Beliefs in Network Games

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Abstract

Networks can have an important effect on economic outcomes. Given the complexity of many of these networks, agents will generally not know their structure. We study the sensitivity of game-theoretic predictions to the specification of players' (common) prior on the network in a setting where players play a fixed game with their neighbors and only have local information on the network structure. We show that two priors are close in a strategic sense if and only if (i) the priors assign similar probabilities to all events that involve a player and his neighbors, and (ii) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are close under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are close, for any number of iterations.

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1 Introduction

In many contexts, an agent's well-being primarily depends on his own action and on the actions of those with whom he has a direct relationship, rather than on the behavior of the population at large. Indeed, Goolsbee and Klenow (2002) and Tucker (2006) find that an individual's decision to adopt a particular communication technology is primarily influenced by the adoption decisions of those with whom he interacts directly, rather than by the overall adoption level. Also, an agent's connections provide access to various resources such as information, knowledge and capital. For instance, a key success factor for a firm in a high tech sector such as the biotechnology industry is its position in a network of R&D partnerships (Powell et al., 1996).¹ Hence, in a variety of settings, the networks formed by agents' relations are important in determining economic outcomes. These networks are generally large and complex, and evolve rapidly over time.² This suggests that agents often will not know the exact structure of the network they belong to.³ At the same time, it is unclear what beliefs agents have about their networks.⁴ We consider a setting in which agents interact strategically with their neighbors in a network under incomplete information on the network structure. We study the sensitivity of game-theoretic predictions in such games to the specification of players' beliefs on the network.

More specifically, suppose that players are located on a network and play a fixed game with their neighbors. Payoffs only depend on a player's own action and characteristics and on the actions and characteristics of his neighbors. Players have a common prior over the network, i.e., a probability distribution over a fixed set of networks. In addition, they have some local information: each player knows the number of neighbors he has in the network, i.e., a player's type is his *degree*. We define a function that for any two priors over a set of networks gives their strategic distance. Loosely speaking, the *strategic distance* between two

¹Other empirical studies that highlight the role of networks include Coleman et al. (1966) and Conley and Udry (2005) on the diffusion of new technologies in medicine and agriculture, respectively, Granovetter (1974) on job search, Tucker (2005) on adoption decisions, and Fafchamps and Lund (2003) on informal insurance networks in developing countries.

²For instance, several empirical emphasize the flexibility of R&D collaborations, with firms having many short term projects with many different partners (e.g. Hagedoorn, 2002; Powell et al., 2005).

³Indeed, Krackhardt and Hanson (1993) report that informal networks are mostly unobservable to senior executives. Also, Powell et al. (1996, p.120) observe that in R&D collaborations in biotechnology, "beneath most formal ties [...] lies a sea of informal relations".

⁴Evidence suggests that agents use simple heuristics (Janicik and Larrick, 2005), and that their perception of the network is biased (e.g. Kumbasar et al., 1994), even in an environment with strong incentives (Johnson and Orbach, 2002).

priors is small if for any game with bounded payoff functions in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under both equilibria. If that is the case, players can obtain approximately the same ex ante expected payoffs under both priors, and we say that the two priors are *close in a strategic sense*. We study the necessary and sufficient conditions for two priors to be close in a strategic sense. We thus consider a type of lower hemicontinuity of the correspondence of (interim) approximate equilibria in network games (see Engl, 1995, for a discussion of different continuity concepts).

Our main result (Theorem 5.3) shows that two priors are close in a strategic sense if and only if (i) the priors assign similar probabilities to all local events, i.e., events that involve a player and his neighbors, and (ii) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are close under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are close, and so on, for any number of iterations.

This result can be interpreted as follows. On the one hand, we can analyze a network game as a set of overlapping "local games" as far as ex ante beliefs are concerned: priors only need to assign similar probabilities to local events. On the other hand, these local games are interlaced through players' *conditional* beliefs: players need to form beliefs on the beliefs of his neighbors about the beliefs of their neighbors, and so on. This means that events that have small probability ex ante can have a large effect on outcomes through players' conditional beliefs: even if with high probability, each player has a type such that his conditional beliefs are close under the two priors, it may be the case that with high probability, a player thinks it is likely, given his type, that his neighbors think it is likely, given their type,... the conditional beliefs of their neighbors are very different. Players' higher order beliefs can thus have a large impact on outcomes if condition (*ii*) is not satisfied.

Interestingly, condition (ii) can also be formulated in terms of correlations among types. An equivalent formulation of (ii) is that the set of types for which conditional beliefs are close under the two priors has to have high probability, and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on. Compare this result to the findings of Morris (2000) on contagion on networks. Morris studies a setting with complete information on the network structure. He finds that, starting from a finite set of players X, behavior does *not* spread contagiously by myopic best-reply dynamics on a network with a countably infinite number of players if and only if the network of players not belonging to X contains a large group of players Y that is sufficiently cohesive, in the sense that players from Y interact mostly with other players from Y, who in turn interact primarily with other players from Y, and so on.

Condition (ii) is a direct stochastic analogue of this result of Morris. Rather than a *fixed* network of players, as in Morris (2000), we consider a *random* network of players, which induces a *fixed* interaction structure on the players' types. The situation we consider is the following. Suppose that there is a set of types with small prior probability for whom conditional beliefs are very different under two priors (so that they may take different actions under the two priors). We ask under what conditions these types do not "infect" a large (in terms of ex ante probability) set of types through players' higher order beliefs. This is the case precisely when there is a group of types with high prior probability which is sufficiently cohesive. We thus map a random network of *players* to a fixed interaction structure of *types*. This allows us to use the formal equivalence between games on a fixed network with complete information and games with incomplete information established by Morris (1997, 2000), which in turn enables us to use ideas from literature on higher order beliefs.

Our result thus sheds light on the relation between incomplete information games, network games with complete information, and network games in which players have incomplete information about the network structure. Our results are also of practical importance. The class of games we introduce, the class of network games of incomplete information, allows for uncertainty over the network size and for arbitrary correlations among player types. So far, the literature⁵ has mostly focused on network games in which the size of the network is commonly known and player types are independent. We show that assumptions on players' information on the network size and on the correlation among player types can have large ramifications. When there is uncertainty about the network size, and when players believe that there is nonzero correlation among types, a prior is *sensitive to small probability events*. That is, an event that has small prior probability can have a large effect on outcomes through players' conditional beliefs: a player may think it is likely, given his type, that his neighbors think it is likely, given their types... that the small probability event is true. When the size of the network is commonly known or when players believe that types are independent. this is ruled out. We show that in those cases, closeness of the two priors in terms of the prior probabilities assigned to local events (condition (i)) implies that there is a sufficiently large set of players whose conditional beliefs are close (condition (ii)). Hence, to explore the full range of strategic outcomes, one needs to go beyond network games with a complete information about the network size and independent types. The class of network games of incomplete information provides a flexible framework to analyze the effects of different

 $^{^5\}mathrm{See}$ below for a discussion of this literature.

assumptions on players' priors.

The current paper is related to three distinct literatures. Firstly, it is related to a recent literature that studies games on networks when players have incomplete information on the network structure (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Kets, 2007b; Sundararajan, 2005). We refer to the games studied in this literature as *Bayesian network games*, to distinguish this class from the class of games that we introduce. In Bayesian network games, the size of the network is commonly known. Moreover, it is often assumed that players' types are (asymptotically) independent.⁶ By contrast, we allow for uncertainty about the network size, and for arbitrary correlations among player types. Bayesian network games are not a subclass of the class of network games of incomplete information, nor is the converse the case. In Bayesian network games, the player set is common knowledge, which is not the case for network games of incomplete information. As we discuss in Section 3, this has implications for the way in which strategies are modeled in both classes of games.

Allowing for uncertainty about the network size and for correlation among player types is both important and natural. It is important because, as we have argued above, the assumptions on players' beliefs about the network size and the correlations among player types can have a qualitative effect on game-theoretic predictions. It is natural because agents will often be uncertain about the extent of their networks, and may well believe that types of his neighbors are correlated. As for uncertainty on the network size, the observation of Myerson (1998) that in some contexts, it is reasonable to assume that players are uncertain about the number of other players in the game holds a *fortiori* for network games, as in these games, players only interact with a small subset of players and have no direct information about the players they do not interact with. As for players' beliefs on the correlation among player types, there is ample evidence that many social and economic networks display positive assortativity, meaning that agents with a high (low) degree tend to be linked primarily with agents with a high (low) degree (see Jackson, 2007, and references therein). Moreover, evidence from social psychology suggests that individuals believe their networks to be highly clustered, i.e., that their networks contain a large number of small cycles (e.g. Crockett, 1982; Krackhardt and Kilduff, 1999). Hence, it is natural to assume that players believe that there is nonzero correlation among neighbor types.

Secondly, the current paper is related to the literature on games with population uncertainty. Games with population uncertainty in which all players interact directly have been studied by a number of authors (e.g. Kalai, 2004; McAfee and McMillan, 1987; Milchtaich, 2004; Myerson, 1998). In these games, players do not know how many players they interact

 $^{^{6}}$ Galeotti et al. (2006) and Kets (2007b) are exceptions.

with. By contrast, we consider a setting in which players interact locally, i.e., they only interact directly with a subset of players, and in which each player knows the number of players he interacts with. However, a player does not know the number of players his *neighbors* interact with. Hence, population uncertainty plays a distinctly different role here than in games with global interactions.

Finally, the current work builds on a literature that relates higher order beliefs to the equilibria of incomplete information games, in particular Monderer and Samet (1989) and Kajii and Morris (1998), and we use extensively concepts and techniques from this literature. Kajii and Morris (1998) study lower hemicontinuity of the approximate equilibrium correspondence in Bayesian games with a (fixed) finite player set and a countably infinite state space.⁷ Kajii and Morris show that two priors over this state space are strategically close if and only if the prior probabilities of events are similar under the two priors and with high probability, it is approximate common knowledge that all players attach similar conditional probabilities to all events, i.e., with high probability, each player believes with high conditional probability that the conditional beliefs of all players are similar under the two priors and that all players believe with high conditional probability that the conditional beliefs of all players are similar, and that all players believe with high conditional probability that all players believe with high conditional probability... that the conditional beliefs of all players are similar under the two priors (for any number of iterations). Our result can thus be seen as a "spatial" analogue of this result: rather than requiring that all players believe that all players believe... that the conditional beliefs of all players are similar, we require that a player believes that his neighbors believe that their neighbors believe... that the conditional beliefs of their neighbors are similar.⁸

Although we study the same issues as Kajii and Morris (1998), and follow their line of argument in our proofs,⁹ conceptually, there are marked differences. We introduce the *local p-belief operator*, a belief operator in the sense of Monderer and Samet (1989). The local *p*-belief operator associates with each set of types a set of types that with conditional probability at least *p* interact exclusively with types in that set. It thus provides a measure of the "cohesiveness" of a set of types. We show that this operator quantifies players'

⁷Monderer and Samet (1996) study the related question under what conditions two information partitions are close in a strategic sense. That is, they fix the probability distribution over the states and vary players' information partitions. Milgrom and Weber (1985) study upper hemicontinuity of the Bayesian equilibrium correspondence.

 $^{^{8}}$ Kets (2007b) studies the same question as Kajii and Morris (1998) for the class of Bayesian network games, and obtains results that are analogous to those of Kajii and Morris (1998).

⁹Also see Rothschild (2005).

higher order beliefs regarding local events in network games, i.e., a player's beliefs about his neighbors' beliefs about their neighbors' beliefs, and so on.

The local *p*-belief operator is closely related to the *p*-belief operator of Monderer and Samet (1989), which quantifies players' higher order beliefs in Bayesian games. While the *p*-belief operator of Monderer and Samet (1989) can be used to characterize players' higher order beliefs over the global structure of the network, the local p-belief operator is well suited to characterize players' higher order beliefs over local events. Interestingly, the local *p*-belief operator is also related to the neighborhood operator of Morris (1997, 2000). Morris introduces the neighborhood operator in the context of games on a *fixed* network. For a given network, the neighborhood operator assigns to each subset of players the set of players in that subset for whom at least proportion p of their interactions is only with players in that subset. That is, the neighborhood operator relates to the cohesiveness of a group of players, just like the local *p*-belief operator relates to the cohesiveness of a set of types. Hence, the local *p*-belief operator shares features of both the *p*-belief operator of Monderer and Samet (1989) and the neighborhood operator of Morris (1997, 2000). Like the p-belief operator, the local *p*-belief operator pertains to players' (higher order) beliefs in incomplete information games. Like the neighborhood operator, the local *p*-belief operator refers to the local interactions of players.

The outline of this paper is as follows. Preliminaries are discussed in Section 2. In Section 3, we introduce the class of network games of incomplete information. The local *p*-belief operator and players' higher order beliefs in network games are discussed in Section 4. Section 5 contains our main result and a discussion of its implications. Section 6 concludes. Appendix A contains the proofs which are not included in the main text.

2 Preliminaries

A network g is a pair consisting of a finite, nonempty set V(g) of vertices and a finite set E(g) of edges, with an edge being an unordered pair of two distinct vertices. Let g be a network. If $\{v, w\} \in E(g)$, where $v, w \in V, v \neq w$, then v and w are neighbors in g; alternatively, we say that v and w are adjacent in g. For ease of notation, an edge $\{v, w\} \in E(g)$ is sometimes denoted by vw.

We consider a setting where the network is drawn from a class of networks according to some probability distribution. Let $n \in \mathbb{N}$, and let $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with vertex set $V^{(n)}$ and let

$$\mathcal{G}:=igcup_{n\in\mathbb{N}}\mathcal{G}^{(n)}$$

be the countable set of all networks with a finite vertex set. Define

$$\mathcal{V} := \bigcup_{n \in \mathbb{N}} V^{(n)} = \mathbb{N}$$

Let \mathscr{F} be the σ -algebra generated by the set of singletons of \mathcal{G} . Let \mathcal{M} denote the set of all probability measures on $(\mathcal{G}, \mathscr{F})$, and let $\mu \in \mathcal{M}$. The probability space $(\mathcal{G}, \mathscr{F}, \mu)$ is a random network model (Jackson, 2007; Vega-Redondo, 2007). Example 2.1 gives a simple example of a random network model with a random number of vertices; for a particularly elegant model of a random network with a random number of vertices, see Bollobás et al. (2007).

Example 2.1 Suppose that a population evolves in (discrete) generations, indexed by $m \in \{0, 1, \ldots\}$. Each member of the *m*th-generation gives birth to a family (possibly empty) of members of the (m+1)th generation. The number of offspring that each individual produces is a random variable, and is independent of the number of offspring of all other individuals. The probability distribution of the number of offspring is the same for each individual. This is a simple branching process (e.g. Grimmett and Stirzaker, 1992). If we associate a vertex with each individual and if we interpret ancestry relations as (undirected) edges, this random process gives rise to a network with a random number of vertices.

In the current framework, random network models represent players' beliefs. Throughout this paper, we therefore refer to a random network model as a *network belief system*.

We are interested in the local environment of vertices. Let \mathcal{Q} be the (countable) set of all finite subsets of \mathbb{N} . Let $v \in \mathcal{V}$, and define the function $N_v : \mathcal{G} \to \mathcal{Q}$ by:

$$\forall g \in \mathcal{G} : \quad N_v(g) := \{ w \in V(g) \mid vw \in E(g) \}$$

Hence, $N_v(g)$ is the set of neighbors of vertex v in network g. We refer to the measurable function N_v as the *neighborhood of* v, and to $N_v(g), g \in \mathcal{G}$, as the *neighborhood of* v in g. Also, define the function $D_v : \mathcal{G} \to \mathbb{N} \cup \{0\}$ by:

$$\forall g \in \mathcal{G} : \quad D_v(g) := |N_v(g)|.$$

That is, $D_v(g)$ is the number of neighbors of vertex v in network g. We refer to $D_v(g)$ as the degree of v in g, and to the random variable D_v as the degree of v. Note that the degree of



Figure 2.1: (a) The network g of Example 2.2; (b) A network isomorphic to g. To see that this network is isomorphic to g, note that there are two permutations of the vertex set $V^{(4)} = \{1, 2, 3, 4\}$ that renders g into this network: $\pi(i) = 5 - i$ for each $i \in V^{(4)}$, or $\pi'(1) = 4, \pi'(2) = 3, \pi'(3) = 1, \pi'(4) = 2.$

v in g can be 0 for two distinct reasons. It could be that v is a vertex in the network, but does not have any neighbors, or that v is not a vertex of the network.

We also consider the number of neighbors the neighbors of a given vertex have. Loosely speaking, the neighbor degree profile of a vertex in a given network is a list of the degrees of the neighbors of the vertex, in a non-increasing order. For $t \in \mathbb{N}$, let

$$\Omega_K^t := \{ (k_1, \dots, k_t) \in \mathbb{N}^t \mid k_1 \ge k_2 \ge \dots \ge k_{t-1} \ge k_t \}.$$

For t = 0, let $\Omega_K^t := \{0\}$, and define

$$\Omega_K := \bigcup_{t \in \mathbb{N} \cup \{0\}} \Omega_K^t.$$

Let \mathscr{F}_K be the σ -field generated by the set of singletons of Ω_K . For $g \in \mathcal{G}$ and $v \in \mathcal{V}$ such that $D_v(g) = 0$, we set $K_v(g) := 0$. Otherwise, define

$$N_1 := N_v(g),$$

$$j(1) := \max\{w \in N_1 \mid D_w(g) \ge D_z(g) \text{ for all } z \in N_1\},$$

$$K_{v,1}(g) := D_{j(1)}(g),$$

and for $\ell = 2, \ldots, D_v(g)$:

$$N_{\ell} := N_{\ell-1} \setminus \{ j(\ell-1) \},\$$

$$j(\ell) := \max\{ w \in N_{\ell} \mid D_w(g) \ge D_z(g) \text{ for all } z \in N_{\ell} \},\$$

$$K_{v,\ell}(g) := D_{j(\ell)}(g).$$

Then, $K_v(g) := (K_{v,1}(g), \ldots, K_{v,D_v(g)}(g))$ is the neighbor degree profile of v in g, and the function $K_v : \mathcal{G} \to \Omega_K$ is the neighbor degree profile of v.

Example 2.2 Suppose we draw network g in Figure 2.1 from the set \mathcal{G} . Its vertex set is $V(g) = \{1, 2, 3, 4\}$, and its edge set is $E(g) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}$. The neighborhood of vertex 1 in g is $N_1(g) = \{2, 3, 4\}$, and its degree in g is $D_1(g) = 3$. The neighbordegree profile of vertex 1 in g is $K_1(g) = (D_4(g), D_3(g), D_2(g)) = (2, 2, 1)$.

The following definition will be useful when specifying players' beliefs in the next section. Let $n \in \mathbb{N}$. Two networks $g, g' \in \mathcal{G}^{(n)}$ are *isomorphic* if there is a permutation π of $V^{(n)}$ such that $\{i, j\} \in E(g)$ if and only if $\{\pi(i), \pi(j)\} \in E(g')$. This defines an equivalence relation; hence, the set of all networks with n vertices $\mathcal{G}^{(n)}$ can be partitioned into a finite number of *isomorphism classes*, i.e., sets of isomorphic networks. Let $\mathscr{C}^{(n)}$ be the collection of isomorphism classes of $\mathcal{G}^{(n)}$, and let $\mathscr{C} := \bigcup_{n \in \mathbb{N}} \mathscr{C}^{(n)}$ be the collection of isomorphism classes of \mathcal{G} . Figure 2.1(a) and (b) depict two networks that are isomorphic.

Throughout this paper, we make the following two assumptions on network belief systems:

Assumption 1 (Finite expected number of vertices) The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that the expected number of vertices is finite, i.e.,

$$\sum_{n\in\mathbb{N}}n\mu(\mathcal{G}^{(n)})<\infty.$$

Assumption 2 (No isolated vertices) The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that with probability 1, each vertex has at least one neighbor. That is,

$$\mu(\{g \in \mathcal{G} \mid D_i(g) > 0 \text{ for all } i \in V(g)\}) = 1.$$

Assumption 2 is for notational convenience only and can easily be relaxed.

3 Network games of incomplete information

A network game of incomplete information is a game on a network, in which players are associated with a vertex in the network, and each player's payoff depends on the types and actions of himself and his neighbors. Players have incomplete information on the network: they have a common prior over the class \mathcal{G} of all finite networks, and they know the number of neighbors they have, i.e., their degree. In particular, they may not know the number of players in the network. Here we introduce the class of network games of incomplete information.

3.1 Game

Let $(\mathcal{G}, \mathscr{F}, \mu)$ be a network belief system satisfying Assumptions 1 and 2. A network $g \in \mathcal{G}$ is drawn according to $(\mathcal{G}, \mathscr{F}, \mu)$. Each vertex in the set V(g) represents a player, and we refer to a player by his vertex label. Players do not know their vertex label, however.¹⁰ Each player $i \in V(g)$ knows the number of neighbors he has in the network: his *type* is his degree. Hence, the *type set* is $T = \mathbb{N} \cup \{0\}$. Henceforth, we will speak of type and *neighbor type profile*, rather than of degree and neighbor degree profile. Each player is endowed with a finite, nonempty set A of pure strategies or *actions*. For each $t \in T$, the *payoffs* of a player of type t are given by a function v_t . For t = 0, v_t is a real function on A, i.e., the payoffs to an isolated player only depend on his own action. For t > 0, v_t is a function from $A \times A^t \times T^t$ to \mathbb{R} that is symmetric in A^t and T^t , i.e., for all permutations π on $\{1, \ldots, t\}$, for all $a \in A, (a_1, \ldots, a_t) \in A^t, (\theta_1, \ldots, \theta_t) \in T^t$,

$$v_t(a, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t)) = v_t(a, (a_{\pi(1)}, \ldots, a_{\pi(t)}), (\theta_{\pi(1)}, \ldots, \theta_{\pi(t)}))$$

with $v_t(a, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t))$ the payoffs to a player of type t with neighbor type profile $(\theta_1, \ldots, \theta_t)$ when he chooses action $a \in A$, and his neighbors play according to the action profile (a_1, \ldots, a_t) .

Definition 3.1 A network game of incomplete information is a tuple

$$\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T} \rangle$$

with its elements defined as above.

We fix the action set A. A network game of incomplete information is then fully characterized by the common prior on $(\mathcal{G}, \mathscr{F})$ and its profile of payoff functions. We henceforth denote a network game of incomplete information $\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T} \rangle$ by the pair (μ, v) , where $v := (v_t)_{t \in T}$.

Let $B \in \mathbb{R}$. A profile v of payoff functions is bounded by B if for all $t \in T, t \neq 0, \theta \in \Omega_K^t$ and for all $a, a' \in A^{t+1}$,

$$\max\{|v_t(a,\theta) - v_t(a',\theta)|, |v_t(a,\theta)|\} \le B.$$

If there exists $B \in \mathbb{R}$ such that the profile v is bounded by B, we say that it is *bounded*.

¹⁰The vertex labelling is introduced merely to be able to define random variables such as the degree of vertices. However, the labelling is completely arbitrary and carries no meaning.



Figure 3.1: The networks of Example 3.2. (a) The network $g^{(3)}$; (b) The network $g^{(300)}$.

As in games with population uncertainty and random-player games, the player set is not commonly known, so that players are not aware of the particular identities of the other players in the game. Hence, we cannot assign a separate strategy to each individual player. Rather, a strategy can only depend on a player's type (cf. Myerson, 1998; Milchtaich, 2004). Hence, for each type $t \in T$, let σ_t be a real function defined on A which satisfies

$$\sigma_t(a) \ge 0$$

for all $a \in A$, and

$$\sum_{a \in A} \sigma_t(a) = 1,$$

with $\sigma_t(a)$ the probability that a player of type t chooses action a. The set of all probability distributions on A is denoted by Σ . An element $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots) \in \Sigma^T$ is referred to as a strategy function.

3.2 Beliefs

To calculate expected payoffs, we need to specify players' beliefs. There are two notable issues. Firstly, as in games with population uncertainty and random-player games, players condition on their type, as well as on the fact that they are selected to play. That is, from a player's perspective, even if all networks in the support of μ have equal probability ex ante, he believes that he is more likely to belong to a network with many players: there are simply more vertices to be associated with in large networks (cf. Myerson, 1998; Milchtaich, 2004). This is illustrated in Example 3.2.

Secondly, a player cannot distinguish between networks in a given isomorphism class, as he does not know his vertex label or the vertex labels of his opponents. Hence, to calculate players' beliefs that they have a given neighbor type profile, we need to consider the probability distribution over isomorphism classes induced by μ , and for each isomorphism class, we need to take into account the number of vertices with that neighbor type profile in the isomorphism class. **Example 3.2** Suppose that the network belief system assigns probability $\frac{1}{2}$ to the network $g^{(3)}$ consisting of a triangle of three players, and probability $\frac{1}{2}$ to the network $g^{(300)}$ consisting of 300 players, connected in a cycle (see Figure 3.1). Though the prior probability of the two networks is $\frac{1}{2}$, from the perspective of a player, it is much more likely that network $g^{(300)}$ is realized, as to each "player position" in $g^{(3)}$, there are 100 player positions in $g^{(300)}$. Using Bayes' rule, a player's belief that $g^{(300)}$ is realized, given that he is a player in the network, is

$$\frac{300 \cdot \frac{1}{2}}{3 \cdot \frac{1}{2} + 300 \cdot \frac{1}{2}} = \frac{300}{303}.$$

 \triangleleft

Formally, recall that \mathscr{C} is the collection of isomorphism classes of \mathcal{G} , and that \mathscr{F}_K is the σ -field associated with the set of all neighbor type profiles Ω_K . For each $C \in \mathscr{C}$, and each $F \in \mathscr{F}_K$, let $n_C(F)$ be the number of vertices in a network in C with their neighbor type profile in F. Note that $n_C(F)$ is well defined: for any two networks $g, g' \in C$, the number of vertices with their neighbor type profile in F is identical. Let

$$\bar{n} := \sum_{n \in \mathbb{N}} n \mu \big(\mathcal{G}^{(n)} \big)$$

be the expected number of players in the network belief system. By Assumption 1, \bar{n} is finite. Consider a player who is called upon to play, but who does not know his type yet. The probability that the neighbor type profile of such a player lies in the set F is

$$q_{\mu}(F) = \frac{1}{\bar{n}} \sum_{C \in \mathscr{C}} \mu(C) n_{C}(F),$$

where we recall that $\mu(C)$ is the prior probability that a network from the isomorphism class C is realized. In words, $q_{\mu}(F)$ is equal to the expected fraction of players with a neighbor type profile in F. We refer to $q_{\mu}(F)$ as the *prior probability* that a player's neighbor type profile is in F. In particular, for each $t \in T$,

$$q_{\mu}(t) := q_{\mu}(\Omega_K^t)$$

denotes the prior probability that a player's type is t. It can be readily checked from the definitions that q_{μ} is indeed a probability measure on the measurable space $(\Omega_K, \mathscr{F}_K)$ of neighbor type profiles:

(a)
$$q_{\mu}(\emptyset) = 0$$
, and $q_{\mu}(\Omega_K) = 1$;



Figure 3.2: The networks representing the isomorphism classes of Example 3.3 that have positive probability.

(b) q_{μ} satisfies σ -additivity: for A_1, A_2, \ldots a collection of disjoint members of \mathscr{F}_K ,

$$q_{\mu}\left(\bigcup_{k\in\mathbb{N}}A_{k}\right)=\sum_{k\in\mathbb{N}}q_{\mu}(A_{k}).$$

Example 3.3 demonstrates how players' beliefs are calculated in the current framework.

Example 3.3 Suppose that a network belief system assigns positive probability only to the networks g_1, g_2, \ldots, g_5 in Figure 3.2 or to networks isomorphic to them. Suppose that all isomorphism classes associated with the networks in Figure 3.2 have equal probability, i.e., for each isomorphism class $C \in \mathscr{C}$ of \mathcal{G} , $\mu(C) = \frac{1}{5}$ if there is a network $g \in \{g_1, g_2, \ldots, g_5\}$ such that $g \in C$, and $\mu(C) = 0$ otherwise. To calculate a player's prior belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$, we now simply need to count the number of vertices in g_1, \ldots, g_5 with their neighbor type profile in F, and compare this to the total number of vertices in g_1, \ldots, g_5 . For instance, a player's prior belief that his neighbor type profile is $\theta = (2, 2)$ is given by

$$q_{\mu}(\theta) = \frac{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3}{4 \cdot \frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 4} = \frac{8}{24}$$

and a player's prior belief that his type is t = 2 is $q_{\mu}(t) = \frac{9}{24}$. This is intuitive: from a player's perspective, he is equally likely to be associated with each of the vertices in g_1, \ldots, g_5 .

Conditional probabilities can be calculated in the usual way. Let $t \in T$ be such that $q_{\mu}(t) > 0$. A player's belief that his neighbor type profile is in the set $F \in \mathscr{F}_K$ given that his type is t is given by

$$q_{\mu}(F|t) := \frac{q_{\mu}(F \cap \Omega_{K}^{t})}{q_{\mu}(\Omega_{K}^{t})}$$
$$= \frac{\sum_{C \in \mathscr{C}} \mu(C) n_{C}(F \cap \Omega_{K}^{t})}{\sum_{C \in \mathscr{C}} \mu(C) n_{C}(\Omega_{K}^{t})}.$$

With minor abuse of notation, we write $q_{\mu}(\theta|t)$ to denote $q_{\mu}(\{\theta\}|t)$ for $\theta \in \Omega_{K}$. We refer to $q_{\mu}(F|t)$ as the *conditional belief* of (a player of) type t that his neighbor type profile is in F.

Example 3.3 (continued) To calculate a player's conditional belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$ given that his type is $t \in T$, we need to count the number of vertices in g_1, \ldots, g_5 with type t and neighbor type profile in F, and compare this to the total number of vertices in g_1, \ldots, g_5 with type t. For instance, a player's conditional belief that his neighbor type profile is $\theta = (2, 2)$ given that his type is t = 2 is

$$q_{\mu}(\theta|t) = \frac{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3}{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 1} = \frac{8}{9}.$$

Indeed, eight out of the nine vertices in g_1, \ldots, g_5 with type t = 2 have neighbor type profile $\theta = (2, 2)$.

Remark 3.4 Tacitly we have assumed that there is some pool of candidate players from which (actual) players are drawn. We have not specified this pool, nor have we specified the method by which players are selected. There is no need to specify this, however, as we are solely interested in players' beliefs *given* that they have been selected to play. Hence, the probability measure q_{μ} gives the probability that an arbitrary player has a certain neighbor type profile. Also see Myerson (1998, pp. 382–384) on this point.

3.3 Payoffs and equilibrium

Now that we have calculated players' beliefs, we can define expected payoffs. Let $t \in T, t \neq 0, \theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$, and define $\sigma_{(\theta)} := (\sigma_{\theta_1}, \ldots, \sigma_{\theta_t}) \in \Sigma^t$. Let

$$v_t(a,\sigma_{(\theta)}) := \sum_{a^{(t)} \in A^t} \left(\prod_{\ell=1}^t \sigma_{\theta_\ell}(a^{(t)}_\ell) \right) v_t(a,a^{(t)},\theta).$$

For each type $t \in T$ such that $q_{\mu}(t) > 0$, the *expected payoffs* to a player of type t of an action $a \in A$ when the other players play according to the strategy function $\sigma \in \Sigma^T$ are

$$\varphi_t(a,\sigma;\mu) := \sum_{\theta \in \Omega_K^t} q_\mu(\theta|t) v_t(a,\sigma_{(\theta)},\theta).$$
(3.1)

For $t \in T$ such that $q_{\mu}(t) = 0$, set $\varphi_t(a, \sigma; \mu) := 0$ for all $a \in A$ and $\sigma \in \Sigma^T$. Also, for each $t \in T$ and $\sigma \in \Sigma^T$, let

$$\varphi_t(\sigma;\mu) := \sum_{a \in A} \sigma_t(a) \varphi_t(a,\sigma;\mu).$$
(3.2)

The type-averaged (expected) payoffs of strategy function $\sigma \in \Sigma^T$ are

$$\Phi(\sigma;\mu) := \sum_{t \in T} q_{\mu}(t)\varphi_t(\sigma;\mu).$$
(3.3)

The type-averaged payoff of a strategy function $\sigma \in \Sigma^T$ is the weighted average of the expected payoffs of the different types under the strategy function σ , and gives the expected payoff of a player who is called upon to play the game, but does not know his type yet. Hence, the expected payoffs of a type correspond to the interim expected payoffs of a player in standard Bayesian games, while the type-averaged payoffs correspond to the ex ante expected payoffs in Bayesian games.

Definition 3.5 Let $\varepsilon \geq 0$. A strategy function $\sigma \in \Sigma^T$ is an ε -equilibrium of a network game of incomplete information (μ, v) if for each $t \in T$ such that $q_{\mu}(t) > 0$, for each action $a \in A$ such that $\sigma_t(a) > 0$,

$$\varphi_t(a,\sigma;\mu) \ge \varphi_t(b,\sigma;\mu) - \varepsilon$$

for all $b \in A$. We refer to a 0-equilibrium as an equilibrium.

Proposition 3.6 Let (μ, v) be a network game of incomplete information. If the profile of payoff functions v is bounded, the game has an equilibrium.

Proof. See Appendix A.

Let (μ, v) be a network game of incomplete information. Then, $\mathcal{N}^{\varepsilon}(\mu, v)$ denotes the set of ε -equilibria of (μ, v) . In particular, $\mathcal{N}^{0}(\mu, v)$ denotes the set of equilibria of (μ, v) .

4 The local *p*-belief operator and higher order beliefs

Let $\mu \in \mathcal{M}$, and let $p \in [0, 1]$. The *local p-belief operator* B^p_{μ} associates with each set of types the subset of types that with conditional probability at least p interact exclusively with types in that set (whenever they have positive probability). Formally, let $S \subseteq T$. Then,

$$B^{p}_{\mu}(S) := \{ t \in S \mid q_{\mu}(t) > 0 \Rightarrow q_{\mu}(S^{t}|t) \ge p \}.$$
(4.1)

Note that $B^p_{\mu}(S)$ includes the types in S that have zero probability. By definition, $B^p_{\mu}(S) \subseteq S$. If also

$$B^p_{\mu}(S) \supseteq S, \tag{4.2}$$

we say that the set of types S is *p*-closed (under μ).¹¹ If a set of types is *p*-closed, then each type in the set interacts with high conditional probability only with types in that set, who in turn interact with high conditional probability only with types in that set, and so on.

¹¹We follow the convention in the literature on higher order beliefs of making the one-sided implications explicit, as it is the one-sided implication in (4.2) that captures the nature of a set being p-closed.

The local *p*-belief operator can be iterated any finite number of times. For instance, $B^p_{\mu}(B^p_{\mu}(S))$ is the set of types $t \in B^p_{\mu}(S)$ such that with conditional probability at least *p*, they interact exclusively with types in $B^p_{\mu}(S)$, that is, with types in *S* that with conditional probability at least *p* interact exclusively with types in *S*. Define $[B^p_{\mu}]^1(S) := B^p_{\mu}(S)$ and, for each $\ell \in \mathbb{N}$, let $[B^p_{\mu}]^{\ell+1} = B^p_{\mu} \circ [B^p_{\mu}]^{\ell}$. Let

$$C^p_{\mu}(S) := \bigcap_{\ell \in \mathbb{N}} \left[B^p_{\mu} \right]^{\ell} (S)$$

be the set of types that with conditional probability at least p interact exclusively with types that with conditional probability at least p...interact exclusively with types in S, for any number of iterations.

Example 3.3 (continued) Let $S := \{1, 2, 3\}$. It is easy to check that the conditional belief of a player with type t = 1 or t = 2 that he interacts exclusively with players with types in S is $q_{\mu}(S^t|t) = 1$, while the conditional belief of a player with type t = 3 that he interacts exclusively with players with types in S is $q_{\mu}(S^3|3) = \frac{1}{3}$. Hence, for $p \in [0, \frac{1}{3}]$, we have $B^p_{\mu}(S) = S$, while for $p \in (\frac{1}{3}, 1]$, it holds that $B^p_{\mu}(S) = \{1, 2\}$. Now consider the conditional beliefs of players with types in the set $B^p_{\mu}(S)$ that they only interact with players with a type in $B^p_{\mu}(S)$. For instance, for $p \in (\frac{1}{3}, 1]$, it is easy to check that $q_{\mu}(B^p_{\mu}(S)|1) = \frac{2}{3}$, while $q_{\mu}((B^p_{\mu}(S))^2|2) = 1$. Hence, for $p \in (\frac{1}{3}, \frac{2}{3})$, it holds that $B^p_{\mu}(B^p_{\mu}(S)) = \{1, 2\}$, while for $p \in (\frac{2}{3}, 1]$, we have $B^p_{\mu}(B^p_{\mu}(S)) = \{2\}$.

The local *p*-belief operator satisfies the following desirable properties:¹²

Monotonicity: For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then $B^p_\mu(T') \subseteq B^p_\mu(T'')$.

Continuity: Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k \in \mathbb{N}}$ is a (weakly) decreasing sequence and $\bigcap_{k \in \mathbb{N}} T_k = S$, then $B^p_\mu(T_k) \downarrow B^p_\mu(S)$.

Continuity in *p*: If $p_k \uparrow p$, then, for any $S \subseteq T$, $B^{p_k}_{\mu}(S) \downarrow B^p_{\mu}(S)$.

For proofs, see Appendix A.

The following two results, which we will use later on, have well-known counterparts in the literature on higher order beliefs (Monderer and Samet, 1989, Prop. 3).

Lemma 4.1 Let $S \subseteq T$, and let $p \in [0,1]$. The set of types $C^p_{\mu}(S)$ is p-closed, i.e.,

$$B^p_\mu(C^p_\mu(S)) = C^p_\mu(S).$$

¹²See Monderer and Samet (1989, 1996) for a discussion. Note that the axiom of monotonicity implies the axiom of subpotency in the current context: for all $S \subseteq T$, $B^p_{\mu}(B^p_{\mu}(S)) \subseteq B^p_{\mu}(S)$.

Lemma 4.2 Let $p \in [0,1]$. Let $t \in T$, and let $T' \subseteq T$. We have that $t \in C^p_\mu(T')$ if and only if there exists a subset of types $S \subseteq T'$ that is p-closed such that $t \in S$ and $S \subseteq B^p_\mu(T')$.

The proofs of Lemmas 4.1 and 4.2 can be found in Appendix A.

Though at first sight the local *p*-belief operator seems to refer primarily to the "cohesiveness" of a set of types, we can use the local *p*-belief operator to characterize players' higher order beliefs, i.e., the beliefs players have over the beliefs of other players over the beliefs of other players, and so on. For instance, consider the set $B^p_{\mu}(S)$ of types for some $S \subseteq T$. We have said that with conditional probability at least p, a player with type $t \in B^p_{\mu}(S)$ interacts exclusively with players whose types lie in S. An alternative formulation is that a player with a type $t \in B^p_{\mu}(S)$ believes, given his type, that with probability at least p, all his neighbors have their types in the set S. That is, the local *p*-belief operator is a belief operator in the sense of Monderer and Samet (1989) restricted to events of the form "the types of all neighbors of an arbitrary player are in a given set". For a discussion of the relation between the *p*-belief operator of Monderer and Samet and the local *p*-belief operator, see Kets (2007a).

A player with a type in $B^p_{\mu}(S)$ for some $S \subseteq T$ believes that with conditional probability at least p, the types of all his neighbors are in S. When the local p-belief operator is iterated, we obtain statements about players' higher order beliefs. When $t \in B^p_{\mu}(B^p_{\mu}(S))$, a player with type t believes (with conditional probability at least p) that his neighbors believe that their neighbors' types are in S, i.e., the set $B^p_{\mu}(B^p_{\mu}(S))$ characterizes a player's beliefs about his neighbors' beliefs about their neighbors (see Figure 4.1(a)). Similarly, when $t \in B^p_{\mu}(B^p_{\mu}(B^p_{\mu}(S)))$, a player believes that his neighbors believe that their neighbors' types are in S. That is, the set $B^p_{\mu}(B^p_{\mu}(B^p_{\mu}(S)))$ characterizes a player's beliefs about his neighbors' beliefs about their neighbors believe that their neighbors believe that their neighbors' types are in S. That is, the set $B^p_{\mu}(B^p_{\mu}(B^p_{\mu}(S)))$ characterizes a player's beliefs about his neighbors' beliefs about their neighbors' beliefs about their neighbors (see Figure 4.1(b)).

The local *p*-belief operator also allows us to characterize a player's beliefs over others' beliefs about himself and his beliefs. Indeed, a player is a neighbor of his neighbors, so that when a player believes (with high conditional probability) that his neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_{\mu}(B^p_{\mu}(S))$), then he believes that the players he interacts with believe that his type is in S. Similarly, if a player believes that his neighbors believe that their neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_{\mu}(B^p_{\mu}(B^p_{\mu}(S)))$), then he believes that his neighbors believe that he believes that their neighbors believe that his neighbors believe that he believes that their types are in S.

We will use the local *p*-belief operator extensively in the next section to analyze players' beliefs in network games of incomplete information.



Figure 4.1: Higher order beliefs in a network. (a) Suppose player *i* has a type in $B^p_{\mu}(B^p_{\mu}(S))$. Then, with conditional probability at least *p*, he believes that his neighbors have a type in *S*, and that with conditional probability at least *p*, they believe that their neighbors' types lie in *S*. (b) Suppose player *i* has a type in $B^p_{\mu}(B^p_{\mu}(B^p_{\mu}(S)))$. Then, with conditional probability at least *p*, he believes that his neighbors have a type in *S*, and that with conditional probability at least *p*, he believes that his neighbors have a type in *S*, and that with conditional probability at least *p*, they believe that their neighbors have a type in *S*, and that with conditional probability at least *p*, they believe that their neighbors have a type in *S*, and that with conditional probability at least *p*, they believe that their neighbors have a type in *S*.

5 Strategic convergence

5.1 Main result

We want to quantify the extent to which priors are similar in a strategic sense. To that aim, we define a measure on the set of priors such that if two priors are close according to this measure, then, for each network game of incomplete information, for each equilibrium of the game in which beliefs are given by one of these priors, there exists an approximate equilibrium of the game with the other prior, such that type-averaged payoffs are close in both equilibria. If that is the case, then, for each possible profile of payoff functions, each player who is called upon to play can obtain approximately the same payoffs (in an ex ante sense) under both priors: from a player's (ex ante) perspective, the two priors are similar. We want to find the weakest conditions that guarantees that the above holds.

Formally, let $\mu, \mu' \in \mathcal{M}$, and let $v := (v_t)_{t \in T}$ be a profile of payoff functions. For each $\varepsilon \geq 0$, define

$$\chi(\mu,\mu';v,\varepsilon) := \sup_{\sigma \in \mathcal{N}^0(\mu,v)} \inf_{\sigma' \in \mathcal{N}^\varepsilon(\mu',v)} |\Phi(\sigma;\mu) - \Phi(\sigma';\mu')|,$$

where Φ is the type-averaged payoff given profile v of payoff functions. Hence, $\chi(\mu, \mu'; v, \varepsilon)$

is a measure of the difference in outcomes under μ and μ' in terms of type-averaged payoffs. That is, for a given $\varepsilon \geq 0$, for each equilibrium under μ , we first find an ε -equilibrium under μ' which minimizes the (absolute) difference in type-averaged payoffs under both equilibria, and we then look for the equilibrium under μ which maximizes this difference. This formalizes the idea that for *each* equilibrium of the network game of incomplete information with one prior, there exists *some* approximate equilibrium of the network game of incomplete information with the other prior, such that type-averaged payoffs are similar under both equilibria. However, the function $\chi(\mu, \mu'; v, \varepsilon)$ is not symmetric in μ and μ' , as we would want. To obtain a symmetric function of μ and μ' , let

$$\chi^*(\mu,\mu';v,\varepsilon) := \max\left\{\chi(\mu,\mu';v,\varepsilon),\chi(\mu',\mu;v,\varepsilon)\right\}.$$

We refer to $\chi^*(\mu, \mu'; v, \varepsilon)$ as the *strategic distance* between μ and μ' for the profile v given ε . The supremum of $\chi^*(\mu, \mu'; v, \varepsilon)$ over v is called the *strategic distance* between μ and μ' given ε .

Note that when ε increases, the set of approximate equilibria weakly increases, as more and more strategies will satisfy the equilibrium criterion, and the (absolute) difference in type-averaged expected payoffs will decrease weakly. Hence, the interesting case is when ε comes arbitrarily close to 0. This leads us to the following definition (cf. Kajii and Morris, 1998):

Definition 5.1 Take any $\mu \in \mathcal{M}$, and consider a sequence $(\mu^k)_{k \in \mathbb{N}}$ in \mathcal{M} . The sequence $(\mu^k)_{k \in \mathbb{N}}$ converges strategically to μ if for each profile v of payoff functions that is bounded, for each $\varepsilon > 0$, we have that

$$\lim_{k \to \infty} \chi^*(\mu, \mu^k; v, \varepsilon) = 0.$$

A natural requirement for strategic convergence is that priors attach similar probabilities to the event that a player has a neighbor type profile in a certain set, i.e., that priors converge in the weak topology on Ω_K . Hence, define

$$d_0(\mu, \mu') := \sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\mu'}(F)|.$$
(5.1)

We also need to consider players' conditional beliefs, i.e., the beliefs they have over their neighbors' types and beliefs, given their own type. For $\delta \in [0, 1]$, let

$$T_{\mu,\mu'}^{\delta} := \left\{ t \in T \mid \begin{array}{c} q_{\mu}(t) > 0 \\ q_{\mu'}(t) > 0 \end{array} \right\} \Rightarrow \sup_{F \in \mathscr{F}_{K}} |q_{\mu}(F|t) - q_{\mu'}(F|t)| \le \delta \right\}$$
(5.2)

be the set of types such that players' conditional beliefs on their neighbors' types are within δ , whenever the type has positive probability under μ and μ' . If δ is small, the conditional beliefs of a player with a type $t \in T^{\delta}_{\mu,\mu'}$ over the types of his neighbors are close under μ and μ' . If a player has a type $t \notin T^{\delta}_{\mu,\mu'}$, then his optimal actions under μ and μ' may differ substantially, as he believes that his local environment is very different under μ and μ' .

However, even if with high (prior) probability, a player has a type such that his conditional beliefs on his neighbors' types are similar under μ and μ' (i.e., $t \in T^{\delta}_{\mu,\mu'}$), outcomes can be very different under the two priors. The reason is that a player may believe with high conditional probability that the conditional beliefs of some of his neighbors on their neighbors' types are very different under μ and μ' (i.e., $t \notin B^p_{\mu}(T^{\delta}_{\mu,\mu'})$ for some $p \in [0,1]$), or that some of his neighbors believe with high conditional probability that the conditional beliefs of some of their neighbors are very different under μ and μ' (i.e., $t \notin B^p_{\mu}(B^p_{\mu}(T^{\delta}_{\mu,\mu'})))$, and so on. Hence, we need to require that with high probability, a player has a type in the set $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$, for some large $p \in [0, 1]$. In that case, a player's conditional beliefs are similar under μ and μ' , and, he believes with high conditional probability that the conditional beliefs of his neighbors are similar under the two priors and that his neighbors believe with high conditional probability that the conditional beliefs of their neighbors are similar under the two priors, and so on. This makes that the actions that are optimal for a player of type $t \in C^p_{\mu}(T^{\delta}_{\mu,\mu'})$ under μ will be (almost) optimal under μ' , as he expects his neighbors to behave similarly under μ and μ' (as his neighbors expect their neighbors to behave similarly, as the neighbors of his neighbors expect their neighbors...).

Formally, for each $S \subseteq T$, let

$$\Theta(S) := \bigcup_{t \in S} \Omega_K^t$$

be the set of neighbor type profiles in which the type of the "central" player belongs to the set S. Then, define

$$d_1(\mu, \mu') := \inf \left\{ \delta \in [0, 1] \mid q_\mu \left(\Theta(C^{1-\delta}_\mu(T^{\delta}_{\mu, \mu'})) \right) \ge 1 - \delta \right\}.$$
(5.3)

If $d_1(\mu, \mu')$ is small, then, with high prior probability (under μ), a player has a type such that his conditional beliefs are similar under μ and μ' , and with high conditional probability, he interacts exclusively with players whose conditional beliefs are close, and who, with high conditional probability, interact exclusively with players whose conditional beliefs are close, and so on.

Remark 5.2 One may think that requiring that with high prior probability, a player has a type in $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$ may not be sufficient: Even if a player believes, given his type, that with

high probability his neighbors will choose the same actions under μ and μ' (allowing for ε -best responses), they may not do so if in fact their type is not in $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$! That is, if with high probability, some of the neighbors of a player have a type $t \notin C^p_{\mu}(T^{\delta}_{\mu,\mu'})$, the payoff to a player with type $t \in C^p_{\mu}(T^{\delta}_{\mu,\mu'})$ can be very different under μ and μ' .¹³ However, Lemma 5.4 below shows that, if the probability is high that a player has a type in the set $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$, then in fact also the probability that his neighbors have a type in $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$ will be high. Hence, it is sufficient to require that with high probability, a player has a type in $C^p_{\mu}(T^{\delta}_{\mu,\mu'})$.

We can combine (5.1) and (5.3) to obtain

$$d^*(\mu, \mu') := \max\left\{ d_0(\mu, \mu'), d_1(\mu, \mu'), d_1(\mu', \mu) \right\}.$$
(5.4)

It is immediate that d^* is nonnegative and symmetric. Moreover, $d^*(\mu, \mu') = 0$ if and only if $\mu = \mu'$. However, d^* need not satisfy the triangle inequality, so that it is not a metric. However, d^* generates a topology on the set \mathcal{M} of probability measures on $(\mathcal{G}, \mathscr{F})$: a sequence $(\mu^k)_{k \in \mathbb{N}}$ converges to μ if and only if for any $\varepsilon > 0$, there exists $K_{\varepsilon} \in \mathbb{N}$ such that $d^*(\mu^k, \mu) \leq \varepsilon$ for all $k > K_{\varepsilon}$.

We are now ready to state our main result.

Theorem 5.3 Let $\mu \in \mathcal{M}$ and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{M} . Then, $(\mu^k)_{k \in \mathbb{N}}$ converges strategically to μ if and only if

$$\lim_{k \to \infty} d^*(\mu, \mu^k) = 0.$$

Theorem 5.3 follows from Proposition 5.6 - 5.8. Proposition 5.6 uses Lemma 5.4 and Lemma 5.5.

Lemma 5.4 Let $\mu \in \mathcal{M}$, and fix $\alpha, p \in [0,1]$. For each $S \subseteq T$, if the probability that a player has a type in the set $C^p_{\mu}(S)$ is at least α , i.e., if

$$q_{\mu}\left(\Theta(C^{p}_{\mu}(S))\right) \geq \alpha,$$

then the probability that this player and his neighbors have their types in $C^p_{\mu}(S)$ is at least αp :

$$q_{\mu}\left(\bigcup_{t\in C^{p}_{\mu}(S)} \left(C^{p}_{\mu}(S)\right)^{t}\right) \geq \alpha p.$$

Proof. See Appendix A.

¹³Indeed, Kajii and Morris (1998) require that the prior probability that *all* players have close conditional beliefs should be high.

Lemma 5.5 Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Let v be a profile of payoff functions. If $\sigma \in \Sigma^T$ is an equilibrium of the game (μ, v) and if v is bounded by B, then there exists a $5\delta B$ -equilibrium σ' of the game (μ', v) , with $\sigma'_t = \sigma_t$ for all $t \in C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})$.

Proof. For ease of notation, define $Q := C_{\mu'}^{1-\delta}(T_{\mu,\mu'}^{\delta})$. For each $t \in Q$, set $\sigma'_t = \sigma_t$. For $t \notin Q$ such that $q_{\mu'}(t) > 0$, let σ'_t be such that $(\sigma'_t)_{t\in T}$ is an equilibrium of the reduced game where each player with a type $t \in Q$ is required to play $\sigma'_t = \sigma_t$. Such an equilibrium exists by Proposition 3.6. By construction, σ'_t is a best response to σ' for $t \notin Q$. Hence, it remains to show that σ'_t is a $5\delta B$ -best response for a type $t \in Q$. Hence, let $t \in Q$ such that $q_{\mu}(t) > 0$ and $q_{\mu'}(t) > 0$. By Lemma 4.1,

$$q_{\mu'}(Q^t|t) \ge 1 - \delta. \tag{5.5}$$

Furthermore, by the definition of $Q = C_{\mu'}^{1-\delta}(T_{\mu,\mu'}^{\delta})$, for each $F \in \mathscr{F}_K$,

$$|q_{\mu}(F|t) - q_{\mu'}(F|t)| \le \delta.$$
(5.6)

Let $a \in A$ such that $\sigma_t(a) > 0$, and let $b \in A$. Then,

$$|\varphi_t(a,\sigma';\mu') - \varphi_t(b,\sigma';\mu')| \leq \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta|t) |v_t(a,\sigma'_{(\theta)}) - v_t(b,\sigma'_{(\theta)})| + \sum_{\theta \in Q^t} q_{\mu'}(\theta|t) |v_t(a,\sigma'_{(\theta)}) - v_t(b,\sigma'_{(\theta)})|.$$
(5.7)

The first sum in (5.7) can be evaluated directly. Using (5.5) and that v is bounded by B,

$$\sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta|t) \left| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \right| < \delta B.$$
(5.8)

To evaluate the second sum in (5.7), first note that for $\theta \in Q^t$, all neighbors play according to σ . As σ is an equilibrium of (μ, v) ,

$$\sum_{\theta \in Q^t} q_{\mu}(\theta|t) \big| v_t(a,\sigma_{(\theta)}) - v_t(b,\sigma_{(\theta)}) \big| \le \sum_{\theta \in \Omega^t_K \setminus Q^t} q_{\mu}(\theta|t) \big| v_t(a,\sigma_{(\theta)}) - v_t(b,\sigma_{(\theta)}) \big|$$
(5.9)

Also, by (5.5) and (5.6), we have that

$$q_{\mu}\left(\Omega_{K}^{t} \setminus Q^{t} \middle| t\right) \leq 2\delta.$$
(5.10)

Combining (5.9) and (5.10), we obtain

$$\sum_{\theta \in Q^t} q_{\mu}(\theta|t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \le 2\delta B.$$
(5.11)

Let $P^t := \{\theta \in Q^t \mid q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \ge 0\}$ be the set of neighbor type profiles θ in Q^t such that the conditional probability of θ under μ' is at least as high as under μ . Then, by (5.6),

$$\sum_{\theta \in Q^{t}} \left| \left(q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \right) \left(v_{t}(a, \sigma_{(\theta)}) - v_{t}(b, \sigma_{(\theta)}) \right) \right| = \sum_{\theta \in P^{t}} \left(q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \right) \left| v_{t}(a, \sigma_{(\theta)}) - v_{t}(b, \sigma_{(\theta)}) \right| + \sum_{\theta \in Q^{t} \setminus P^{t}} \left(q_{\mu}(\theta|t) - q_{\mu'}(\theta|t) \right) \left| v_{t}(a, \sigma_{(\theta)}) - v_{t}(b, \sigma_{(\theta)}) \right| \le 2\delta B. \quad (5.12)$$

Combining (5.11) and (5.12), we obtain

$$\sum_{\theta \in Q^t} q_{\mu'}(\theta|t) \big| v_t(a,\sigma_{(\theta)}) - v_t(b,\sigma_{(\theta)}) \big| \le \sum_{\theta \in Q^t} q_{\mu}(\theta|t) \big| v_t(a,\sigma_{(\theta)}) - v_t(b,\sigma_{(\theta)}) \big| + \sum_{\theta \in Q^t} \big| q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \big| \big| v_t(a,\sigma_{(\theta)}) - v_t(b,\sigma_{(\theta)}) \big| \le 4\delta B. \quad (5.13)$$

Combining (5.7), (5.8) and (5.13) gives

$$|\varphi_t(a,\sigma';\mu') - \varphi_t(b,\sigma';\mu')| \le 5\delta B.$$

Proposition 5.6 establishes the sufficiency of the condition in Theorem 5.3.

Proposition 5.6 Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Let v be a profile of payoff functions. Suppose that $d^*(\mu, \mu') \leq \delta$. Then, if σ is an equilibrium of the game (μ, v) and v is bounded by B, then there exists a $5\delta B$ -equilibrium σ' of the game (μ', v) such that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| \le (4-\delta)\delta B$$

Proof. For ease of notation, define $Q := C_{\mu'}^{1-\delta}(T_{\mu,\mu'}^{\delta})$. As $d^*(\mu,\mu') \leq \delta$,

$$|q_{\mu}(F) - q_{\mu'}(F)| \le \delta$$
(5.14)

for all $F \in \mathscr{F}_K$, and

$$q_{\mu'}(\Theta(Q)) \ge 1 - \delta. \tag{5.15}$$

Let $\sigma \in \Sigma^T$ be an equilibrium of (μ, v) . By Lemma 5.5, there exists a $5\delta B$ -equilibrium $\sigma' \in \Sigma^T$ of (μ', v) such that $\sigma'_t = \sigma_t$ for all $t \in Q$. Hence, using (5.15) and Lemma 5.4 (with

$$\begin{split} \alpha &= p = 1 - \delta), \\ \left| \Phi(\sigma'; \mu') - \Phi(\sigma; \mu') \right| &\leq \sum_{\substack{t \in Q: \\ q_{\mu'}(T) > 0}} q_{\mu'}(t) \sum_{\theta \in Q^t} q_{\mu'}(\theta | t) \sum_{a \in A} |\sigma'_t(a) v_t(a, \sigma'_{(\theta)}) - \sigma_t(a) v_t(a, \sigma'_{(\theta)})| + \\ &\sum_{\substack{t \in Q: \\ q_{\mu'}(T) > 0}} q_{\mu'}(t) \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta | t) \sum_{a \in A} |\sigma'_t(a) v_t(a, \sigma'_{(\theta)}) - \sigma_t(a) v_t(a, \sigma'_{(\theta)})| + \\ &\sum_{\substack{t \in T \setminus Q: \\ q_{\mu'}(T) > 0}} q_{\mu'}(t) \sum_{\theta \in \Omega_K^t} q_{\mu'}(\theta | t) \sum_{a \in A} |\sigma'_t(a) v_t(a, \sigma'_{(\theta)}) - \sigma_t(a) v_t(a, \sigma'_{(\theta)})| \\ &< 0 + (1 - (1 - \delta)^2) B \\ &= (2 - \delta) \delta B. \end{split}$$

Define the function $\zeta : \Omega_K \to T$ by $\zeta(\theta) = t$ whenever $\theta \in \Omega_K^t$. That is, the function ζ gives the type of a player for each possible neighbor type profile he may have. Let $P := \{\theta \in \Omega_K \mid q_{\mu'}(\zeta(\theta)) - q_{\mu}(\zeta(\theta)) \ge 0\}$. Then,

$$\begin{split} \left| \Phi(\sigma; \mu') - \Phi(\sigma; \mu) \right| &\leq \sum_{\theta \in P} \left(q_{\mu'}(\zeta(\theta)) - q_{\mu}(\zeta(\theta)) \right) \sum_{a \in A} \sigma_{\zeta(\theta)} |v_{\zeta(\theta)}(a, \sigma_{(\theta)})| + \\ &\sum_{\theta \in \Omega_K \setminus P} \left(q_{\mu}(\zeta(\theta)) - q_{\mu'}(\zeta(\theta)) \right) \sum_{a \in A} \sigma_{\zeta(\theta)} |v_{\zeta(\theta)}(a, \sigma_{(\theta)})|, \\ &\leq 2\delta B. \end{split}$$

Combining (5.16) and (5.16) gives the desired result.

We now establish necessity. Proposition 5.7 establishes that $d_0(\mu, \mu')$ should be small for strategic outcomes to be similar (in the sense defined above).

Proposition 5.7 Let $\delta \in [0, 1]$, and let $\mu, \mu' \in \mathcal{M}$. If

$$d_0(\mu, \mu') > \delta,$$

then there exists a profile v of payoff functions with bound B = 1 and an equilibrium σ of the game (μ, v) such that for any δ -equilibrium σ' of (μ', v) , it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta.$$

Proof. If $d_0(\mu, \mu') > \delta$, there exists a set of neighbor type profiles $F \in \mathscr{F}_K$ such that $|q_\mu(F) - q_{\mu'}(F)| > \delta$. For each $t \in T, t > 0, a \in A, a^{(t)} \in A^t$ and $\theta \in \Omega_K^t$, let

$$v_t(a, a^{(t)}, \theta) = \begin{cases} 1 & \text{if } \theta \in F, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta$$

for any two strategy functions $\sigma, \sigma' \in \Sigma^T$.

Proposition 5.8 establishes that strategic outcomes can be very different if $d_1(\mu, \mu')$ is large.

Proposition 5.8 Let $\delta \in [0, 1]$, and let $\mu, \mu' \in \mathcal{M}$. If

$$d_1(\mu, \mu') > \delta,$$

then there exists a profile v of payoff functions with bound B = 3 and an equilibrium σ of the game (μ, v) such that for any δ -equilibrium σ' of the game $(\mu'v)$, it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2.$$

Proof. As $d_1(\mu, \mu') > \delta$, we have

$$q_{\mu} \big(\Theta(C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})) \big) > 1 - \delta$$

or

$$q_{\mu'} \left(\Theta(C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})) \right) > 1 - \delta.$$
(5.16)

Without loss of generality, assume that (5.16) holds. Recall that for each $t \notin T^{\delta}_{\mu,\mu'}$, there exists a set of neighbor type profiles $F_t \in \mathscr{F}_K$ such that

$$q_{\mu'}(F_t|t) - q_{\mu}(F_t|t) > \delta.$$

Write $A = \{b^1, b^2, \dots, b^m\}$, where $m \in \mathbb{N}$, and let payoffs be defined as follows.¹⁴ For each $t \in T$, $a^{(t)} \in A^t$ and $\theta \in \Omega_K^t$, let

$$\begin{aligned} v_t(b^1, a^{(t)}, \theta) &:= 0, \\ v_t(b^2, a^{(t)}, \theta) &:= \begin{cases} 2 & \text{if } t \in T^{\delta}_{\mu,\mu'} \text{ and } a^{(t)}_j = b^2 \text{ for some } j \in \{1, \dots, t\}, \\ -\delta & \text{if } t \in T^{\delta}_{\mu,\mu'} \text{ and } a^{(t)}_j = b^1 \text{ for all } j \in \{1, \dots, t\}, \\ 1 - q_{\mu}(F_t|t) & \text{if } t \notin T^{\delta}_{\mu,\mu'} \text{ and } \theta \in F_t, \\ -q_{\mu}(F_t|t) & \text{if } t \notin T^{\delta}_{\mu,\mu'} \text{ and } \theta \notin F_t, \end{cases} \end{aligned}$$

and for $\ell \in \{3, \ldots, m\}$, let

$$v_t(b^\ell, a^{(t)}, \theta) := -2.$$

 $^{^{14}}$ This game is based on the "infection game" of Kajii and Morris (1998).

Hence, action b^1 always gives a payoff of 0, regardless of the actions and types of a player and his neighbors. For players with type $t \in T^{\delta}_{\mu,\mu'}$, action b^2 is only profitable if there is at least one neighbor who also takes action b^2 . By contrast, the payoffs of b^2 to players with type $t \notin T^{\delta}_{\mu,\mu'}$ only depends on their neighbor type profile θ : action b^2 is profitable only if θ belongs to F_t . All other actions than b^1 and b^2 are strictly dominated.

Consider the network game of incomplete information (μ, v) . In this game, there is an equilibrium $\sigma \in \Sigma^T$ in which all types $t \in T$ choose action b^1 with probability 1. For each type t, expected payoffs are 0, so that type-averaged payoffs are 0. Now consider the game (μ', v) . By definition, for each type $t \notin T^{\delta}_{\mu,\mu'}, q_{\mu'}(F_t|t) - q_{\mu}(F_t|t) > \delta$. The interim expected payoffs of playing b^2 are then

$$\varphi_t(b^2, \sigma; \mu') = q_{\mu'}(F_t|t) \big(1 - q_{\mu}(F_t|t) \big) - \big(1 - q_{\mu'}(F_t|t) \big) q_{\mu}(F_t|t) > \delta$$

for any strategy function $\sigma \in \Sigma^T$. Hence, in any δ -equilibrium, players with type $t \notin T^{\delta}_{\mu,\mu'}$ will play action b^2 . Let $\hat{T}^{\delta}_{\mu,\mu'} := \{t \in T^{\delta}_{\mu,\mu'} \mid q_{\mu}(t) > 0\}$ be the set of types in $T^{\delta}_{\mu,\mu'}$ that have positive probability under μ , and let $t \in \hat{T}^{\delta}_{\mu,\mu'}$. If $q_{\mu}((T^{\delta}_{\mu,\mu'})^t|t) < 1-\delta$, then, with conditional probability at least δ , a player with type t has at least one neighbor who plays b^2 . Hence, the interim expected payoffs of b^2 to such a type are at least

$$\delta \cdot 2 - (1 - \delta) \cdot \delta > \delta,$$

so that in any δ -equilibrium, players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\left((T^{\delta}_{\mu,\mu'})^{t}|t\right) < 1 - \delta$ will play b^{2} . By a similar argument, players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\left(\left(B^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\right)^{t}|t\right) < 1 - \delta$ will play b^{2} in any δ -equilibrium. This argument can be iterated any finite number of times. Consequently, all players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\left(\left(C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\right)^{t}|t\right) < 1 - \delta$ will play b^{2} in any δ -equilibrium.

By (5.16), the probability that a player has a type $t \notin C_{\mu'}(T^{\delta}_{\mu,\mu'})$ is greater than δ . As by Lemma 4.1 the set $C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})$ is $(1-\delta)$ -closed, the probability that a player has a type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\left(\left(C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\right)^t|t\right) < 1-\delta$ is greater than δ . Hence, in any δ -equilibrium $\sigma' \in \Sigma^T$ of (μ', v) , type-averaged expected payoffs are greater than δ^2 , so that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2.$$

We can now prove Theorem 5.3.

Proof. (If) Let v be a profile of payoff functions. By Proposition 5.6, for v bounded by B, and for $k \in \mathbb{N}$ such that $5Bd^*(\mu, \mu^k) \leq \varepsilon$,

$$\chi^*(\mu,\mu^k;v,\varepsilon) \le \left(4 - d^*(\mu,\mu^k)\right) d^*(\mu,\mu^k) B.$$

Hence, for all profiles of payoff functions v that are bounded and for all $\varepsilon > 0$, if $d^*(\mu, \mu^k) \to 0$, then $\chi^*(\mu, \mu^k; v, \varepsilon) \to 0$.

(Only if) Let $\mu, \mu' \in \mathcal{M}$. For $\delta \in [0, 1)$, if $d_0(\mu, \mu') > \delta$ or $d_1(\mu, \mu') > \delta$, then, by Propositions 5.7 and 5.8, there exists a profile of payoff functions v bounded by B = 3and an equilibrium $\sigma \in \Sigma^T$ of (μ, v) such that for any δ -equilibrium $\sigma' \in \Sigma^T$ of (μ', v) , $|\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')| > \delta^2$.

Remark 5.9 In the current setting, all players with the same payoff function independently implement the same strategy, i.e., strategies do not depend on a player's identity. This does not drive our results. We study the continuity of a given equilibrium correspondence; whether the equilibrium is defined in terms of deviations of individual players or of types, is irrelevant for the question we study. Secondly, Kets (2007b) shows that a counterpart of Theorem 5.3 holds for Bayesian network games (where the player set is fixed and strategies may depend on a player's identity) when one defines strategic convergence in terms of symmetric Bayesian ε -equilibria (see Kajii and Morris (1998) for a similar result for Bayesian games).

Remark 5.10 Our definition of strategic closeness requires that type-averaged expected payoffs be close in equilibria under two priors. An alternative notion would require that with high probability, a player and his neighbors follow the same strategies under the two priors (cf. Monderer and Samet, 1996). Indeed, from the proof of Proposition 5.6, it follows that for two priors $\mu, \mu' \in \mathcal{M}$, if $d^*(\mu, \mu') \leq \delta$ for some $\delta \in [0, 1]$, and $\sigma \in \Sigma^T$ is an equilibrium of (μ, v) for a profile v bounded by B, then there is a $5\delta B$ -equilibrium of (μ', v) such that the prior probability (either under μ or μ') that a player or his neighbors have a type $t \in T$ such that $\sigma'_t \neq \sigma_t$ is at most $\delta(2-\delta)$. However, this alone does not imply that the two priors μ and μ' give similar outcomes from a player's ex ante perspective: one should also consider the difference in prior probabilities under μ and μ' . This is done in the last step of the proof of Proposition 5.6. Hence, the appropriate definition of strategic closeness in the current setting considers differences in type-average expected payoffs.

Remark 5.11 Our definition of strategic convergence requires that players choose approximate best responses given their type. If, alternatively, we would only have required that they choose approximate best responses before learning their type, i.e., if we would have considered some ex ante or type-averaged notion of approximate equilibrium, then convergence in the weak topology on Ω_K (i.e., $d_0(\mu, \mu^k) \to 0$) is necessary and sufficient for strategic convergence, see Theorem A.9 in Appendix A. **Remark 5.12** We allow for a player's payoff to depend on the types of his neighbors. Obviously, for the subclass of games in which a player's payoffs do not depend on his neighbors' types, the conditions we derive for strategic convergence is still sufficient, though they may not be necessary. Our conjecture is that the current conditions cannot be weakened substantially for this subclass of games.

Remark 5.13 The assumption that a player's payoffs only depend on the actions and types of his direct neighbors is not essential. Under some suitable modifications and some additional technical assumptions, one could obtain similar results for games in which players' payoffs depend on the actions and types of players that are less than k steps away from them in the network, for arbitrary $k \in \mathbb{N}$.

We now proceed to discuss the implications of Theorem 5.3 in more detail.

5.2 Conditional beliefs and strategic convergence

Theorem 5.3 shows that it is not sufficient if two priors assign similar (prior) probabilities to all events in the space of neighbor type profiles for them to be strategically close. In addition, it needs to hold that with high probability, a player has a type such that his conditional beliefs are similar under the two priors, and that he thinks it is likely, given his type, the conditional beliefs of his neighbors are close, and that they think it is likely, given their type, ... that the conditional beliefs of their neighbors are similar, for any number of iterations. In the current section we investigate when this latter condition will be binding.

To shed some light on this, we first investigate when this condition plays no role. We adopt the following definition from Kajii and Morris (1998):

Definition 5.14 A prior $\mu \in \mathcal{M}$ is insensitive to small probability events if for each sequence $(\mu^k)_{k \in \mathbb{N}}$ in \mathcal{M} ,

$$\lim_{k \to \infty} d_0(\mu, \mu^k) = 0 \Rightarrow \lim_{k \to \infty} d^*(\mu, \mu^k) = 0.$$

In words, a prior $\mu \in \mathcal{M}$ is insensitive to small probability events if a necessary and sufficient condition for strategic convergence of any sequence $(\mu^k)_{k\in\mathbb{N}}$ in \mathcal{M} to μ is that $d_0(\mu, \mu^k)$ converges to zero when k goes to ∞ . The next proposition establishes that a necessary and sufficient condition for a prior to be insensitive to small probability events is that it can be approximated on a finite subset of T that is sufficiently closed: **Proposition 5.15** A prior $\mu \in \mathcal{M}$ is insensitive to small probability events if and only if for each $\varepsilon > 0$, there exists a finite set of types $S_{\varepsilon} \subseteq T$ that is $(1 - \varepsilon)$ -closed under μ such that the probability that a player has a type in S_{ε} is at least $1 - \varepsilon$, i.e.,

$$q_{\mu}(\Theta(S_{\varepsilon})) \ge 1 - \varepsilon$$

The proof can be found in Appendix A.

It is easy to see that the following conditions are sufficient for a prior μ to be insensitive to small probability events:

- Finite support: The set of types that have positive probability under μ is finite, i.e., $|\{t \in T \mid q_{\mu}(t) > 0\}| < \infty;$
- **Independent types:** Players' types are independent, i.e., for all $t \in T$, all $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$, $q_\mu(\theta|t) \propto \prod_{\ell=1}^t \theta_\ell q_\mu(\theta_\ell)$.
- **Perfect correlation over types:** Players only interact with players of their own type, i.e., for all $t \in T$ such that $q_{\mu}(t) > 0$, $q_{\mu}((t, \ldots, t)|t) = 1$, where (t, \ldots, t) is a vector in T of length t.

One case of interest in which a prior has finite support is when the number of players is fixed, as in Bayesian network games. An example of a network belief system with an unbounded number of players and independent types is given in Example 2.1. Finally, network belief systems in which types are perfectly correlated are studied by e.g. Ellison (1993).

Proposition 5.15 also gives some insight into the question under which conditions a prior is most sensitive to small probability events. Consider two priors $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Suppose that with probability at least $1 - \delta$, a player has a type $t \in T^{\delta}_{\mu,\mu'}$, i.e., a type such that his conditional beliefs under μ and μ' are within δ . Let $\Theta_0 \subseteq T^{\delta}_{\mu,\mu'}$ be the (possibly empty) set of types in $T^{\delta}_{\mu,\mu'}$ that with high conditional probability interact with types that do not belong to $T^{\delta}_{\mu,\mu'}$, and, for $\ell = 1, 2, \ldots$, let $\Theta_{\ell} \subseteq (T^{\delta}_{\mu,\mu'} \setminus \Theta_{\ell-1})$ be the set of types in $T^{\delta}_{\mu,\mu'}$, $\Theta_{\ell-1}$ that interact with high conditional probability with types that do not belong to $T^{\delta}_{\mu,\mu'} \setminus \Theta_{\ell-1}$. If a player has a type in one of the sets Θ_{ℓ} , his own conditional beliefs are close under the two priors, but, with high conditional probability, he interacts with types whose conditional beliefs are very different under μ and μ' , or who, with high conditionally probability, interact with types whose conditional beliefs are very different under μ and μ' , or who, with high conditionally probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ will be small, as illustrated in Figure 5.1. In that case, there is contagion



Figure 5.1: Even if with high probability, a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ may be small.

among types: a player whose conditional beliefs are similar under μ and μ' may be induced to follow a different strategy under μ' than under μ because he thinks it is likely that his neighbors' beliefs are different, or that they think that their neighbors' beliefs are different, and so on. Note that contagion is not physical here: players do not have to interact directly to be "infected" by others' behavior. It suffices that players *believe* (given their type) that it is likely that their neighbors believe that ... their neighbors have a certain type.

This situation is ruled out under the following two conditions. The first condition is that the set $T^{\delta}_{\mu,\mu'}$ is sufficiently cohesive, in the sense that all types in $T^{\delta}_{\mu,\mu'}$ interact (with high conditional probability) only with types in $T^{\delta}_{\mu,\mu'}$, who in turn interact only with types in $T^{\delta}_{\mu,\mu'}$, and so on. In that case, if it is a high probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, it will be a high probability event that a player has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$.

The second condition is that players' types are independent. In that case, players' conditional beliefs play no role: if priors assign similar prior probabilities to all events, then players' conditional beliefs will also be similar. Hence, when there is some correlation among types, but the set $T^{\delta}_{\mu,\mu'}$ is not sufficiently cohesive, players' conditional beliefs play an important role so that small probability events can have a large effect on outcomes.

This means that one should be careful in defining the game. In particular, it is often assumed in the literature on network games that the size of the network is fixed and that types are independent. The current analysis shows that these assumptions are not innocuous. If players believe that there is some correlation among types and there is uncertainty about the size of the network, then priors may be sensitive to small probability events, which is not the case when the number of players is fixed or when types are independent.

6 Conclusions

Given the complexity of many networks, it is important to study whether game-theoretic predictions are robust to assumptions on players' beliefs and information. We have studied the robustness of game-theoretic predictions to assumptions on players' (common) prior in network games of incomplete information. We have asked under what conditions on two priors it holds that for any bounded network game of incomplete information in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the game with the other prior such that ex ante expected payoffs are close. Our main result (Theorem 5.3) shows that two priors are close in a strategic sense if and only if they assign similar prior probabilities to all events involving a player and his neighbors, and, in addition, the set of types for which conditional beliefs are similar has high probability, and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on. This latter condition can also be formulated in terms of players' higher order beliefs: with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are similar, for any number of iterations.

An important motivation for this work comes from the realization that networks are often large and complex. This suggests that is natural to assume that players on a network have incomplete information about its structure, thus motivating the study of the robustness of game-theoretic predictions to the specification of players' beliefs on their network. There seems to be some tension between this motivation and our results. On the one hand, we assume that players are subject to information constraints. Yet, our results are derived in a setting where players use sophisticated arguments to form expectations over their opponents' behavior. However, it can be shown that the same results are obtained (in the limit) when the game is played repeatedly and to the behavior of his neighbors in the last period (cf. Jackson and Yariv, 2007; Morris, 2000).

To establish our results, we have used ideas and concepts from the literature on higher order beliefs. There are other important questions in the setting of network games of incomplete information that can be answered using ideas from this literature. One important question is how sensitive game-theoretic predictions are to the assumptions on players' information about the network structure. As in much of the literature on network games, we have assumed that players only know their degree. Indeed, Friedkin (1983) finds that the "observational horizon" of individuals is limited in communication networks in organizations: individuals only know their local environment in the network. However, there is a large variability among individuals. In addition, players can also represent entities like firms or countries, whose horizon is likely to be larger. For these reasons, it is important to investigate the sensitivity of predictions to informational assumptions. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) study the effect of gradually varying players' information about the network in a specific setting. Their results indicate that informational assumptions can have an important effect on results. However, to date, there is no systematic investigation how assumptions players' information affects results. The link with the literature on higher order beliefs may also be helpful here, as this literature contains numerous results on the effect of perturbing information structures. The current results suggest that such robustness questions are important to study in network games of incomplete information, and they illustrate how one can utilize ideas from the literature on higher order beliefs to answer such questions.

Appendix A Proofs

A.1 Proof of Proposition 3.6

Proposition 3.6 uses Lemma A.1.

Lemma A.1 Let (μ, v) be a network game of incomplete information such that the profile v of payoff functions is bounded. For each $t \in T$, let the function $\varphi_t(\cdot; \mu)$ on Σ^T be defined as in (3.2). Then, $\varphi_t(\cdot; \mu)$ is continuous on the (topological) product space Σ^T .

Proof. For each $t \in T$ and $n \in \mathbb{N}$, let

$$\Omega_K^{t,n} := \{ (k_1, \dots, k_t) \in \{1, \dots, n\}^t \mid k_1 \ge k_2 \ge \dots \ge k_{t-1} \ge k_t \}$$

be the set of neighbor type profiles of a player of type t such that the type of each neighbor is at most n. Clearly, $\Omega_K^{t,n}$ is a finite subset of the countable set Ω_K^t . For each $t \in T$ and $\sigma \in \Sigma^T$, define

$$\varphi_t^{(n)}(\sigma;\mu) := \begin{cases} \sum_{a \in A} \sigma_t(a) \sum_{\theta \in \Omega_K^{t,n}} q_\mu(\theta|t) v_t(a,\sigma_{(\theta)},\theta) & \text{if } q_\mu(t) > 0\\ 0 & \text{otherwise.} \end{cases}$$

For $t \in T$ such that $q_{\mu}(t) = 0$, it holds that $\varphi_t^{(n)}(\sigma; \mu) = \varphi_t(\sigma; \mu) = 0$ for all $\sigma \in \Sigma^T$. Let $t \in T$ such that $q_{\mu}(t) > 0$. By the triangle inequality, for each $\sigma \in \Sigma^T$,

$$\left|\varphi_t(\sigma;\mu) - \varphi_t^{(n)}(\sigma;\mu)\right| \leq \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta|t) \left| v_t(a,\sigma_{(\theta)},\theta) \right|.$$

As v is bounded, there exists $B \ge 0$ such that

$$\sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_{\mu}(\theta|t) \left| v_t(a,\sigma_{(\theta)},\theta) \right| \le B \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_{\mu}(\theta|t)$$

for all $\sigma \in \Sigma^T$. Moreover,

$$\lim_{n \to \infty} \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta|t) = 0$$

Hence, for each $\varepsilon > 0$, there exists $N_{\varepsilon} \in \mathbb{N}$ such that for all $\sigma \in \Sigma^T$,

$$\left|\varphi_t(\sigma;\mu) - \varphi_t^{(n)}(\sigma;\mu)\right| \le \varepsilon$$
 (A.1)

for all $n > N_{\varepsilon}$. That is, for each $t \in T$, the sequence $(\varphi_t^{(n)}(\cdot; \mu))_{n \in \mathbb{N}}$ converges uniformly on Σ^T to $\varphi_t(\cdot; \mu)$. As for each $n \in \mathbb{N}$, the function $\varphi_t^{(n)}(\cdot; \mu)$ is continuous on Σ^T , the function $\varphi_t(\cdot; \mu)$ is continuous on Σ^T .

We are now ready to prove Proposition 3.6. Consider a network game of incomplete information (μ, v) such that v is bounded, and fix some strategy function $\tau \in \Sigma^T$. Let $n \in \mathbb{N}$, and let $T^{(n)} := \{1, \ldots, n\}$. Recall the definition of the function $\varphi_t(\cdot; \mu)$ on Σ^T in (3.2).

Consider the strategic game $G^{(n)} = \langle T^{(n)}, \Sigma, (\tilde{\varphi}_t^{(n)}(\cdot; \mu))_{t \in T^{(n)}} \rangle$, where for each $t \in T^{(n)}$, $\tilde{\varphi}_t^{(n)}(\cdot; \mu)$ is a real-valued payoff function on Σ^n defined by

$$\tilde{\varphi}_t^{(n)}(\sigma^{(n)};\mu) = \varphi_t(\sigma_1^{(n)},\ldots,\sigma_n^{(n)},\tau_{n+1},\tau_{n+2},\ldots;\mu)$$

for all $\sigma^{(n)} \in \Sigma^n$. That is, the payoff of a player $t \in T^{(n)}$ in the game $G^{(n)}$ is the expected payoff of a player of type t in the original game (μ, v) , given that players with type $t \in T \setminus T^{(n)}$ play according to τ . The set Σ is a nonempty, convex, compact subset of a finite-dimensional Euclidean space, and for each $t \in T^{(n)}$, $\tilde{\varphi}_t^{(n)}(\cdot; \mu)$ is a continuous real-valued function on Σ^n that is quasi-concave in σ_t on Σ . Hence, the best-response correspondence $b_t : \Sigma^n \rightrightarrows \Sigma^n$ of each player $t \in T^{(n)}$ is nonempty, convex-valued, and upper-hemicontinuous, so that by Kakutani's fixed point theorem, a Nash equilibrium $(\bar{\sigma}_1^{(n)}, \ldots, \bar{\sigma}_n^{(n)}) \in \Sigma^n$ exists for $G^{(n)}$.

For each $n \in \mathbb{N}$, define

$$\bar{\sigma}^{(n)} := (\bar{\sigma}_1^{(n)}, \dots, \bar{\sigma}_n^{(n)}, \tau_{n+1}, \tau_{n+2}, \dots)$$

The set Σ is compact; hence, by the Cantor diagonal method (e.g. Ok, 2007, pp. 197– 198), there exists a subsequence $(\bar{\sigma}^{(n_j)})_{j\in\mathbb{N}}$ of the sequence $(\bar{\sigma}^{(n)})_{n\in\mathbb{N}}$ that converges to some $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots) \in \Sigma^T$. We claim that $\bar{\sigma}$ is an equilibrium of the original game (μ, v) . Suppose not. Then there exists $t \in T$ and $\sigma_t \in \Sigma$ such that

$$\varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_{t-1}, \bar{\sigma}_t, \bar{\sigma}_{t+1}, \dots; \mu) < \varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_{t-1}, \sigma_t, \bar{\sigma}_{t+1}, \dots; \mu).$$

By Lemma A.1, φ_t is continuous on the topological product space Σ^T . Hence, there exists $j \in \mathbb{N}$ such that $n_j \geq t$ and

$$\varphi_t(\bar{\sigma}_1^{(n_j)}, \dots, \bar{\sigma}_t^{(n_j)}, \dots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \dots; \mu) < \varphi_t(\bar{\sigma}_1^{(n_j)}, \dots, \sigma_t, \dots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \dots; \mu).$$

But this contradicts that $(\bar{\sigma}_1^{(n_j)}, \ldots, \bar{\sigma}_{n_j}^{(n_j)})$ is a Nash equilibrium of the game $G^{(n_j)}$.

A.2 Properties of the local *p*-belief operator

In this section, we prove the properties of the local p-belief operator as listed in Section 4, and we prove Lemma 4.1 and 4.2.

Lemma A.2 (Continuity) Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k\in\mathbb{N}}$ is a decreasing sequence and $\bigcap_{k\in\mathbb{N}} T_k = S$, then $B^p_\mu(T_k) \downarrow B^p_\mu(S)$.

Proof. First note that $B^p_{\mu}(T_{k+1}) \subseteq B^p_{\mu}(T_k)$ for all $k \in \mathbb{N}$, i.e., $(B^p_{\mu}(T_k))_{k \in \mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k\in\mathbb{N}} B^p_\mu(T_k) = B^p_\mu(\bigcap_{k\in\mathbb{N}} T_k).$$

First suppose $t \in \bigcap_{k \in \mathbb{N}} B^p_{\mu}(T_k)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. We need to distinguish two cases. First suppose that $q_{\mu}(t) = 0$. Then, by definition, $t \in B^p_{\mu}(\bigcap_{k \in \mathbb{N}} T_k)$. So suppose $q_{\mu}(t) > 0$. Then, $q_{\mu}(T^t_k|t) \ge p$ for all $k \in \mathbb{N}$. Furthermore, $(T^t_k)_{k \in \mathbb{N}}$ is a decreasing sequence, and $\bigcap_{k \in \mathbb{N}} T^t_k = S^t$. Hence (e.g. Grimmett and Stirzaker, 1992, Lemma 1.3.5),

$$\lim_{k \to \infty} q_{\mu}(T_k^t | t) = q_{\mu} \Big(\bigcap_{k \in \mathbb{N}} T_k^t | t \Big).$$

Combining these results gives

$$q_{\mu}\left(\bigcap_{k\in\mathbb{N}}T_{k}^{t}\big|t\right)\geq p,$$

and hence $t \in B^p_{\mu}(\bigcap_{k \in \mathbb{N}} T^t_k)$.

Secondly, suppose that $t \in B^p_{\mu}(\bigcap_{k \in \mathbb{N}} T_k)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. Again, we need to consider two cases. If $q_{\mu}(t) = 0$, then it follows directly from the definition of B^p_{μ} that $t \in B^p_{\mu}(T_k)$ for all $k \in \mathbb{N}$, and therefore $t \in \bigcap_{k \in \mathbb{N}} B^p_{\mu}(T_k)$. So suppose $q_{\mu}(t) > 0$. Then, $q_{\mu}(S^t|t) \ge p$ implies that $q_{\mu}(T_k|t) \ge p$ for all $k \in \mathbb{N}$. Hence, $t \in B^p_{\mu}(T_k)$ for all $k \in \mathbb{N}$, and $t \in \bigcap_{k \in \mathbb{N}} B^p_{\mu}(T_k)$.

Lemma A.3 (Monotonicity) For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then $B^p_{\mu}(T') \subseteq B^p_{\mu}(T'')$.

Proof. If $T' \subseteq T''$, then $T' \cap T'' = T'$. Hence,

$$B^{p}_{\mu}(T') = B^{p}_{\mu}(T' \cap T'') = B^{p}_{\mu}(T') \cap B^{p}_{\mu}(T'') \subseteq B^{p}_{\mu}(T'').$$

Lemma A.4 (Continuity in p) If $p_k \uparrow p$, then, for any $S \subseteq T$, $B^{p_k}_{\mu}(S) \downarrow B^p_{\mu}(S)$.

Proof. Let $S \subseteq T$. It follows directly from the definition of the local *p*-belief operator that $(B^{p_k}_{\mu}(S))_{k\in\mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k \in \mathbb{N}} B^{p_k}_{\mu}(S) = B^p_{\mu}(S)$$

Suppose $t \in \bigcap_{k \in \mathbb{N}} B^{p_k}_{\mu}(S)$. If $q_{\mu}(t) = 0$, then it follows directly from the definition that $t \in B^p_{\mu}(S)$. So suppose $q_{\mu}(t) > 0$. Then, $q_{\mu}(S^t|t) \ge p_k$ for all $k \in \mathbb{N}$, and therefore $q_{\mu}(S^t|t) \ge p$. Hence, $t \in B^p_{\mu}(S)$.

Conversely, suppose $t \in B^p_{\mu}(S)$. If $q_{\mu}(t) = 0$, then it follows directly that $t \in \bigcap_{k \in \mathbb{N}} B^{p_k}_{\mu}(S)$. So suppose $q_{\mu}(t) > 0$. Then, $q_{\mu}(S^t) \ge p$, and hence $q_{\mu}(S^t) \ge p_k$ for all $k \in \mathbb{N}$. We conclude that $t \in B^{p_k}_{\mu}(S)$ for all k, and hence $t \in \bigcap_{k \in \mathbb{N}} B^{p_k}_{\mu}(S)$.

Finally, we present the proofs of Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. By definition, $B^p_{\mu}(C^p_{\mu}(S)) \subseteq C^p_{\mu}(S)$. It remains to show that $B^p_{\mu}(C^p_{\mu}(S)) \supseteq C^p_{\mu}(S)$. Obviously, $([B^p_{\mu}]^{\ell}(S))_{k\in\mathbb{N}}$ is a weakly decreasing sequence, and, by definition, $\bigcap_{\ell\in\mathbb{N}} [B^p_{\mu}]^{\ell}(S) = C^p_{\mu}(S)$. Hence, using that the local *p*-belief operator is continuous,

$$C^p_{\mu}(S) = \bigcap_{\ell \in \mathbb{N}} \left[B^p_{\mu} \right]^{\ell}(S) \subseteq \bigcap_{\substack{\ell \in \mathbb{N}: k \ge 2\\ \mu \in \mathcal{O}}} \left[B^p_{\mu} \right]^{\ell}(S) = B^p_{\mu} \left(\bigcap_{\ell \in \mathbb{N}} \left[B^p_{\mu} \right]^{\ell}(S) \right) = B^p_{\mu} \left(C^p_{\mu}(S) \right). \qquad \Box$$

Proof of Lemma 4.2. Suppose $t \in C^p_{\mu}(T')$. By Lemma 4.1, the set $C^p_{\mu}(T')$ is *p*-closed. Also, by definition, $C^p_{\mu}(T') \subseteq B^p_{\mu}(T')$. Hence, we can set $S = C^p_{\mu}(T')$, and the statement follows.

Conversely, let $S \subseteq T$ be such that $t \in S$, and

$$S \subseteq B^p_{\mu}(S), \tag{A.2}$$

$$S \subseteq B^p_{\mu}(T'). \tag{A.3}$$

We show by induction on ℓ that $S \subseteq [B^p_{\mu}]^{\ell}(T')$ for all $\ell \in \mathbb{N}$, from which it follows that $t \in C^p_{\mu}(T')$. By (A.3), $S \subseteq [B^p_{\mu}]^1(T')$. For each $\ell \in \mathbb{N}$, if $S \subseteq [B^p_{\mu}]^{\ell}(T')$, then by (A.2) and by monotonicity of the local *p*-belief operator,

$$S \subseteq B^p_{\mu}(S) \subseteq B^p_{\mu}\left(\left[B^p_{\mu}\right]^{\ell}(T')\right) = \left[B^p_{\mu}\right]^{\ell+1}(T').$$

A.3 Proof of Lemma 5.4

By Lemma 4.1, $C^p_{\mu}(S)$ is *p*-closed. Hence, for all $t \in C^p_{\mu}(S)$ such that $q_{\mu}(t) > 0$, $q_{\mu}((C^p_{\mu}(S))^t | t) \ge p$. This yields:

$$q_{\mu} \left(\bigcup_{t \in C^{p}_{\mu}(S)} \left(C^{p}_{\mu}(S) \right)^{t} \right) = \sum_{\substack{t' \in C^{p}_{\mu}(S): \\ q_{\mu}(t') > 0}} q_{\mu} \left(\bigcup_{t \in C^{p}_{\mu}(S)} \left(C^{p}_{\mu}(S) \right)^{t} | t' \right) q_{\mu}(t')$$
$$= \sum_{\substack{t' \in C^{p}_{\mu}(S): \\ q_{\mu}(t') > 0}} q_{\mu} \left((C^{p}_{\mu}(S))^{t'} | t' \right) q_{\mu}(t')$$
$$\geq p \sum_{\substack{t' \in C^{p}_{\mu}(S) \\ t' \in C^{p}_{\mu}(S)}} q_{\mu}(t')$$
$$\geq \alpha p.$$

Remark A.5 Note that Lemma 5.4 can be generalized: we can replace $C^p_{\mu}(S)$ in the lemma by any subset of T that is p-closed. We have presented it in its current form for expositional reasons.

A.4 Continuity of the type-averaged equilibrium correspondence

In (approximate) equilibrium, players are required to choose best responses given their type, i.e., equilibrium is defined in terms of expected payoffs. Alternatively, we could define equilibrium in terms of type-averaged expected payoffs, allowing types with low prior probability to follow strategies that are suboptimal. In standard Bayesian games, lower hemicontinuity of the ex ante ε -equilibrium has been studied by Engl (1995). He shows that the weak topology is sufficient for lower hemicontinuity of the ex ante ε -equilibrium in countable state spaces. Here, we derive the analogous result for the type-averaged ε -equilibrium correspondence (see below for a precise definition). We show that the weak topology is sufficient (and also necessary) to guarantee lower-hemicontinuity of this correspondence.

First we need some definitions. Recall the definition of type-averaged expected payoffs from Section 3.

Definition A.6 Let $\varepsilon \ge 0$. A strategy function $\sigma \in \Sigma^T$ is a type-averaged ε -equilibrium of a network game of incomplete information (μ, v) if

$$\Phi(\sigma;\mu) \ge \Phi(\sigma';\mu) - \varepsilon$$

for all $\sigma' \in \Sigma$. We refer to a type-averaged 0-equilibrium as a type averaged equilibrium.

Proposition A.7 Let (μ, v) be a network game of incomplete information. If the profile of payoff functions v is bounded, the game has a type-averaged equilibrium.

The proof follows directly from the proof of Proposition 3.6. Let $\mathcal{N}^{\varepsilon}_{\tau}(\mu, v)$ denote the set of type-averaged ε -equilibria of (μ, v) .

We define a notion of strategic convergence for the current setting. Let $\mu, \mu' \in \mathcal{M}$, and let v be a profile of payoff functions. For $\varepsilon \geq 0$, define

$$\chi_{\tau}(\mu,\mu';v,\varepsilon) := \sup_{\sigma \in N^0_{\tau}(\mu,v)} \inf_{\sigma' \in N^{\varepsilon}_{\tau}(\mu',v)} |\Phi(\sigma;\mu) - \Phi(\sigma';\mu')|,$$

and

$$\chi_{\tau}^{*}(\mu,\mu;v,\varepsilon) := \max\left\{\chi_{\tau}(\mu,\mu';v,\varepsilon),\chi_{\tau}(\mu',\mu;v,\varepsilon)\right\}$$

Definition A.8 Take any $\mu \in \mathcal{M}$, and consider a sequence $(\mu^k)_{k \in \mathbb{N}}$ in \mathcal{M} . The sequence $(\mu^k)_{k \in \mathbb{N}}$ converges strategically in a type-averaged sense to μ if for each profile v of payoff functions that is bounded, for each $\varepsilon > 0$, we have that

$$\lim_{k \to \infty} \chi^*_{\tau}(\mu, \mu^k; v, \varepsilon) = 0.$$

Recall the definition of d_0 from Section 5, and note that the metric d_0 generates the weak topology.

Theorem A.9 Let $\mu \in \mathcal{M}$ and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{M} . Then, $(\mu^k)_{k \in \mathbb{N}}$ converges strategically in a type-averaged sense to μ if and only if

$$\lim_{k \to \infty} d_0(\mu, \mu^k) = 0.$$

The proof follows from Proposition A.10 and A.11.

Proposition A.10 establishes that the weak topology is sufficient.

Proposition A.10 Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Let v be a profile of payoff functions. Suppose that $d_0(\mu, \mu') \leq \delta$. Then, if $\sigma \in \Sigma^T$ is a type-averaged equilibrium of the game (μ, v) and if v is bounded by B, then σ is a type-averaged $4\delta B$ -equilibrium of the game (μ', v) , and

$$|\Phi(\sigma;\mu) - \Phi(\sigma;\mu')| \le 2\delta B.$$

Proof. Let $t \in T$ be such that $q_{\mu}(t) > 0$, and let $\sigma'_t \in \Sigma$. As σ is a type-averaged equilibrium of (μ, v) ,

$$\Phi(\sigma';\mu) - \Phi(\sigma;\mu) \le 0$$

Hence,

$$\Phi(\sigma;\mu') - \Phi(\sigma';\mu') \ge \Phi(\sigma;\mu') - \Phi(\sigma;\mu) + \Phi(\sigma';\mu) - \Phi(\sigma';\mu').$$
(A.4)

As $d_0(\mu, \mu') \leq \delta$,

$$\Phi(\sigma;\mu) - \Phi(\sigma;\mu') \ge -2\delta B,\tag{A.5}$$

$$\Phi(\sigma';\mu) - \Phi(\sigma';\mu') \ge -2\delta B. \tag{A.6}$$

Substituting (A.5) and (A.6) in (A.4), we find

$$\Phi(\sigma; \mu') \ge \Phi(\sigma'; \mu') - 4\delta B,$$

proving the first claim. The second claim follows directly from (A.5).

The next proposition shows that the topology generated by d_0 is also necessary.

Proposition A.11 Let $\delta \in [0, 1]$, and let $\mu, \mu' \in \mathcal{M}$. If

$$d_0(\mu, \mu') > \delta$$

then there exists a profile v of payoff functions with bound B = 1 and an equilibrium σ of the game (μ, v) such that for any δ -equilibrium σ' of (μ', v) , it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta.$$

The proof follows directly from the proof of Proposition 5.7.

We can now prove Theorem A.9.

Proof. (If) Let v be a profile of payoff functions. By Proposition A.10, for v bounded by B, and for $k \in \mathbb{N}$ such that $4Bd_0(\mu, \mu^k) \leq \varepsilon$,

$$\chi_{\tau}^*(\mu,\mu^k;v,\varepsilon) \le 2d_0(\mu,\mu^k)B.$$

Hence, for all profiles of payoff functions v that are bounded and for all $\varepsilon > 0$, if $d_0(\mu, \mu^k) \to 0$, then $\chi^*_{\tau}(\mu, \mu^k; v, \varepsilon) \to 0$.

(Only if) Let $\mu, \mu' \in \mathcal{M}$. For $\delta \in [0, 1)$, if $d_0(\mu, \mu') > \delta$, then, by Proposition A.11, there exists a profile of payoff functions v bounded by B = 1 and a type-averaged equilibrium $\sigma \in \Sigma^T$ of (μ, v) such that for any type-averaged δ -equilibrium $\sigma' \in \Sigma^T$ of $(\mu', v), |\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')| > \delta$.

A.5 Proof of Proposition 5.15

Proposition 5.15 uses Lemma A.12.

Lemma A.12 Let $\mu \in \mathcal{M}$, and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{M} . If

$$\lim_{k \to \infty} d_0(\mu, \mu^k) = 0 \qquad and \qquad \lim_{k \to \infty} d_1(\mu, \mu^k) = 0,$$

then

$$\lim_{k \to \infty} d_1(\mu^k, \mu) = 0$$

Proof. Let $\varepsilon > 0$. By assumption, there exists $K \in \mathbb{N}$ such that for all k > K,

$$\sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\mu^k}(F)| \le \frac{\varepsilon}{2},\tag{A.7}$$

and

$$\inf\left\{\delta \in [0,1] \mid q_{\mu}\left(\Theta(C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu^{k}}))\right)\right\} \leq \frac{\varepsilon}{2}.$$
(A.8)

Let k > K. Recall that for $t \in T_{\mu,\mu^k}^{\varepsilon/2}$ such that $q_{\mu}(t) > 0$ and $q_{\mu^k}(t) > 0$,

$$\sup_{F \in \mathscr{F}_K} |q_{\mu}(F|t) - q_{\mu^k}(F|t)| \le \frac{\varepsilon}{2},\tag{A.9}$$

and define

$$\hat{T}^{\varepsilon/2}_{\mu,\mu^k} := \{ t \in T^{\varepsilon/2}_{\mu,\mu^k} \mid q_\mu(t) > 0 \}.$$

Note that, unlike $T_{\mu,\mu^k}^{\varepsilon/2}$, the set $\hat{T}_{\mu,\mu^k}^{\varepsilon/2}$ is not symmetric in μ and μ^k , i.e., $\hat{T}_{\mu^k,\mu}^{\varepsilon/2} \neq \hat{T}_{\mu,\mu^k}^{\varepsilon/2}$. Using (A.9) and the fact that the local *p*-belief operator is monotonic, we obtain

$$B^{1-\varepsilon/2}_{\mu}(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}) \subseteq B^{1-\varepsilon}_{\mu^k}(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}) \subseteq B^{1-\varepsilon}_{\mu^k}(\hat{T}^{\varepsilon}_{\mu,\mu^k}).$$

Hence,

$$C^{1-\varepsilon/2}_{\mu}(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}) \subseteq C^{1-\varepsilon}_{\mu^k}(\hat{T}^{\varepsilon}_{\mu,\mu^k}).$$

Using this and (A.8), we obtain

$$\begin{aligned} q_{\mu} \Big(\Theta(C^{1-\varepsilon}_{\mu^{k}}(T^{\varepsilon}_{\mu,\mu^{k}})) \Big) &\geq q_{\mu} \Big(\Theta(C^{1-\varepsilon}_{\mu^{k}}(\hat{T}^{\varepsilon}_{\mu,\mu^{k}})) \Big) \geq \\ q_{\mu} \Big(\Theta(C^{1-\varepsilon/2}_{\mu}(\hat{T}^{\varepsilon/2}_{\mu,\mu^{k}})) \Big) &= q_{\mu} \Big(\Theta(C^{1-\varepsilon/2}_{\mu}(T^{\varepsilon/2}_{\mu,\mu^{k}})) \Big) \geq 1 - \frac{\varepsilon}{2} \end{aligned}$$

so that by (A.7),

$$q_{\mu^k} \left(\Theta(C^{1-\varepsilon}_{\mu^k}(T^{\varepsilon}_{\mu,\mu^k})) \right) \ge 1 - \varepsilon.$$

Combining these results gives

$$\inf \left\{ \delta \in [0,1] \mid q_{\mu^k} \left(\Theta(C_{\mu^k}^{1-\delta}(T_{\mu,\mu^k}^{\delta})) \right) \ge 1-\delta \right\} \le \varepsilon.$$

We can now prove Proposition 5.15

(If) Let $\varepsilon > 0$, and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{M} . Suppose that $S_{\varepsilon} \subseteq T$ is such that

$$|S_{\varepsilon}| < \infty, \tag{A.10}$$

$$S_{\varepsilon} \subseteq B^{1-\varepsilon}_{\mu}(S_{\varepsilon}), \tag{A.11}$$

$$q_{\mu}(\Theta(S_{\varepsilon})) \ge 1 - \varepsilon. \tag{A.12}$$

By Lemma A.12, if $d_0(\mu, \mu^k) \to 0$ and $d_1(\mu, \mu^k) \to 0$, then also $d_1(\mu^k, \mu) \to 0$. Hence, it is sufficient to show that $d_1(\mu, \mu^k) \to 0$ whenever $d_0(\mu, \mu^k) \to 0$.

Let $\hat{S}_{\varepsilon} := \{t \in S_{\varepsilon} \mid q_{\mu}(t) > 0\}$ be the set of types in S_{ε} that have positive probability under μ . By (A.10), there exists c > 0 such that $q_{\mu}(t) = q_{\mu}(\Omega_K^t) \ge c$ for all $t \in \hat{S}_{\varepsilon}$. Then, for all $k \in \mathbb{N}$, for all $t \in \hat{S}_{\varepsilon}$,

$$\sup_{F \in \mathscr{F}_{K}} |q_{\mu}(F|t) - q_{\mu^{k}}(F|t)| = \sup_{F \in \mathscr{F}_{K}} \left| \frac{q_{\mu}(F \cup \Omega_{K}^{t})}{q_{\mu}(\Omega_{K}^{t})} - \frac{q_{\mu^{k}}(F \cup \Omega_{K}^{t})}{q_{\mu}(\Omega_{K}^{t})} + \frac{q_{\mu^{k}}(F \cup \Omega_{K}^{t})}{q_{\mu}(\Omega_{K}^{t})} - \frac{q_{\mu^{k}}(F \cup \Omega_{K}^{t})}{q_{\mu^{k}}(\Omega_{K}^{t})} \right|,$$

$$\leq \sup_{F \in \mathscr{F}_{K}} \frac{1}{q_{\mu}(t)} |q_{\mu}(F \cup \Omega_{K}^{t}) - q_{\mu^{k}}(F \cup \Omega_{K}^{t})| + \frac{\sup_{F \in \mathscr{F}_{K}} \frac{q_{\mu^{k}}(F|t)}{q_{\mu}(t)}}{q_{\mu}(t)} |q_{\mu}(\Omega_{K}^{t}) - q_{\mu^{k}}(\Omega_{K}^{t})|,$$

$$\leq \left(\frac{2}{c}\right) \cdot \sup_{F \in \mathscr{F}_{K}} |q_{\mu}(F) - q_{\mu^{k}}(F)|.$$
(A.13)

Suppose that $\lim_{k\to\infty} d_0(\mu, \mu^k) = 0$. Then there exists $K \in \mathbb{N}$ such that for all k > K,

$$\sup_{F \in \mathscr{F}_K} |q_{\mu}(F) - q_{\mu^k}(F)| \le \left(\frac{c}{2}\right) \cdot \varepsilon.$$

Let k > K. Then, by (A.13), for all $t \in \hat{S}_{\varepsilon}$ such that $q_{\mu^k}(t) > 0$, it holds that

$$\sup_{F\in\mathscr{F}_K} |q_{\mu}(F|t) - q_{\mu^k}(F|t)| \le \varepsilon,$$

so that $S_{\varepsilon} \subseteq T^{\varepsilon}_{\mu,\mu^k}$. By monotonicity of the local *p*-belief operator and (A.11),

$$S_{\varepsilon} = B^{1-\varepsilon}_{\mu}(S_{\varepsilon}) \subseteq B^{1-\varepsilon}_{\mu}(T^{\varepsilon}_{\mu,\mu^k})$$

Using Lemma 4.2 and (A.11), we obtain

$$t \in S_{\varepsilon} \Rightarrow t \in C^{1-\varepsilon}_{\mu} (T^{\varepsilon}_{\mu,\mu^k}),$$

so that (using (A.12))

$$q_{\mu} \Big(\Theta(C^{1-\varepsilon}_{\mu}(T^{\varepsilon}_{\mu,\mu^k})) \Big) \ge q_{\mu} \Big(\Theta(S_{\varepsilon})) \Big) \ge 1-\varepsilon.$$

Hence, $d_1(\mu, \mu^k) \leq \varepsilon$ whenever $d_0(\mu, \mu^k) \leq \left(\frac{c}{2}\right)\varepsilon$.

(Only if) Suppose that

$$\lim_{k \to \infty} d_0(\mu, \mu^k) = 0 \Rightarrow \lim_{k \to \infty} d_1(\mu, \mu^k).$$

First we show that there exists a sequence $(\nu^k)_{k\in\mathbb{N}}$ in \mathcal{M} such that

- (a) for each $k \in \mathbb{N}$, the set of types $\{t \in T \mid q_{\nu^k}(t) > 0\}$ that have positive probability under ν^k is finite;
- (b) $(\nu^k)_{k\in\mathbb{N}}$ converges to μ in the weak topology on Ω_K :

$$\lim_{k \to \infty} \sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| = 0.$$

The sequence $(\nu^k)_{k\in\mathbb{N}}$ is easy to construct. If μ has finite support in T, i.e., if the set $\{t \in T \mid q_{\mu}(t) > 0\}$ is finite, then simply set $\nu^k = \mu$ for all $k \in \mathbb{N}$. Otherwise, we construct $(\nu^k)_{k\in\mathbb{N}}$ as follows. For each $k \in \mathbb{N}$, define

$$\mathcal{G}^{(k)} := \{ g \in \mathcal{G} \mid \forall i \in V(g), D_i(g) \le k \}$$

to be the set of networks in which the maximum degree is k. Note that the sequence $(\mathcal{G}^{(k)})_{k\in\mathbb{N}}$ is increasing. For each $g\in\mathcal{G}$, let

$$\nu^{k}(g) \begin{cases} \frac{\mu(g)}{\mu(\mathcal{G}^{(k)})} & \text{if } g \in \mathcal{G}^{(k)} \text{ and } \mu(\mathcal{G}^{(k)}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that (a) is satisfied. To see that (b) is also satisfied, first recall that $\mathscr{C}^{(k)}$ is the collection of isomorphism classes in $\mathcal{G}^{(k)}$. For each $k \in \mathbb{N}$ such that $\mu(\mathcal{G}^{(k)}) > 0$, we have

$$\begin{split} \sup_{F\in\mathscr{F}_{K}}|q_{\mu}(F)-q_{\nu^{k}}(F)| &= \sup_{F\in\mathscr{F}_{K}}\left|\frac{\sum_{C\in\mathscr{C}}\mu(C)n_{C}(F)}{\sum_{C\in\mathscr{C}}\mu(C)n_{C}(\Omega_{K})} - \frac{\sum_{C\in\mathscr{C}}\nu^{k}(C)n_{C}(R)}{\sum_{C\in\mathscr{C}}\nu^{k}(C)n_{C}(\Omega_{K})}\right| \\ &= \sup_{F\in\mathscr{F}_{K}}\left|\frac{\sum_{C\in\mathscr{C}}\mu(C)n_{C}(F)}{\sum_{C\in\mathscr{C}}\mu(C)n_{C}(\Omega_{K})} - \frac{\sum_{C\in\mathscr{C}^{(k)}}\nu^{k}(C)n_{C}(\Omega_{K})}{\sum_{C\in\mathscr{C}^{(k)}}\nu^{k}(C)n_{C}(\Omega_{K})}\right| \\ &\leq \frac{1}{\bar{n}}\sup_{F\in\mathscr{F}_{K}}\left|\sum_{C\in\mathscr{C}}\mu(C)n_{C}(F) - \sum_{C\in\mathscr{C}^{(k)}}\mu(C)n_{C}(F)\right| + \\ &\left(\frac{1}{\bar{n}} - \frac{1}{\sum_{C\in\mathscr{C}^{(k)}}\mu(C)n_{C}(\Omega_{K})}\right)\sup_{F\in\mathscr{F}_{K}}\sum_{C\in\mathscr{C}^{(k)}}\mu(C)n_{C}(F) \\ &\leq \frac{1}{\bar{n}}\sup_{F\in\mathscr{F}_{K}}\left(\sum_{C\in\mathscr{C}\setminus\mathscr{C}^{(k)}}\mu(C)n_{C}(F)\right) + 1 - \frac{\bar{n}}{\sum_{C\in\mathscr{C}^{(k)}}\mu(C)n_{C}(\Omega_{K})}. \end{split}$$

As for all $F \in \mathscr{F}_K$,

$$\lim_{k \to \infty} \sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(F) = \sum_{C \in \mathscr{C}} \mu(C) n_C(F),$$

it follows that (b) holds.

Since μ is insensitive to small probability events, we also have that $d_1(\mu, \nu^k) \to 0$. Hence, for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all k > K,

$$\sup_{F \in \mathscr{F}_K} |q_{\mu}(F) - q_{\nu^k}(F)| \le \frac{\varepsilon}{3}$$
(A.14)

and

$$\inf\left\{\delta \in [0,1] \mid q_{\mu}\left(\Theta(C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\nu^{k}}))\right) \ge 1-\delta\right\} \le \frac{\varepsilon}{3}.$$
(A.15)

Let k > K, and define

$$\hat{T}^{\varepsilon}_{\mu,\nu^k} := \{ t \in T^{\varepsilon}_{\mu,\nu^k} \mid q_{\nu^k}(t) > 0 \}$$

to be the set of types in $T^{\varepsilon}_{\mu,\nu^k}$ that have positive probability under ν^k . By (A.15) and using that the local *p*-belief operator is monotonic and continuous in *p*,

$$q_{\mu} \Big(\Theta(C^{1-\varepsilon}_{\mu}(T^{\varepsilon}_{\mu,\nu^{k}})) \Big) \ge q_{\mu} \Big(\Theta(C^{1-\varepsilon/3}_{\mu}(T^{\varepsilon}_{\mu,\nu^{k}})) \Big) \ge q_{\mu} \Big(\Theta(C^{1-\varepsilon/3}_{\mu}(T^{\varepsilon/3}_{\mu,\nu^{k}})) \Big) \ge 1 - \frac{\varepsilon}{3},$$

so that by (A.14),

$$q_{\nu^k} \left(\Theta(C^{1-\varepsilon}_{\mu}(\hat{T}^{\varepsilon}_{\mu,\nu^k})) \right) = q_{\nu^k} \left(\Theta(C^{1-\varepsilon}_{\mu}(T^{\varepsilon}_{\mu,\nu^k})) \right) \ge 1 - \frac{2\varepsilon}{3},$$

and hence (using (A.14) again),

$$q_{\mu} \big(\Theta(C^{1-\varepsilon}_{\mu}(\hat{T}^{\varepsilon}_{\mu,\nu^{k}})) \big) \ge 1 - \varepsilon.$$

By definition, $\hat{T}^{\varepsilon}_{\mu,\nu^{k}}$, and hence $C^{1-\varepsilon}_{\mu}(\hat{T}^{\varepsilon}_{\mu,\nu^{k}})$, is finite. Moreover, by Lemma 4.1, $C^{1-\varepsilon}_{\mu}(\hat{T}^{\varepsilon}_{\mu,\nu^{k}})$ is $(1-\varepsilon)$ -closed. Hence, by setting

$$S_{\varepsilon} = C_{\mu}^{1-\varepsilon} \left(\hat{T}_{\mu,\nu^k}^{\varepsilon} \right)$$

we obtain the desired result.

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