis is true then $[0,1]$ can be covered by ω_1 many Lebesgue null sets, denoting by ω_1 the least uncountable cardinal number. Hence Corollary 1 implies:

Corollary 2. *If the continuum hypothesis holds then given any super-atomless probability spaces (X, Σ, μ) and (Y, T, ν), the product measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension.*

Recall that a weakening of the continuum hypothesis is given by Martin's axiom, but that Martin's axiom still implies that the union of fewer than $\mathfrak{c}$ many Lebesgue null sets in $[0,1]$ is a Lebesgue null set, where $\mathfrak{c}$ is the cardinal of the continuum. Thus under Martin's axiom the hypothesis on $[0,1]$ in Corollary 1 holds for $\kappa = \mathfrak{c}$. Hence Corollary 1 implies:

Corollary 3. *Suppose Martin's axiom is true. Then given any probability spaces (X, Σ, μ) and (Y, T, ν) such that μ is α-super-atomless with $\alpha \geq \mathfrak{c}$, and ν is β-super-atomless with $\beta \geq \mathfrak{c}$, the product measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension.*

The final result of this note is:

Theorem 5. *Let X and Y be Polish spaces, μ an atomless Borel probability measure on X, and ν an atomless Borel probability measure on Y. Then there is a super-atomless probability measure μ' on X which extends μ, and a super-atomless probability measure ν' on Y which extends ν, such that the product measure on $X \times Y$ corresponding to μ' and ν' has a rich Fubini extension.*

4 Proofs

4.1 Lemmata

Lemma 1. *Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $(X \times Y, \Lambda, \lambda)$ the corresponding product probability space. Suppose there is a sequence $\langle H^i \rangle_{i \in \mathbb{N}}$ of subsets of $X \times Y$ such that:*

(a) *There is a null set N in X such that for each $x \in X \setminus N$, the x-section H^i_x is in T with $\nu(H^i_x) = 1/2$ for all $i \in \mathbb{N}$.*

(b) *There is a null set N in Y such that for each $y \in Y \setminus N$, the y-section H^i_y is in Σ with $\mu(H^i_y) = 1/2$ for all $i \in \mathbb{N}$.*

(c) *For each $B \in \mathrm{T}$ there is null set N_B in X such that for each $x \in X \setminus N_B$, B and the sections H^i_x, $i \in \mathbb{N}$, form a stochastically independent family in T.*

(d) *For each $A \in \Sigma$ there is null set N_A in Y such that for each $y \in Y \setminus N_A$, A and the sections H^i_y, $i \in \mathbb{N}$, form a stochastically independent family in Σ.*

Then λ has a rich Fubini extension $\overline{\lambda}$ such that the domain of $\overline{\lambda}$ contains all the sets H^i, $i \in \mathbb{N}$, and such that a function $f \colon X \times Y \to \{0, 1\}^{\mathbb{N}}$ which witnesses richness of $\overline{\lambda}$ is given by setting, for each $(x, y) \in X \times Y$ and $i \in \mathbb{N}$,

$$f^i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in H^i \\ 0 & \text{if } (x, y) \notin H^i. \end{cases}$$

Proof. Let $\mathcal{F}$ denote the set of all subsets $F \subset X \times Y$ such that the integrals $\int_X \nu(F_x)\, d\mu(x)$ and $\int_Y \mu(F_y)\, d\nu(y)$ are well defined and equal. Then $\mathcal{F}$ is a Dynkin class (i.e. $\varnothing \in \mathcal{F}$ and $\mathcal{F}$ is closed against forming complements and unions of disjoint sequences) as may easily be checked. Also, (a) to (d) imply that whenever $A_1 \times B_1, \ldots, A_n \times B_n$ are finitely many measurable rectangles in $X \times Y$ and $H^{i_1}, \ldots, H^{i_m}$ is a finite subfamily of $\langle H^i \rangle_{i \in \mathbb{N}}$, then the intersection

$$(A_1 \times B_1) \cap \cdots \cap (A_n \times B_n) \cap H^{i_1} \cap \cdots \cap H^{i_m}$$

belongs to $\mathcal{F}$. Therefore, by the monotone class theorem, there is a σ-algebra $\Lambda' \subset \mathcal{F}$ which contains all measurable rectangles in $X \times Y$ and all the sets H^i, $i \in \mathbb{N}$. Define $\lambda' \colon \Lambda' \to \mathbb{R}$ by setting $\lambda'(F) = \int_X \nu(F_x)\, d\mu(x)$ for $F \in \Lambda'$. Using the monotone convergence theorem, it follows that λ' is a probability measure on $X \times Y$. Let $\overline{\lambda}$ be its completion, and $\overline{\Lambda}$ the domain of $\overline{\lambda}$. Then since $\mathcal{F}$ contains all measurable rectangles in $X \times Y$, we have $\overline{\Lambda} \supset \Lambda$. By construction, the Fubini property holds for the characteristic functions of the elements of Λ', which in particular implies that if N is a λ'-null set in $X \times Y$, then for μ-almost every $x \in X$, the x-section of N is a ν-null set in Y, and for ν-almost every $y \in Y$, the y-section of N is a μ-null set in X. Consequently, the Fubini property holds for the characteristic functions of the elements of $\overline{\Lambda}$. In particular, $\overline{\lambda}$ coincides with λ on Λ. Thus $\overline{\lambda}$ is a Fubini extension of λ such that the domain $\overline{\Lambda}$ of $\overline{\lambda}$ contains all the sets H^i, $i \in \mathbb{N}$. Note that we have $\overline{\lambda}(H^i) = 1/2$ for all $i \in \mathbb{N}$.

Now consider the function $f\colon X \times Y \to \{0,1\}^{\mathbb{N}}$ defined in the statement of the lemma. Since $H^i \in \overline{\Lambda}$ for each $i \in \mathbb{N}$, f is measurable for $\overline{\Lambda}$ and the Borel sets of $\{0,1\}^{\mathbb{N}}$.

It remains to show that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent, and that for almost every $x \in X$, $f(x,\cdot)$ is inverse measure preserving for ν and $\nu^B_{\mathbb{N}}$. To this end, for each $x \in X$ let T_x denote the σ-algebra on Y generated by the family $\langle H^i_x\rangle_{i\in\mathbb{N}}$, and let $\overline{N}$ be a null set in X chosen according condition (a). In particular, then, for each $x \in X\setminus\overline{N}$, T_x is a sub-σ-algebra of T. Also, in view of (c), we may assume that for each $x \in X\setminus\overline{N}$, the family $\langle H^i_x\rangle_{i\in\mathbb{N}}$ is stochastically independent (applying (c) e.g. to $B = Y$ and replacing $\overline{N}$ by a larger null set, if necessary).

Fix any $\overline{x} \in X\setminus\overline{N}$. Applying (c) to each finite intersection of elements of the family $\langle H^i_{\overline{x}}\rangle_{i\in\mathbb{N}}$, we may see that there is a null set $N_{\overline{x}}$ in X such that for each $x \in X\setminus N_{\overline{x}}$, the family of all the sets H^i_x, $i \in \mathbb{N}$, and $H^i_{\overline{x}}$, $i \in \mathbb{N}$, is a stochastically independent family in T. But this implies that for each $x \in X\setminus N_{\overline{x}}$, the σ-algebras $\mathrm{T}_{\overline{x}}$ and T_x are stochastically independent. Now the definition of f implies that for each $x \in X$, $f(x,\cdot)$ is measurable for T_x and the Borel sets of $\{0,1\}^{\mathbb{N}}$, and it follows that for each $x \in X\setminus N_{\overline{x}}$, $f(x,\cdot)$ and $f(\overline{x},\cdot)$ are stochastically independent.

Since this argument applies to each fixed $\overline{x} \in X\setminus\overline{N}$, it follows that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent. Finally, note that if $x \in X\setminus\overline{N}$, then since $\langle H^i_x\rangle_{i\in\mathbb{N}}$ is stochastically independent for such an x, $f(x,\cdot)$ is inverse measure preserving for ν and $\nu^B_{\mathbb{N}}$, by the definition of f and since $\nu(H^i_x) = 1/2$ for all $i \in \mathbb{N}$ and all $x \in X\setminus\overline{N}$. This completes the proof. □

Lemma 2. *Let (X,Σ,μ) be a κ-super-atomless probability space. Then there is a stochastically independent family $\langle E_\xi\rangle_{\xi<\kappa}$ in Σ, with $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$, such that for each $A \in \Sigma$ there is a countable set $D_A \subset \kappa$ such that A and the sets E_ξ, $\xi \in \kappa\setminus D_A$, form a stochastically independent family in Σ.*

Proof. Suppose first that μ is Maharam-type-homogeneous, and let $(\mathfrak{A},\hat{\mu})$ denote the measure algebra of μ. Then by Maharam's theorem, there is a measure algebra isomorphism between $(\mathfrak{A},\hat{\mu})$ and the measure algebra of the usual measure ν_κ of $\{0,1\}^\kappa$. Denote this latter measure algebra by $(\mathfrak{C}_\kappa,\hat{\mu}_\kappa)$. For each $\xi < \kappa$ let $F_\xi = \{x \in \{0,1\}^\kappa\colon x(\xi) = 1\}$. Then $\langle F_\xi\rangle_{\xi<\kappa}$ is a stochastically independent family in the domain of ν_κ, with $\nu_\kappa(F_\xi) = 1/2$ for each $\xi < \kappa$. Thus the family $\langle F^\bullet_\xi\rangle_{\xi<\kappa}$, where $F^\bullet_\xi$ is the element in $\mathfrak{C}_\kappa$ determined by F_ξ, is a stochastically independent family in $\mathfrak{C}_\kappa$, with $\hat{\nu}_\kappa(F^\bullet_\xi) = 1/2$ for each $\xi < \kappa$. By a standard fact, the set $\{F^\bullet_\xi\colon \xi < \kappa\}$ completely generates $\mathfrak{C}_\kappa$. Consequently, since $(\mathfrak{A},\hat{\mu})$ and $\mathfrak{C}_\kappa$ are isomorphic as measure algebras, there is a stochastically independent family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$, with $\hat{\mu}(a_\xi) = 1/2$ for each $\xi < \kappa$, such that the set $\{a_\xi\colon \xi < \kappa\}$ completely generates $\mathfrak{A}$. For each $\xi < \kappa$ select an element E_ξ in Σ which determines a_ξ. Then $\langle E_\xi\rangle_{\xi<\kappa}$ is a stochastically independent family in Σ, with $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$. Now pick any $A \in \Sigma$. Let $A^\bullet$ be the element in $\mathfrak{A}$ determined by A. Since the set $\{a_\xi : \xi < \kappa\}$ completely generates $\mathfrak{A}$, there is a countable set $D_A \subset \kappa$ such that $A^\bullet$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by the set $\{a_\xi : \xi \in D_A\}$.[6] But this subalgebra of $\mathfrak{A}$ and the closed subalgebra of $\mathfrak{A}$ generated by the set $\{a_\xi : \xi \in \kappa \setminus D_A\}$ are stochastically independent, because the family $\langle a_\xi \rangle_{\xi<\kappa}$ is stochastically independent.[7] It follows that $A^\bullet$ and the elements a_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in $\mathfrak{A}$, whence A and the sets E_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in Σ.

Now suppose μ is not Maharam-type-homogeneous. Since μ is a probability measure, Maharam's theorem implies that there is a countable partition $\langle S_i \rangle_{i\in I}$ of X, with $S_i \in \Sigma$ and $\mu(S_i) > 0$ for each $i \in I$, such that, denoting by μ_i the subspace measure on S_i induced by μ, μ_i is Maharam-type-homogeneous for each $i \in I$. Let κ_i be the Maharam type of μ_i and note that $\kappa = \min\{\kappa_i : i \in I\}$ (by the definition of "κ-super-atomless"). For each $i \in I$, let Σ_i denote the domain of μ_i (i.e. Σ_i is the trace of Σ on S_i) and let $\overline{\mu}_i$ denote the normalization of μ_i so that $\overline{\mu}_i$ is a probability measure. (Thus $\overline{\mu}_i$ is the measure on S_i given as $\overline{\mu}_i = \frac{1}{\mu_i(S_i)}\mu_i$.)

Now for each $i \in I$, considering $(S_i, \Sigma_i, \overline{\mu}_i)$ as a probability space in its own right, let $\langle E^i_\xi \rangle_{\xi<\kappa_i}$ be a family in Σ_i, constructed according to the first part of this proof. Recalling that $\kappa = \min\{\kappa_i : i \in I\}$, for each $i \in I$ let $\langle E^i_\xi \rangle_{\xi<\kappa}$ be a subfamily of the family $\langle E^i_\xi \rangle_{\xi<\kappa_i}$, and then let $\langle E_\xi \rangle_{\xi<\kappa}$ be the family in Σ defined by setting $E_\xi = \bigcup_{i\in I} E^i_\xi$ for each $\xi < \kappa$. Note that we must have $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$.

Let $E_{\xi_1}, \dots, E_{\xi_n}$ be any finite subfamily of $\langle E_\xi \rangle_{\xi<\kappa}$. Then, by choice of the families $\langle E^i_\xi \rangle_{\xi<\kappa}$, $i \in I$, and since $\langle S_i \rangle_{i\in I}$ is a partition of X,

$$\begin{aligned}
\mu(E_{\xi_1} \cap \cdots \cap E_{\xi_n}) &= \sum_{i\in I} \mu_i(E^i_{\xi_1} \cap \cdots \cap E^i_{\xi_n}) \\
&= \sum_{i\in I} \mu_i(S_i)\overline{\mu}_i(E^i_{\xi_1} \cap \cdots \cap E^i_{\xi_n}) \\
&= \sum_{i\in I} \mu_i(S_i) 2^{-n} \\
&= 2^{-n} \\
&= \prod_{j=1}^{n} \mu(E_{\xi_j}).
\end{aligned}$$

Thus the family $\langle E_\xi \rangle_{\xi<\kappa}$ is stochastically independent.

Consider any $A \in \Sigma$. Set $A_i = A \cap S_i$ for each $i \in I$. By choice of the families $\langle E^i_\xi \rangle_{\xi<\kappa}$, $i \in I$, for each $i \in I$ there is a countable set $D^i_A \subset \kappa$ such that A_i and the sets E^i_ξ, $\xi \in \kappa \setminus D^i_A$, form a stochastically independent family in Σ_i for $\overline{\mu}_i$. Set $D_A = \bigcup_{i\in I} D^i_A$ and consider any finite subfamily $E_{\xi_1}, \dots, E_{\xi_n}$ of $\langle E_\xi \rangle_{\xi<\kappa}$ with

[6]See Fremlin (2002, 331G(d) and 331G(e)).
[7]See Fremlin (2002, 325X(e) and 325X(f)).

$\xi_j \notin D_A$ for $j = 1, \ldots, n$. Then

$$\begin{aligned}
\mu(A \cap E_{\xi_1} \cap \cdots \cap E_{\xi_n}) &= \sum_{i \in I} \mu_i(A_i \cap E^i_{\xi_1} \cap \cdots \cap E^i_{\xi_n}) \\
&= \sum_{i \in I} \mu_i(S_i)\overline{\mu}_i(A_i \cap E^i_{\xi_1} \cap \cdots \cap E^i_{\xi_n}) \\
&= \sum_{i \in I} \mu_i(S_i)\overline{\mu}(A_i)2^{-n} \\
&= \Big(\sum_{i \in I} \mu_i(A_i)\Big)2^{-n} \\
&= \mu(A)\prod_{j=1}^{n} \mu(E_{\xi_j}).
\end{aligned}$$

Thus, A and the sets E_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in Σ. □

Lemma 3. *Consider $\{0,1\}^{\mathbb{N}}$ as endowed with its usual measure $\nu_{\mathbb{N}}$, let X be an uncountable set, and let $\overline{\nu}$ be the product measure on $(\{0,1\}^{\mathbb{N}})^X$. Further, for each $i \in \mathbb{N}$ and each $x \in X$, let*

$$K^i_x = \left\{y \in (\{0,1\}^{\mathbb{N}})^X \colon y^i(x) = 1\right\}.$$

Finally, let Y be a subset of $(\{0,1\}^{\mathbb{N}})^X$ with full outer measure for $\overline{\nu}$, and let ν be the subspace measure on Y induced by $\overline{\nu}$. Then:

(i) *Let T be the domain of ν and set $H^i_x = K^i_x \cap Y$ for $i \in \mathbb{N}$ and $x \in X$. Then:*

  (1) *For each $i \in \mathbb{N}$ and each $x \in X$, $H^i_x \in \mathrm{T}$ with $\nu(H^i_x) = 1/2$.*

  (2) *Given any $B \in \mathrm{T}$, there is countable set $J_B \subset X$ such that B and the sets H^i_x, $i \in \mathbb{N}$, $x \in X \setminus J_B$, form a stochastically independent family in T.*

(ii) *Let ν' be the image measure of ν under the inclusion of Y into $(\{0,1\}^{\mathbb{N}})^X$, and T' its domain. Then:*

  (1) *For each $i \in \mathbb{N}$ and each $x \in X$, $K^i_x \in \mathrm{T}'$ with $\nu'(K^i_x) = 1/2$.*

  (2) *Given any $B \in \mathrm{T}'$, there is countable set $J_B \subset X$ such that B and the sets K^i_x, $i \in \mathbb{N}$, $x \in X \setminus J_B$, form a stochastically independent family in T'.*

Proof. Write $\overline{\mathrm{T}}$ for the domain of $\overline{\nu}$. For each $i \in \mathbb{N}$ and each $x \in X$, we have $K^i_x \in \overline{\mathrm{T}}$ by the definition of a product measure, whence $H^i_x \in \mathrm{T}$ and thus $K^i_x \in \mathrm{T}'$. Also by the definition of a product measure, $\overline{\nu}(K^i_x) = 1/2$ for all $i \in \mathbb{N}$ and $x \in X$. Since Y has full outer measure, it follows that $\nu(H^i_x) = 1/2$ for all $i \in \mathbb{N}$ and $x \in X$, and from this that $\nu'(K^i_x) = 1/2$ for all $i \in \mathbb{N}$ and $x \in X$. Thus (i)(1) and (ii)(1) hold.

For each $x \in X$ let $\overline{\mathrm{T}}_x$ be the sub-σ-algebra of $\overline{\mathrm{T}}$ generated by the x-th coordinate projection $\pi_x \colon (\{0,1\}^{\mathbb{N}})^X \to \{0,1\}^{\mathbb{N}}$. By definition of a product measure,

$\langle \overline{\mathrm{T}}_x \rangle_{x \in X}$ is a stochastically independent family of sub-σ-algebras of $\overline{\mathrm{T}}$. Since the family of coordinate projections $\pi^i \colon \{0,1\}^{\mathbb{N}} \to \{0,1\}$, $i \in \mathbb{N}$, is stochastically independent for $\nu_{\mathbb{N}}$, it follows that $\langle K^i_x \rangle_{i \in \mathbb{N}, x \in X}$ is a stochastically independent family in $\overline{\mathrm{T}}$. For each $J \subset X$, let $\overline{\mathrm{T}}_J$ be the smallest sub-σ-algebra of $\overline{\mathrm{T}}$ which includes $\overline{\mathrm{T}}_x$ for each $x \in J$.

Consider any $C \in \overline{\mathrm{T}}$. For some countable $J \subset X$, there is a $C' \in \overline{\mathrm{T}}_J$ which differs from C by a null set. The fact that $\langle \overline{\mathrm{T}}_x \rangle_{x \in X}$ is stochastically independent implies that $\overline{\mathrm{T}}_J$ and $\overline{\mathrm{T}}_{X \setminus J}$ are stochastically independent. Since $K^i_x \in \overline{\mathrm{T}}_x$ for each $x \in X$ and $i \in \mathbb{N}$, it follows that C' and the sets K^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in $\overline{\mathrm{T}}$. Since C' differs from C by a null set, the same is true for C and the sets K^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$.

Now consider any $B \in \mathrm{T}$. For some $C \in \overline{\mathrm{T}}$ we have $B = C \cap Y$ and $\overline{\nu}(C) = \nu(B)$. From the previous paragraph, there is a countable set $J \subset X$ such that C and the sets K^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in $\overline{\mathrm{T}}$. Let L be any non-empty finite subset of $X \setminus J$, and for each $l \in L$ let M_l be any non-empty finite subset of $\mathbb{N}$. Consider the finite subfamily $\langle H^m_l \rangle_{l \in L, m \in M_l}$ of the family $\langle H^i_x \rangle_{x \in X \setminus J, i \in \mathbb{N}}$. Using the fact that Y has full outer measure for $\overline{\nu}$, we may see that

$$\begin{aligned}
\nu\Big(B \cap \bigcap_{l \in L, m \in M_l} H^m_l\Big) &= \nu\Big((C \cap Y) \cap \bigcap_{l \in L, m \in M_l} (K^m_l \cap Y)\Big) \\
&= \nu\Big(\Big(C \cap \bigcap_{l \in L, m \in M_l} K^m_l\Big) \cap Y\Big) \\
&= \overline{\nu}\Big(C \cap \bigcap_{l \in L, m \in M_l} K^m_l\Big) \\
&= \overline{\nu}(C) \prod_{l \in L, m \in M_l} \overline{\nu}(K^m_l) \qquad \text{because } L \subset X \setminus J \\
&= \nu(B) \prod_{l \in L, m \in M_l} \nu(H^m_l).
\end{aligned}$$

It follows that B and the sets H^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T. Thus (i)(2) holds.

Finally, consider any $B \in \mathrm{T}'$. Let ι_Y denote the inclusion of Y into $\big(\{0,1\}^{\mathbb{N}}\big)^X$, and recall that ν' was defined to be the image measure of ν under ι_Y. From the previous paragraph, there is a countable set $J \subset X$ such that $\iota_Y^{-1}(B)$ and the sets H^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T. Since ν' is the image measure of ν under ι_Y, and since $H^i_x = \iota_Y^{-1}(K^i_x)$ by the definition of H^i_x, it follows that B and the sets K^i_x, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T'. Thus (ii)(2) holds. □

4.2 Proof of Theorem 1

Since (X, Σ, μ) is super-atomless, and since for any infinite cardinal κ there is a bijection between κ and $\kappa \times \mathbb{N}$, Lemma 2 implies that we may select an uncountable cardinal κ and a stochastically independent family $\langle E^i_\xi \rangle_{\xi < \kappa, i \in \mathbb{N}}$ in Σ, with $\mu(E^i_\xi) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $A \in \Sigma$ there is countable set $J_A \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_A$, A and the sets E^i_ξ, $i \in \mathbb{N}$, form a stochastically independent family in Σ.

For each $\xi < \kappa$, define a function y_ξ from X to $\{0,1\}^{\mathbb{N}}$ by setting

$$y^i_\xi(x) = \begin{cases} 1 & \text{if } x \in E^i_\xi \\ 0 & \text{if } x \notin E^i_\xi \end{cases}$$

for $i \in \mathbb{N}$ and $x \in X$. Attach a countably infinite subset $D_\xi \subset X$ to each $\xi < \kappa$ in such a way that for each countably infinite subset $D \subset X$ there is a $\xi < \kappa$ such that $D \cap D_\xi = \emptyset$. (Since X is uncountable, this is possible. Indeed, X being uncountable implies that we may select a disjoint family $\langle D_i \rangle_{i \in I}$ of countably infinite subsets of X such that $\#(I) = \omega_1$. Now since κ is uncountable, there is a surjection from κ onto I, say ϕ. Let $D_\xi = D_{\phi(\xi)}$.)

Now for each $\xi < \kappa$ let

$$N_\xi = \Big\{ y \in \big(\{0,1\}^{\mathbb{N}}\big)^X : \text{there is a null set } N \subset X \text{ such that } y \restriction X \setminus N = y_\xi \restriction X \setminus N \text{ and } N \cap D_\xi = \emptyset \Big\}$$

and then let $Y = \bigcup_{\xi < \kappa} N_\xi$. Let $\overline{\nu}$ be the product measure on $(\{0,1\}^{\mathbb{N}})^X$ when each factor $\{0,1\}^{\mathbb{N}}$ is given its usual measure $\nu_{\mathbb{N}}$. Note that for each $\xi < \kappa$, N_ξ is a $\overline{\nu}$-null set, since all of its elements agree on the infinite set D_ξ. On the other hand, Y has full outer measure for $\overline{\nu}$. Indeed, let W be any non-negligible $\overline{\nu}$-measurable subset of $(\{0,1\}^{\mathbb{N}})^X$. Then $W \supset W'$ for some non-empty subset W' of $(\{0,1\}^{\mathbb{N}})^X$ which is determined by coordinates in some countable subset of X, say J. By construction, there is a $\xi < \kappa$ such that $J \cap D_\xi = \emptyset$. Since the countable set J is a null set in X, it follows that, for such a ξ, the set

$$\Big\{ y \in \big(\{0,1\}^{\mathbb{N}}\big)^X : y \restriction X \setminus J = y_\xi \restriction X \setminus J \Big\}$$

is included in N_ξ and intersects the set W'. Thus Y intersects every non-negligible $\overline{\nu}$-measurable subset of $(\{0,1\}^{\mathbb{N}})^X$, i.e., Y has full outer measure for $\overline{\nu}$.

Let ν be the subspace measure on Y induced by $\overline{\nu}$, and T its domain. Then since Y has full outer measure for $\overline{\nu}$, (Y, T, ν) is a probability space. Note also that for each $\xi < \kappa$, N_ξ is a ν-null set in Y. Hence for any $A \in \Sigma$, $\bigcup_{\xi \in J_A} N_\xi$ is a ν-null set in Y since J_A is countable.

For each $i \in \mathbb{N}$ let

$$H^i = \{(x, y) \in X \times Y : y^i(x) = 1\}.$$

Then Lemma 1 applies to the sequence $\langle H^i \rangle_{i \in \mathbb{N}}$. Indeed, note first that we may assume that Σ is complete. Next, note that by construction, for any $y \in Y$ there is a $\xi < \kappa$ such that for each $i \in \mathbb{N}$ the y-section H^i_y differs from E^i_ξ by a null set. By the choice of the family $\langle E^i_\xi \rangle_{\xi < \kappa, i \in \mathbb{N}}$, it follows that H^i_y is in Σ with $\mu(H^i_y) = 1/2$ for each $i \in \mathbb{N}$ and each $y \in Y$, and that given any $A \in \Sigma$, if y does not belong to the null set $\bigcup_{\xi \in J_A} N_\xi$ then A and the sets H^i_y, $i \in \mathbb{N}$, form a stochastically independent family in Σ. Thus (b) and (d) of Lemma 1 hold for the family $\langle H^i \rangle_{i \in \mathbb{N}}$. By Lemma 3(i), (a) and (c) of Lemma 1 hold, too. Thus, by Lemma 1, the product measure corresponding to μ and ν has a rich Fubini extension. This completes the proof. □

4.3 Proof of Remark 1

In the proof of Theorem 1, define the sets N_ξ, $\xi < \kappa$, alternatively as

$$N_\xi = \Big\{ y \in \left(\{0,1\}^{\mathbb{N}}\right)^X : \text{there is a countable } D \subset X \text{ such that } y \restriction X \backslash D = y_\xi \restriction X \backslash D \text{ and } D \cap D_\xi = \emptyset \Big\}.$$

Observe that the arguments of the proof of Theorem 1 continue to hold with this new definition of the sets N_ξ. Now if $\#(X) \le \mathfrak{c}$, then the set of all countable subsets of X has cardinal $\le \mathfrak{c}$, which implies $\#(N_\xi) \le \mathfrak{c}$ for each $\xi < \kappa$, under the new definition of N_ξ. Clearly, we may choose κ in the proof of Theorem 1 so as to have $\kappa \le \mathfrak{c}$. But if $\#(N_\xi) \le \mathfrak{c}$ for each $\xi < \kappa$ and $\kappa \le \mathfrak{c}$, then we have $\#(Y) \le \mathfrak{c}$, by the definition of Y as $Y = \bigcup_{\xi < \kappa} N_\xi$. Note that since X must be an uncountable set by hypothesis, we must have, in fact, $\#(Y) = \mathfrak{c}$. □

4.4 Proof of Theorem 2

Construct a subset Y of $(\{0,1\}^{\mathbb{N}})^X$ in the same way as in the proof of Theorem 1, and then define the probability measure ν on Y as in the proof of Theorem 1. Let ν' denote the image measure of ν under the inclusion of Y into $(\{0,1\}^{\mathbb{N}})^X$, and let T' denote the domain of ν'. Observe that ν' is a probability measure on $(\{0,1\}^{\mathbb{N}})^X$ which extends the product measure on $(\{0,1\}^{\mathbb{N}})^X$ that is given when $\{0,1\}^{\mathbb{N}}$ is endowed with its usual measure. For each $i \in \mathbb{N}$ let

$$K^i = \left\{ (x, y) \in X \times \left(\{0,1\}^{\mathbb{N}}\right)^X : y^i(x) = 1 \right\}.$$

Note that Lemma 1—with $\left((\{0,1\}^{\mathbb{N}})^X, \mathrm{T}', \nu'\right)$ in place of (Y, T, ν)—applies to the family $\langle K^i \rangle_{i \in \mathbb{N}}$. To see this, observe that the complement of Y in $(\{0,1\}^{\mathbb{N}})^X$ and the sets N_ξ, $\xi < \kappa$, appearing in the construction of Y are ν'-null sets and conclude from this that (b) and (d) of Lemma 1 hold for the family $\langle K^i \rangle_{i \in \mathbb{N}}$. From Lemma 3(ii) it may be seen that (a) and (c) of Lemma 1 hold. Thus, by Lemma 1, the product measure corresponding to μ and ν' has a rich Fubini extension $\overline{\lambda}$ whose domain $\overline{\Lambda}$ contains the sets K^i, $i \in \mathbb{N}$. In particular, the function f defined in the statement of the theorem is $\overline{\Lambda}$-measurable. Finally, since ν' extends the product measure on $(\{0,1\}^{\mathbb{N}})^X$ that is given when $\{0,1\}^{\mathbb{N}}$ is endowed with its usual measure, it follows that (b) in the statement of the theorem holds. This completes the proof. □

4.5 Proof of Theorem 3

Suppose the product measure corresponding to μ and ν has a rich Fubini extension, with domain $\overline{\Lambda}$ say. We may assume that the σ-algebras Σ and T are complete. Then, by Definitions 1, 2, and 4, there are an element $H \in \overline{\Lambda}$ and null sets $N^X \subset X$ and $N^Y \subset Y$ such that (a) for each $x \in X \setminus N^X$ the x-section H_x is in T with $\nu(H_x) = 1/2$, (b) given any $x \in X \setminus N^X$ we have $\nu(H_x \cap H_{x'}) = 1/4$ for almost all $x' \in X \setminus N^X$, and (c) for each $y \in Y \setminus N^Y$ the y-section H_y is in Σ.

Then by Sun (2006, Theorem 2.8) it follows that given any $A \in \Sigma$, there is a null set $N_A \subset Y$ such that $\mu(H_y \cap A) = (1/2)\mu(A)$ for all $y \in Y \setminus N_A$. In particular, then, given any $A \in \Sigma$ and any $y \in Y \setminus N_A$, there is a null set $N_{y,A} \subset Y$ such that $\mu(H_{y'} \cap (H_y \cap A)) = (1/2)\mu(H_y \cap A)$ for all $y' \in Y \setminus N_{y,A}$. Thus, given $A \in \Sigma$, if $y \in Y \setminus N_A$ and $y' \in Y \setminus N_{A,y}$, then $\mu(H_{y'} \cap H_y \cap A) = (1/4)\mu(A)$.

Setting $A = X$, the previous paragraph show in particular that each $y \in Y$ is contained in some null set of Y, i.e. Y can be covered by some family of null sets. Set $\alpha = \operatorname{cov} \mathcal{N}(\nu)$.

Fix any $A \in \Sigma$ with $\mu(A) > 0$. By transfinite induction, choose a family $\langle y_\xi \rangle_{\xi < \alpha}$ in Y as follows. Let y_0 be an arbitrarily chosen element of $Y \setminus N_A$. Given that $\langle y_\eta \rangle_{\eta < \xi}$ has been chosen, where $\xi < \alpha$, choose a y_ξ in $Y \setminus (N_A \cup \bigcup_{\eta < \xi} N_{y_\eta, A})$. Such a choice is possible for each $\xi < \alpha$ because $\xi < \alpha = \operatorname{cov} \mathcal{N}(\nu)$ implies $Y \setminus (N_A \cup \bigcup_{\eta < \xi} N_{y_\eta, A}) \neq \emptyset$.

Then for any $\xi, \xi' < \alpha$ with $\xi \neq \xi'$, we have

$$\begin{aligned} \mu\big((H_{y_\xi} \cap A) \cap (H_{y_{\xi'}} \cap A)\big) &= \mu(H_{y_\xi} \cap H_{y_{\xi'}} \cap A) \\ &= \frac{1}{4}\mu(A) \\ &= \frac{1}{2}\mu(H_{y_\xi} \cap A) = \frac{1}{2}\mu(H_{y_{\xi'}} \cap A) \end{aligned}$$

whence $\mu\big((H_{y_\xi} \cap A) \triangle (H_{y_{\xi'}} \cap A)\big) = (1/2)\mu(A)$. Thus since $\mu(A) > 0$, writing $(\mathfrak{A}, \hat{\mu})$ for the measure algebra of μ, and $\mathfrak{A}_A$ for the principal ideal in $\mathfrak{A}$ determined by A, $\mathfrak{A}_A$ has a subset that is discrete for the measure metric of $(\mathfrak{A}, \hat{\mu})$ and has cardinal α.[8] In particular, the Maharam type of μ cannot be finite, and hence by Fremlin (2002, 323A(d), and 2005, 524D) it follows, considering $(\mathfrak{A}_A, \hat{\mu} \restriction \mathfrak{A}_A)$ as a measure algebra in its own right, that the Maharam type of $\mathfrak{A}_A$ is, in fact, at least α. Thus (b) of the theorem holds.

As for (a), note that for each $A \in \Sigma$ and $B \in \mathrm{T}$ we have $(A \times B) \cap H \in \overline{\Lambda}$ and hence, by the Fubini property, $\int_A \nu(H_x \cap B)\, d\mu(x) = \int_B \mu(H_y \cap A)\, d\nu(x)$. From the second paragraph of this proof, $\int_B \mu(H_y \cap A)\, d\nu(x) = (1/2)\mu(A)\nu(B)$ for each $A \in \Sigma$ and $B \in \mathrm{T}$. Consequently, for each fixed $B \in \mathrm{T}$,

$$\int_A \nu(H_x \cap B)\, d\mu(x) = \frac{1}{2}\nu(B)\mu(A) \text{ for all } A \in \Sigma.$$

[8] Recall that if (Z, Υ, ρ) is a finite measure space and $(\mathfrak{C}, \hat{\rho})$ its measure algebra, the measure metric on $\mathfrak{C}$ is just the metric that assigns, to every pair $E^\bullet$, $F^\bullet$ of elements of $\mathfrak{C}$, the number $\rho(E \triangle F)$ where E and F are any elements of $\mathcal{T}$ determining $E^\bullet$ and $F^\bullet$, respectively.

Hence, for each $B \in \mathrm{T}$ there is a null set $N_B \subset X$ such that $\nu(H_x \cap B) = (1/2)\nu(B)$ for all $x \in X \setminus N_B$. From this it follows that (a) of the theorem holds, using an argument analogous to that which had led to (b) of the theorem. □

4.6 Proof of Theorem 4

Since both α, β are $\geq \kappa$, and since there is a bijection between κ and $\kappa \times \mathbb{N}$, using Lemma 2 we may select a stochastically independent family $\langle E^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$ in Σ, with $\mu(E^i_\xi) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $A \in \Sigma$ there is countable set $J_A \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_A$, A and the sets E^i_ξ, $i \in \mathbb{N}$, form a stochastically independent family in Σ. Similarly, we may select a stochastically independent family $\langle F^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$ in T, with $\nu(F^i_\xi) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $B \in \mathrm{T}$ there is countable set $J_B \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_B$, B and the sets F^i_ξ, $i \in \mathbb{N}$, form a stochastically independent family in T.

For each $\xi < \kappa$ set $M'_\xi = M_\xi \setminus \bigcup_{\eta<\xi} M_\eta$ and $N'_\xi = N_\xi \setminus \bigcup_{\eta<\xi} N_\eta$. Then $\langle M'_\xi \rangle_{\xi<\kappa}$ is a disjoint family of null sets in X which covers X, and $\langle N'_\xi \rangle_{\xi<\kappa}$ a disjoint family of null sets in Y which covers Y. For each $i \in \mathbb{N}$ set

$$H^i = \Big(\bigcup_{\xi<\kappa} M'_\xi \times (F^i_\xi \setminus N_\xi)\Big) \cup \Big(\bigcup_{\xi<\kappa} (E^i_\xi \setminus M_\xi) \times N'_\xi\Big).$$

We want to see that Lemma 1 applies to the family $\langle H^i \rangle_{i\in\mathbb{N}}$. To this end, for each $x \in X$ let ξ_x be the least ordinal $\xi < \kappa$ such that $x \in M_\xi$. Thus ξ_x is also the uniquely determined ordinal $\xi < \kappa$ such that $x \in M'_\xi$. Observe that for each $x \in X$ and each $i \in \mathbb{N}$ the x-section H^i_x satisfies

$$F^i_{\xi_x} \setminus N_{\xi_x} \subset H^i_x \subset F^i_{\xi_x} \cup N_{\xi_x}\,.$$

Thus for each $x \in X$ and each $i \in \mathbb{N}$, H^i_x differs from $F^i_{\xi_x}$ by a null set. We may assume that T is complete. It then follows that for each $x \in X$ and each $i \in \mathbb{N}$, H^i_x belongs to T and $\nu(H^i_x) = 1/2$. Moreover, by choice of the family $\langle F^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$, it follows that given any $B \in \mathrm{T}$ and $x \in X$, if $\xi_x \notin J_B$—where J_B is the countable subset of κ that has been associated with B at the beginning of this proof—then B and the sets H^i_x, $i \in \mathbb{N}$, form a stochastically independent family in T. That is, if x does not belong to the null set $\bigcup_{\xi\in J_B} M'_\xi$, then B and the sets H^i_x, $i \in \mathbb{N}$, form a stochastically independent family in T. Thus (a) and (c) of Lemma 1 hold. Similarly it follows that (b) and (d) of Lemma 1 hold. Thus, by Lemma 1, the product measure corresponding to μ and ν has a rich Fubini extension. This completes the proof. □

4.7 Proof of Theorem 5

Let $\mathfrak{c}$ denote the cardinal of the continuum. By Theorem 4, it suffices to show that if Z is any Polish space and ν an atomless Borel probability measure on Z, then there is an extension of ν to a measure ν' on Z such that ν' is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$ and such that there is a non-decreasing family $\langle N_\xi \rangle_{\xi<\mathfrak{c}}$ of ν'-null sets in Z which covers Z, i.e. such that $\bigcup_{\xi<\mathfrak{c}} N_\xi = Z$.

To this end, note first that if I is any set with $\#(I) \leq \mathfrak{c}$ then there is a subset $A \subset \{0,1\}^I$, with $\#(A) \leq \mathfrak{c}$, such that A has full outer measure for the usual measure ν_I on $\{0,1\}^I$(see Fremlin, 2005, 523B together with 523D(d)).

Now consider $\{0,1\}^\mathfrak{c}$ with its usual measure $\nu_\mathfrak{c}$. Fix any $\overline{x} \in \{0,1\}^\mathfrak{c}$. For each $\xi < \mathfrak{c}$, let $J_\xi = \{\eta < \mathfrak{c} \colon \eta \leq \xi\}$. By the fact stated in the previous paragraph, for each $\xi < \mathfrak{c}$ we may choose a set $N'_\xi \subset \{0,1\}^\mathfrak{c}$ so that (a) $x \restriction \mathfrak{c} \setminus J_\xi = \overline{x} \restriction \mathfrak{c} \setminus J_\xi$ for each $x \in N'_\xi$, (b) N'_ξ intersects every non-negligible measurable subset of $\{0,1\}^\mathfrak{c}$ which is determined by coordinates in J_ξ, and (c) $\#(N'_\xi) \leq \mathfrak{c}$. For each $\xi < \mathfrak{c}$, let $N_\xi = \bigcup_{\eta \leq \xi} N'_\eta$. Then $\langle N_\xi \rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of subsets of $\{0,1\}^\mathfrak{c}$.

Let $Y = \bigcup_{\xi<\mathfrak{c}} N_\xi$. Then $\#(Y) = \mathfrak{c}$. Since $\mathfrak{c}$ has uncountable cofinality, (b) implies that Y has full outer measure for $\nu_\mathfrak{c}$ (because every non-negligible measurable subset of $\{0,1\}^\mathfrak{c}$ includes a non-negligible measurable subset of $\{0,1\}^\mathfrak{c}$ which is determined by coordinates in some countable set $J \subset \mathfrak{c}$). Finally, because of (a), N_ξ is a $\nu_\mathfrak{c}$-null set in $\{0,1\}^\mathfrak{c}$ for each $\xi < \mathfrak{c}$.

Now let μ be Lebesgue measure on $[0,1]$ and let λ be the product measure on $\{0,1\}^\mathfrak{c} \times [0,1]$ which is determined by $\nu_\mathfrak{c}$ and μ. By Fremlin (2005, 334X(g)), λ is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$. Since $\#(Y) = \mathfrak{c}$ and Y has full outer measure for $\nu_\mathfrak{c}$, the arguments in the proof of Proposition 521P(b) in Fremlin (2005) show that there is a subset $C \subset Y \times [0,1] \subset \{0,1\}^\mathfrak{c} \times [0,1]$ such that

(1) C has full outer measure for λ.

(2) The subspace measure λ_C on C induced by λ is countably separated.

(1) implies that λ_C is a probability measure on C and in particular that the measure algebra of λ_C can be identified with that of λ. Thus since λ is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$, so is λ_C.

Observe that $\langle N_\xi \times [0,1] \rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of λ-null sets in $\{0,1\}^\mathfrak{c} \times [0,1]$ whose union is $Y \times [0,1]$. Thus setting $M_\xi = C \cap (N_\xi \times [0,1])$ for each $\xi < \mathfrak{c}$, we get a non-decreasing family $\langle M_\xi \rangle_{\xi<\mathfrak{c}}$ of λ_C-null sets in C which covers C.

Write Λ_C for the domain of λ_C. (2) above means that we may identify C with a subset of $\mathbb{R}$ such that $B \cap C \in \Lambda_C$ for each Borel set B of $\mathbb{R}$. Under this identification, let $\hat{\mu}$ be the image measure of λ under the inclusion of C into $\mathbb{R}$, and let $\widehat{\mathcal{T}}$ be the domain of $\hat{\mu}$. Thus $\widehat{\mathcal{T}} = \{F \subset \mathbb{R} \colon F \cap C \in \Lambda_C\}$. In particular, $\mathbb{R} \setminus C$ is a $\hat{\mu}$-null. Thus if we set $M'_\xi = M_\xi \cup (\mathbb{R} \setminus C)$ for each $\xi < \mathfrak{c}$, then $\langle M'_\xi \rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of $\hat{\mu}$-null sets which covers $\mathbb{R}$.

Now let Z be any Polish space, and ν an atomless Borel probability measure on Z. The construction in Podczeck (2008, Appendix) yields a bijection $\zeta \colon \mathbb{R} \to Z$ and a probability measure ν' on Z such that ν' extends ν, ν' is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$, and ϕ is inverse measure preserving for ν' and $\hat{\mu}$ in both directions. Thus if we set $M''_x = \phi(M'_\xi)$, then $\langle M''_\xi \rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of ν'-null sets which covers Z. □

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