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# CONSTRAINED ASSET MARKETS

**Guangsug Hahn**

Division of Humanities and Social Sciences

POSTECH, South Korea

**Dongchul Won**[*]

College of Business Adminstration

Ajou University, South Korea

[*]Corresponding Author: Dongchul Won, Ajou University, College of Business Adminstration, San 5 Wonchun-Dong, Yeongtong-Gu, Suwon, KOREA 443-749, Phone: +82-31-219-2708, Email: dcwon@ajou.ac.kr

## Abstract

The risk-sharing role of redundant assets is not yet fully understood in constrained asset markets. For example, the well-known notions of arbitrage may fail to explain the viability property of asset prices when redundant assets are involved in generating a nontrivial linear structure of free portfolios in equilibrium in constrained asset markets. This paper establishes the existence of equilibrium in two-period asset markets which are subject to portfolio constraints. First, we provide a full analysis of complicated equilibrium behavior of constrained portfolios with redundant assets by adopting a new notion of arbitrage. To do this, a technique of portfolio decomposition is developed to identify the linear structure of free portfolios embedded in the aggregate set of constrained portfolios. Second, we present a new condition on the aggregate set of portfolios which is indispensable for the existence of equilibrium in constrained asset markets. The literature assumes that the individual portfolio constraint set or the individual marketed set of income transfers is closed to study the presence of optimal portfolios or viability of asset prices in a partial equilibrium framework. As illustrated later, however, this condition alone fails to be sufficient for the existence of equilibrium. Consequently, we resolve the equilibrium existence issue with two-period constrained asset markets.



KEYWORDS: equilibrium, portfolio constraints, redundant assets, incomplete markets, arbitrage, viability.

JEL Classification: G12, D52, C62, G11

## I. Introduction

Financial derivatives such as call and put options, stock index futures, forward contracts do not create a new opportunity of risk sharing in frictionless markets because their payoffs can be freely replicated by portfolios of available assets. Redundant assets can be priced in frictionless markets by the law of one price.[1] Thus, there will be no room for financial derivatives in the literature of general equilibrium with incomplete markets as far as they are redundant.[2] In the real world, however, financial derivatives are introduced through costly financial innovation. The empirical literature also documents that they are frequently mispriced.[3] This implies that the law of one price may fail and redundant assets can contribute to risk sharing in the real world. Thus, empirical evidences lead to tension between risk sharing rationale for redundant assets and asset pricing theory built upon the law of one price.

Redundant assets may contribute to income spanning under portfolio constraints. Their risk-sharing role, however, is not yet fully understood in constrained asset markets. For example, the well-known notions of arbitrage may fail to explain the viability property of asset prices when redundant assets are involved in generating a nontrivial linear structure of free portfolios in equilibrium in constrained asset markets.[4]

The purpose of this paper is to establish the existence of equilibrium in two-period asset markets which are subject to portfolio constraints. The main consequences of the paper are differentiated from the literature in several respects. First, we provide a full analysis of complicated equilibrium behavior of constrained portfolios with redundant assets by adopting a notion of 'projective arbitrage.' To do this, a technique of portfolio decomposition is developed to identify the linear structure of free portfolios embedded in the aggregate set of constrained portfolios.[5] To our knowledge, this result is new to the literature. The linear structure of free portfolios leads to an unbounded multiplicity of portfolio choices in asset market equilibrium. The multiplicity of optimal portfolio choices is of hybrid type. It is nominal in the sense that unbounded free portfolios do not affect the choices of optimal consumption but can be real in the sense that they matter to risk sharing in general. Moreover, the linear structure of free port-

[1] Formally speaking, an asset is redundant if its return vector is linearly dependent on the return vectors of other assets.

[2] See Cass (1984), Werner (1985), Geanakoplos and Polemarchakis (1987), Gottardi and Hens (1996) among others.

[3] Mispricing refers to the violation of the law of one price. Examples of mispricing are stock index futures (Canina and Figlewski (1995)), primes and scores (Jarrow and O'Hara (1989)), closed-end funds (Pontiff (1996)).

[4] Asset prices are *viable* if they allow agents to find optimal portfolios. See Harrison and Kreps (1979).

[5] Constrained portfolios are a portfolio which satisfies the portfolio constraints.

folios implies that the law of one price can limitedly hold in constrained asset markets. Thus, the 'constrained law of one price' partially mitigates tension between risk sharing rationale and asset pricing theory for redundant assets and provides a yardstick for judging the properness of a notion of arbitrage as a conceptual framework of equilibrium analysis. As shown later, projective arbitrage belongs to the proper class of arbitrages but well-known notions of arbitrage do not.

To be more specific about the constrained law of one price, we consider an asset $A$ and a portfolio $B$ which replicates the contingent payoffs of the asset $A$. A trading strategy of taking long position with the asset $A$ and short position with the portfolio $B$ constitutes the *null-income portfolio* $C$.[6] By the law of one price, the value of the asset $A$ equals the value of the portfolio $B$ and therefore, the null-income portfolio $C$ has null value in equilibrium of frictionless markets. Thus, the portfolio $C$ is totally useless in unconstrained markets. This is not the case with constrained asset markets. When the law of one price fails due to portfolio constraints, the risk sharing role of the null-income portfolio $C$ is obvious because the portfolio constraints are binding at the optimal choice of portfolios. A subtle case occurs when the portfolio $C$ belongs to the linear structure of free portfolios. In this case, it will have null-value in equilibrium whether the portfolio constraints are binding or not at the individual choice of optimal portfolios and thus, the asset $A$ can be priced by the constrained law of one price. To capture asset pricing implications of null-income portfolios in constrained asset markets, we need to identify a special type of null-income portfolios called a *bridge portfolio*. Bridge portfolios are a null-income portfolio in the linear subspace which is spanned by the set of aggregate constrained null-income portfolios.[7] As shown later, bridge portfolios always have null value in equilibrium independently of the allocation of the initial endowments of goods for all agents. Consequently, bridge portfolios constitute the linear structure of free portfolios in the aggregate set of constrained portfolios. The other types of constrained null-income portfolios, however, need not have null value in equilibrium.

To articulate the risk sharing role of redundant assets, the whole space of portfolio choices is decomposed into the linear subspace of bridge portfolios and its orthogonal complement. The projection of a constrained portfolio onto the orthogonal complement is called the *value portfolio*. The value portfolio generates the same income transfers as the pre-projection portfolio

[6] Null-income portfolios are ones which pay nothing in future contingencies. Mathematically speaking, they are in the kernel of the return matrix. For a $m \times n$ matrix $R$, $\{v \in \mathbb{R}^n : R \cdot v = 0\}$ is the kernel of $R$.

[7] Constrained null-income portfolios are a null-income portfolio which satisfies the portfolio constraints.

but need not be feasible. Bridge portfolios have the peculiar property that they are of null value but may be indispensable for risk sharing because they can complement the value portfolios to meet feasibility in constrained markets. Thus, a proper class of notions of arbitrage must be able to capture the asset pricing implications of bridge portfolios.

Second, we present a new condition that the aggregate set of feasible portfolios which yield nonnegative income in each future contingency for all agents be closed in constrained asset markets. This condition differentiates the current work from the literature which studies constrained asset markets in a partial equilibrium framework. For example, Cvitanic and Karatzas (1992, 1993), Luttmer (1995), Jouini and Kallal (1999) among others assume that the individual portfolio constraint set or the individual marketed set of income transfers is closed to study the presence of optimal portfolios or viability of asset prices in constrained asset markets. As illustrated later, however, the closedness condition on the individual choice set alone is no longer sufficient for the existence of equilibrium in constrained asset markets with redundant assets. Thus, the conditions required for viability of asset prices may not be sufficient for the existence of equilibrium. The divergence between viability and equilibrium conditions can arise in the presence of nonzero bridge portfolios which are actively involved in risk sharing. A proper understanding of the relationship between the risk-sharing capability and valuation of constrained null-income portfolios will play a key role in resolving the equilibrium existence problem with bridge portfolios. As later shown, a full exposition of the properties of bridge portfolios relies on sophisticated convex analysis on the portfolio constraint sets.

Instead of imposing the closednesss condition on the aggregate set of feasible portfolios, one might be tempted to replace it by the requirement that the aggregate set of marketed income transfers be closed. It is shown later, however, that the latter may not be a right substitute for the former in ensuring the existence of equilibrium in constrained asset markets with nontrivial bridge portfolios. This means that the closedness condition on both individual and aggregate marketed income sets may fail as equilibrium conditions. Thus, the reduced-form analysis on the marketed income sets which is quite popular in the literature of asset pricing may not be appropriate in characterizing equilibrium asset prices in constrained asset markets with many agents.

Cass, Siconolfi and Villanacci (2001) notice the difficulty with redundant assets in constrained markets as following.

> "In this context, Assumption 1 is not at all innocuous.[8] When their portfolio holdings are constrained, households may very well benefit from the opportunities afforded by the availability of additional bonds whose yields are not linearly independent."

Real complications with redundant assets arise in the case with nonlinear portfolio constraints which allow bridge portfolios to serve risk sharing.[9] Balasko, Cass, and Siconolfi (1990), and Benveniste and Ketterer (1992) study economies with linear constraints where individual portfolios are required to satisfy a systems of linear homogeneous equations. In this case, it is quite straightforward to identify bridge portfolios because they are determined by the return matrix and the linear constraints. In the case with nonlinear convex constraints, however, bridge portfolios may not be directly identified from individual constraint sets alone because of their nonlinearity. The tricky job to extract the linear structure of bridge portfolios from the nonlinear system of individual portfolio constraints is done through a sophisticated process of portfolio decomposition. Another tough problem arises from the fact that bridge portfolios are free in equilibrium but may matter to risk sharing. Since bridge portfolios are free in equilibrium, agents are allowed to hold long or short a sufficiently large of bridge portfolios. But the problem is we do not know *a priori* how much agents need them to meet feasibility in equilibrium. This fact poses a big trouble to the application of fixed point theorems for verifying the existence of equilibrium because the set of bridge portfolios may be unbounded in equilibrium. These difficulties seem to be a main reason that the existence issue of equilibrium with nonlinearly constrained markets has not yet been resolved.[10]

We briefly review the literature on the existence of equilibrium in constrained asset markets. The existence issue of this paper was initially addressed by Siconolfi (1986), which has led to important researches like the indeterminacy problem with portfolio constraints of Cass, Siconolfi and Villanacci (2001). As in this paper, Siconolfi (1986) assumes that asset markets are subject to convex portfolio constraints. Siconolif (1986), however, imposes a severe restriction on the risk-sharing role of redundant assets by prohibiting agents from possessing large constrained null-income portfolios.[11] Specifically, Siconolfi (1986) does not cover the case with nonzero bridge portfolios. Balasko, Cass and Siconolfi (1990), and Benveniste and Ketterer (1992) assume that portfolio constraints are represented by linear homogeneous equations. In

[8] Assumption 1 of Cass, Siconolfi and Villanacci (2001) is a condition which assumes away redundant assets.

[9] Won, D. and G. Hahn (2003) provide interesting examples about the risk sharing role of constrained null-income portfolios.

[10] As reviewed below, the literature with nonlinear portfolio constraints assumes that nonzero bridge portfolios do not exist.

[11] Siconolif (1986) assumes that if $\lambda\theta_i$ for each $\lambda > 0$ is a constrained null-income portfolio for agent $i$, then $\theta_i = 0$.

particular, Balasko, Cass and Siconolfi (1990) develop an ingenious technique to handle the unboundedness problem with bridge portfolios in the case with linear portfolio constraints. But their approach is no longer applicable to the case with nonlinear portfolio constraints. Angeloni and Cornet (2006) provide a general existence theorem of equilibrium for multi-period stochastic exchange economies which allows to cover financial markets with nominal as well as real assets. But, they are not concerned about the case where bridge portfolios contribute to risk sharing.

Constrained asset markets are also differentiated from unconstrained ones in terms of survival conditions. It is well-known that equilibrium exists in unconstrained asset markets under the strong survival condition that agents' initial endowments of goods are in the interior of the consumption set. A great benchmark with survival conditions is Gottardi and Hens (1996) which provide an extensive study of the survival problem with unconstrained incomplete markets. As shown later, however, the strong survival condition with goods markets alone is not enough to make sure the existence of equilibrium in constrained asset markets. Thus, the approach of Gottardi and Hens (1996) is not valid here because they start with the existence theorem for the unconstrained incomplete markets which satisfies the strong survival condition with goods markets.[12]

The rest of this paper is organized as follows. The constrained asset markets under study are described in Section II. The consequences of portfolio decomposition are provided in Section III. In Section IV, the notion of projective arbitrage is presented and its comparative advantage over well-known notions of arbitrage are discussed in terms of the viability property of asset prices. Section V is devoted to proving the existence of equilibrium. To do this, we provide the survival condition with asset markets and characterize the boundary behavior of aggregate demand correspondences. Concluding remarks are made in the last section.

## II. The Economy

[12]They construct a sequence of economies by perturbing the original endowments of goods in a way that the perturbations belong to the interior of the consumption set. The limit of a sequence of equilibria for the perturbed economies becomes a quasi-equilibrium of the economy. It becomes equilibrium of the economy under the cheaper point or irreducibility conditions. As illustrated later, however, equilibrium may not exist under portfolio constraints although the endowments of goods are in the interior of the consumption set for all agents. Thus, the approach of Gottardi and Hens (1996) is not useful for constrained asset markets.

An economy is assumed to persist over two periods. Uncertainty is described by the finite set of events $S = \{1, \ldots, S\}$ and resolved in the second period.[13]Assets are traded in the first period (denoted by 0) and consumptions are allowed only in the second period (denoted by 1).[14] Assets pay in monetary units in the second period.[15] The monetary returns are contingent upon the event $s \in S$. A set of $L$ consumption goods are available in each $s \in S$. Let $I = \{1, 2, \ldots, I\}$ denote the set of agents, $J = \{1, 2, \ldots, J\}$ the set of financial assets, and $L = \{1, 2, \ldots, L\}$ the set of consumption goods. Each agent $i \in I$ has the consumption set $X_i := \mathbb{R}^{SL}_{+}$, an initial endowment of goods $e_i \in X_i$, and the preferences represented by a utility function $u_i : X_i \to \mathbb{R}$. For a collection of points $\{y(1), \ldots, y(S)\}$ in $\mathbb{R}^L$, we set $y = (y(1), \ldots, y(S))$. Utility functions are assumed to satisfy the following properties.

**Assumption 2.1 :** The following hold true.

$(i)$ Each $u_i$ is continuous, strictly increasing, and quasiconcave.[16]

$(ii)$ $e_i(s) > 0$ for each $i$ and $s \in S$, and $\sum_{i \in I} e_i \gg 0$.[17]

The first condition of Assumption 2.1 is quite standard. The second condition is a survival condition with goods markets which state that each agent has a positive endowment of at least one good in each state and the aggregate endowments of every good are positive. As illustrated later, survival conditions with goods markets are not sufficient for the existence of equilibrium. A survival condition with asset markets will be introduced to make up for the insufficiency of $(ii)$ of Assumption 2.1. There has to be a trade-off between the strictness of a 'good' pair of survival conditions with goods and asset markets in that if one condition gets tougher, the other condition must be weakened to make sure the existence of equilibrium. It will be illustrated that such trade-off occurs to the pair of survival conditions chosen here.

Each asset $j \in J$ pays $r_{sj}$ at state $s$. The vector of asset returns in state $s$ is given by a $J$-dimensional row vector $r(s) = (r_{sj})_{j \in J}$ and the return of asset $j$ by a $S$-dimensional column

[13]We use the same symbol to denote a set and also its last element: no confusion should arise.

[14]As discussed later, the model covers the case where consumption arises in both period.

[15]Alternatively we can assume that assets pay units of the numeraire good because nominal assets can be converted into real assets and vice versa. For details, see Magill and Shafer (1991).

[16]The function $u_i$ is strictly increasing if for any $x, x'$ in $X_i$ with $x - x' \in \mathbb{R}^{SL}_{+}$ and $x \neq x'$, $u_i(x) > u_i(x')$.

[17]Let $v$ and $v'$ be vectors in an Euclidean space. Then $v \geq v'$ implies that $v - v' \in \mathbb{R}^{SL}_{+}$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbb{R}^{SL}_{++}$.

vector $r_j = (r_{sj})_{s \in S}$. The asset payoffs are described by an $S \times J$ matrix $R = [(r(s))_{s \in S}]$. Here either $S \geq J$ or $S < J$ holds. This means that it does not matter whether financial markets are potentially complete or not. Market incompleteness is rather represented by the opportunity set $\Theta_i$ of portfolios in $\mathbb{R}^J$. Let $\Theta = \sum_{i \in I} \Theta_i$.

For a given pair $(p, q) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^J$, each agent $i \in I$ chooses $(x_i^*, \theta_i^*) \in X_i \times \Theta_i$ to solve the optimization problem:

$$\max_{(x_i, \theta_i)} u_i(x_i)$$

subject to

$$\begin{aligned}
&q \cdot \theta_i \leq 0, \\
&p(s) \cdot (x_i(s) - e_i(s)) \leq r(s) \cdot \theta_i, \quad \forall s \in S, \\
&\theta_i \in \Theta_i, \ x_i \in X_i
\end{aligned}$$

For simplicity, we use the following notation.

$$p \,\square\, (x_i - e_i) = \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(S) \cdot (x_i(S) - e_i(S)) \end{bmatrix}, \qquad W(q) = \begin{bmatrix} -q \\ R \end{bmatrix}.$$

The budget constraints and demand correspondence for agent $i$ are set as

$$\begin{aligned}
\mathcal{B}_i(p, q) &= \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : \begin{bmatrix} 0 \\ p \,\square\, (x_i - e_i) \end{bmatrix} \leq W(q) \cdot \theta_i \right\}, \\
\xi_i(p, q) &= \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : (x_i, \theta_i) \in \operatorname*{arg\,max}_{(x, \theta) \in \mathcal{B}_i(p, q)} u_i(x) \right\}.
\end{aligned}$$

Let $\mathcal{E} = \langle (X_i, \Theta_i, u_i, e_i)_{i \in I}, R \rangle$ denote the economy described above. Competitive equilibrium of the economy $\mathcal{E}$ is defined as follows.

**Definition 2.2 :** A pair $(p, q, x, \theta) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^J \times (\prod_{i \in I} X_i) \times (\prod_{i \in I} \Theta_i)$ is a *competitive equilibrium* if

(1) $(x_i, \theta_i) \in \xi_i(p, q)$ for every $i \in I$,

(2) $\sum_{i \in I} (x_i - e_i) = 0$,

(3) $\sum_{i \in I} \theta_i = 0$.

Let $V$ denote the subspace spanned by the row vectors of the return matrix $R$ and $V^{\perp}$ be its kernel, *i.e.*, $V^{\perp} = \{\theta \in \mathbb{R}^J : R \cdot \theta = 0\}$. Redundant assets exist if and only if $V^{\perp} \neq \{0\}$. In particular, some assets are redundant if the rank of the return matrix $R$ is less than the minimum of $J$ and $S$. Portfolios in $V^{\perp}$ are called *null-income portfolios*, which generate null-income transfer in each state of the second period. In particular, portfolios in $\Theta_i \cap V^{\perp}$ are called *constrained null-income portfolios* for agent $i$. We make the following assumption for each $i \in I$.

**Assumption 2.2 :** The set $\Theta_i$ is a closed, convex set in $\mathbb{R}^J$ with $0 \in \Theta_i$.

As illustrated in Luttmer (1996), Assumption 2.2 covers market frictions such as short-selling constraints, bid-ask spreads, and proportional transaction costs. This condition is assumed in the literature which studies asset pricing in constrained asset markets.[18] As shown below, however, the condition alone is not sufficient for the existence of equilibrium.

Let $\omega_i$ be a vector in $\mathbb{R}^S$. For each $i \in I$, we set $\Phi_i(\omega_i) = \Theta_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta + \omega_i \geq 0\}$. This set contains portfolios in $\Theta_i$ which generate state-contingent incomes at least as much as $-\omega_i$ for agent $i \in I$. We set $\Phi_i = \Phi_i(0)$ for each $i \in I$. For each $i$, we define the set $\langle R \rangle_i = \{\mu_i \in \mathbb{R}^S : \mu_i = R \cdot \theta_i \text{ for some } \theta_i \in \Theta_i\}$. Instead of Assumption 2.2, one may consider the condition that the marketed set $\langle R \rangle_i$ of income transfers be closed. This condition is assumed in the literature which conducts the reduced-form analysis to examine the relationship between viable and arbitrage-free prices. As shown below, however, the closedness condition on $\langle R \rangle_i$'s is not sufficient for equilibrium to exist either. This is why viability is not enough to ensure the existence of equilibrium in constrained asset markets with many agents. Consequently, the presence of optimal portfolios for individual investors may not guarantee the existence of equilibrium in constrained asset markets. The following condition will make up for the deficiency of Assumption 2.2.

**Assumption 2.3 :** For all $\omega_i \in \mathbb{R}^S$, $\sum_{i \in I} \Phi_i(\omega_i)$ is closed in $\mathbb{R}^J$.

Assumption 2.3 requires that the aggregate set of feasible portfolios which generate state-

[18] See Luttmer (1996) and Cvitanic and Karatzas (1992, 1993) among others. The closedness condition on the individual portfolio constraint set is indispensable for the existence of optimal portfolios in the budget set.

contingent incomes greater than or equal to $-\omega_i$ for each $i \in I$ be closed in $\mathbb{R}^J$. To our knowledge, this is a new condition in the literature. Assumption 2.3 turns out to be a minimal requirement for investigating the existence of equilibrium and capturing the asset pricing implications of bridge portfolios. As shown below, the popular condition of the finance literature that the individual set of marketed incomes $\langle R\rangle_i$ be closed in $\mathbb{R}^S$ alone may not be sufficient for the existence of equilibrium when the economy does not satisfy Assumption 2.3. Moreover, the additional requirement that the aggregate set $\sum_{i\in I}\langle R\rangle_i$ be closed in $\mathbb{R}^S$ fails to cure the problem. Thus, the reduced-form approach of the literature may not be valid for conducting equilibrium analysis in constrained asset markets. Assumption 2.3 trivially holds when either $\text{rank}(R) = J$ or $\Theta_i = \mathbb{R}^J$ for all $i \in I$. As shown later in Example 3.2, Assumption 2.3 finds an interesting application in Balasko, Cass and Siconolfi (1990) where each $\Theta_i$ is a linear subspace of $\mathbb{R}^J$. A class of portfolio constraints which satisfy Assumption 2.3 are discussed in the end of the current section.

**Example 2.1 :** We illustrate that an economy may have no equilibrium only because of the failure of Assumption 2.3 to hold. An interesting point is that the closedness condition on $\langle R\rangle_i$'s fails to ensure the existence of equilibrium. We consider an economy where $L = 1$, $S = 3$, $I = 2$, and $J = 3$. The single good is also used as a numeraire. We assume that the payoff matrix $R$ is given by the $3 \times 3$ matrix[19]

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $V^\perp$ is the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 0, 0)$ and $(0, 0, 1)$.

Both agents have the same endowment of goods and distinct preferences.

$$\begin{aligned} u_1(x) &= x(1) + 0.5x(2) + 0.5x(3), \quad & e_1 &= (1, 1, 1), \\ u_2(x) &= x(1) + x(2) + x(3), & e_2 &= (1, 1, 1). \end{aligned}$$

[19]The choice of null-return assets in $R$ is a matter of convenience. If $R$ is postmultiplied by a nonsingular matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, then it is transformed into $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, which exhibits linear dependence between nonzero return vectors. The same arguments as made below will apply to the economy with the transformed return matrix.

We assume that, for each $i = 1, 2$,

$$\begin{aligned}\Theta_1 &= \{(a,b,c) \in \mathbb{R}^3 : a \geq -2,\ b \geq 1/(a+3) - 1 - c\},\\ \Theta_2 &= \{(a,b,c) \in \mathbb{R}^3 : a \leq 2,\ b \geq -1,\ c \geq -1\}.\end{aligned}$$

Notice that $e_i$ is in the interior of $X_i = \mathbb{R}^3_+$ and $\Theta_i$ contains $0$ in the interior for each $i = 1, 2$. Thus, the economy satisfies the strong survival condition. It is straightforward to see that

$$\begin{aligned}\langle R\rangle_1 &= \{(\tilde{a},\tilde{b},\tilde{c}) \in \mathbb{R}^3 : \tilde{b} = \tilde{c}\},\\ \langle R\rangle_2 &= \{(\tilde{a},\tilde{b},\tilde{c}) \in \mathbb{R}^3 : \tilde{b} = \tilde{c} \text{ and } \tilde{c} \geq -1\}.\end{aligned}$$

and therefore, $\langle R\rangle_i$ is closed for each $i = 1, 2$. Moreover, the aggregate marketed income set $\langle R\rangle_1 + \langle R\rangle_2$ is also closed.

Since $\{\theta \in \mathbb{R}^3 : R \cdot \theta \geq 0\} = \{(a,b,c) \in \mathbb{R}^3 : b \geq 0\}$, we see that for all $i = 1, 2$,

$$\begin{aligned}\Phi_1 &= \{(a,b,c) \in \mathbb{R}^3 : a \geq -2,\ b \geq 0,\ b \geq 1/(a+3) - 1 - c\},\\ \Phi_2 &= \{(a,b,c) \in \mathbb{R}^3 : a \leq 2,\ b \geq 0,\ c \geq -1\}.\end{aligned}$$

Clearly, $\Phi_1$ and $\Phi_2$ are closed.

We claim that $\Phi_1 + \Phi_2$ is not closed in $\mathbb{R}^4$. To show this, we choose vectors $\theta_1^n = (n, 1 - 1/n + 1/(n+3), -2 + 1/n)$ and $\theta_2^n = (-n, 0, -1)$ in $\mathbb{R}^3$ for each $n$. Clearly, we have $\theta_i^n \in \Phi_i$ for each $n$ and $i = 1, 2$. We set $\theta^n = \theta_1^n + \theta_2^n = (0, 1 - 1/n + 1/(n+3), -3 + 1/n)$ for each $n$. Then $\theta^n$ converges to $\theta = (0, 1, -3)$. Since $\theta_i^n \in \Phi_i$ for each $n$ and $i = 1, 2$, we have $\theta^n \in \Phi_1 + \Phi_2$ for each $n$. We have $\theta \notin \Phi_1 + \Phi_2$.[20] Thus, $\Phi_1 + \Phi_2$ is not closed.

Now we show that the economy has no equilibrium. We set $\hat{u}_1(a,b,c) = b + 2$ and $\hat{u}_2(a,b,c) = 2b + 3$. For a price $q = (q_1, q_2, q_3)$, we define the set $\mathcal{A}_i(q) = \{(a,b,c) \in \mathbb{R}^3 : q_1 a + q_2 b + q_3 c \leq 0,\ (a,b,c) \in \Theta_i\}$ for each $i = 1, 2$. The function $\hat{u}_i$ is a reduced-form utility function defined over feasible portfolios and $\mathcal{A}_i(q)$ is the budget set for agent i at the price $q$. Then the utility maximization problem for agent $i = 1, 2$ is reduced to the following relations.

$$\max_{(a,b,c) \in \mathcal{A}_i(q)} \hat{u}_i(a,b,c).$$

Suppose that there exists an equilibrium $\{(q_1,q_2,q_3), (a_1,b_1,c_1), (a_2,b_2,c_2)\}$. We must have $q_1 = 0$. Otherwise, agent 1 or 2 could profit from taking an indefinite size of either short or

[20]Suppose that $\theta \in \Phi_1 + \Phi_2$. Then there exists $(a_i, b_i, c_i) \in \Phi_i$ for each $i = 1, 2$ such that $a_1 + a_2 = 0$, $b_1 + b_2 = 1$, $c_1 + c_2 = -3$, $c_1 \geq 1/(a_2+3) - 1 - b_1$ and $c_2 \geq -1$. Since $a_1 \geq -2$ and $0 \leq b_1 \leq 1$, we have $c_1 \geq 1/(a_2+3) - 1 - b_1 > -2$. On the other hand, $c_1 + c_2 = -3$ and $c_2 \geq -1$ imply $c_1 \leq -2$, which is impossible.

long position with asset 1. We set $q_2 = 1$ by normalizing asset prices. We claim that $q_3 \neq 0$. Otherwise, it follows from the budget constraint of agent 1 that $b_1 = -q_3 c_1 = 0$. Recalling that $(a_1, b_1, c_1) \in \Theta_1$, we have $c_1 \geq 1/(a_1 + 3) - 1 > -1$. By the market clearing condition on asset 3, this implies that $c_2 = -c_1 < -1$, which contradicts the fact that $c_2 \geq -1$ in $\Theta_2$.

By substituting $b_i = -q_3 c_i$ into $\hat{u}_i$ for each $i = 1, 2$, we obtain

$$\begin{aligned} \hat{u}_1(a_1, b_1, c_1) &= 2 - c_1 q_3, \\ \hat{u}_2(a_2, b_2, c_2) &= 3 - 2c_2 q_3. \end{aligned}$$

Since $c_2 \geq -1$, we must have $q_3 > 0$ and $c_2 = -1$ for utility maximization of agent 2. Then it follows from the market clearing condition on asset 3 that $c_1 = 1$. This result is contradictory because $\hat{u}_1(a_1, b_1, c_1) = 2 - q_3 < \hat{u}_1(0, 0, 0) = 2$ and $(0, 0, 0) \in \mathcal{A}_1(q)$. Thus, no equilibrium exists for the economy. $\square$

We introduce more notation for subsequent analysis. Let $A$ be a nonempty convex subset in $\mathbb{R}^m$ for some positive integer $m$. We denote the closure of $A$ by $c\ell(A)$, the interior of $A$ by $int(A)$, and the boundary of $A$ with respect to the relative topology by $\partial A$. The relative interior of $A$, denoted by $ri(A)$, is the interior of $A$ in the smallest affine subspace of $\mathbb{R}^m$ which contains $A$.

The asymptotic cone of $A$ is the set

$$\mathcal{K}(A) = \left\{ v \in \mathbb{R}^m : \exists \{x^n\} \text{ in } A \text{ and } \{a^n\} \text{ in } \mathbb{R} \text{ with } a^n \to 0 \text{ such that } v = \lim_{n \to \infty} a^n x^n \right\}.$$

It is well-known that $\mathcal{K}(A)$ is closed, and $\mathcal{K}(A) = \mathcal{K}(c\ell(A)) = \mathcal{K}(ri(A))$. Moreover, when $A$ is closed, $\mathcal{K}(A)$ coincides with the recession cone of $A$ defined by the set $\{v \in \mathbb{R}^m : A + v \subset A\}$.[21] Thus, $v \in \mathcal{K}(A)$ is a direction of recession of $A$ when $A$ is closed. If $C$ is not closed, its recession need not be closed. In this case, $\mathcal{K}(A)$ may differ from the recession cone of $A$. Clearly, $\mathcal{K}(A) \subset A$ whenever $0 \in A$ and $A$ is closed. For a convex set in $\mathbb{R}^m$ with $0 \in A$, let $\mathcal{L}(A)$ denote the set $\{v \in \mathbb{R}^m : \lambda v \in A \quad \text{for all } \lambda \in \mathbb{R}\}$. Clearly, $\mathcal{L}(A)$ is not empty and a linear subspace of $\mathbb{R}^m$. Notice that $\mathcal{L}(A)$ is the maximal subspace contained in $A$. When $A$ is closed, $\mathcal{L}(A)$ is the lineality space of $\mathcal{K}(A)$.[22]

We set $C_i = \mathcal{K}(\Theta_i)$ and $G_i = \mathcal{K}(\Phi_i)$ for each $i \in I$. By Corollary 8.3.3 of Rockafellar (1970), we have $G_i = C_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$. A point in $C_i \cap V^{\perp}$ is a constrained null-income

[21] When $A$ is obviously closed and convex, we rather prefer to call $\mathcal{K}(A)$ the recession cone of $A$. Rockafellar (1970) is a great reference to the properties of recession cone.

[22] See Rockafellar (1970).

portfolio that constitutes a direction of recession of $\Theta_i$. Let $N$ denote the set $\mathcal{L}\left[\sum_{i\in I}(C_i \cap V^{\perp})\right]$. If $N \neq \{0\}$, there exists a set of nonzero portfolios in $C_i \cap V^{\perp}$ that jointly span $N$. Let $M$ denote the orthogonal complement of $N$ in $V^{\perp}$. For each $i \in I$, let $\hat{\Theta}_i$ and $\hat{C}_i$ denote the projections of $\Theta_i$ and $C_i$ onto $V + M$, respectively. Notice that $\hat{\Theta}_i$ and $\hat{C}_i$ need not be closed. Similarly, let $\hat{\Phi}_i$ and $\hat{G}_i$ denote the projections of $\Phi_i$ and $G_i$ onto $V + M$, respectively. It is clear that $\hat{\Phi}_i = \hat{\Theta}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$ and $\hat{G}_i = \hat{C}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$. Recalling that $V^{\perp} \subset \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$, it is obvious that $\Phi_i \cap V^{\perp} = \Theta_i \cap V^{\perp}$, $G_i \cap V^{\perp} = C_i \cap V^{\perp}$ for all $i \in I$, and $\mathcal{L}\left[\sum_{i\in I}(G_i \cap V^{\perp})\right] = N$.

For each $i \in I$, let $\omega_i$ be a point in $\mathbb{R}^S$ with $\Phi_i(\omega_i) \neq \emptyset$. Since $\{\theta \in \mathbb{R}^J : R \cdot \theta + \omega_i \geq 0\}$ is the translation of the cone $\{\theta \in \mathbb{R}^J : R\cdot\theta \geq 0\}$ by $-\omega_i$, we have $\mathcal{K}\left[\{\theta \in \mathbb{R}^J : R \cdot \theta + \omega_i \geq 0\}\right] = \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$. By Corollary 8.3.3 of Rockafellar (1970), $G_i$ is the recession cone of $\Phi_i(\omega_i)$, *i.e.*, $G_i = \mathcal{K}\left[\Phi_i(\omega_i)\right]$. We set $N(\omega) = \mathcal{L}\left[\sum_{i\in I}(\mathcal{K}(\Phi_i(\omega_i)) \cap V^{\perp})\right]$ and let $M(\omega)$ denote the orthogonal complement of $N(\omega)$ in $V^{\perp}$. It follows that

$$N(\omega) = \mathcal{L}\left[\sum_{i\in I}(\mathcal{K}(\Phi_i(\omega_i)) \cap V^{\perp})\right] = \mathcal{L}\left[\sum_{i\in I}(G_i \cap V^{\perp})\right] = N.$$

These results are summarized as following.

**Lemma 2.1 :** Suppose that $\Phi_i(\omega_i) \neq \emptyset$ for a point $\omega_i \in \mathbb{R}^S$. Then the following hold for each $i \in I$.

$(i)$ $G_i = \mathcal{K}\left[\Phi_i(\omega_i)\right] = C_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$ and $\hat{G}_i = \hat{C}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$.

$(ii)$ $\Phi_i \cap V^{\perp} = \Theta_i \cap V^{\perp}$, $G_i \cap V^{\perp} = C_i \cap V^{\perp}$, and $\hat{G}_i \cap V^{\perp} = \hat{C}_i \cap V^{\perp}$.

$(iii)$ $N(\omega) = N$ and $M(\omega) = M$.

The following lemma presents a condition under which Assumption 2.3 holds.

**Lemma 2.2 :** Suppose that $\{C_i \cap V^{\perp}, i \in I\}$ is positively semi-independent.[23] Then for each $\omega_i \in \mathbb{R}^S$, $\sum_{i\in I} \Phi_i(\omega_i)$ is closed in $\mathbb{R}^J$.

PROOF : See the appendix. ■

[23] A collection $\{T_i, i \in I\}$ of nonempty convex sets in $\mathbb{R}^S$ is positively semi-independent if $v_i \in T_i$ for all $i \in I$ and $\sum_{i\in I} v_i = 0$ implies that $v_i = 0$ for all $i \in I$.

Notice that if $N = \{0\}$, then $\{C_i \cap V^\perp, i \in I\}$ is positively semi-independent. As a corollary of Lemma 2.2, Assumption 2.3 holds trivially when $C_i \cap V^\perp = \{0\}$ for each $i \in I$.[24] Assumption 2.3 is strictly general than the positive semi-independence of $\{C_i \cap V^\perp, i \in I\}$. For example, it will be shown in Example 3.2 that Assumption 2.3 is fulfilled in the constrained asset markets of Balasko, Cass and Siconolfi (1990) where each $\Theta_i$ is a linear subspace of $\mathbb{R}^J$ and $\{C_i \cap V^\perp, i \in I\}$ need not be positively semi-independent.

## III. Portfolio Decomposition

As emphasized in the Preface, it is important to identify the set of bridge portfolios in terms of arbitrage pricing and the existence of equilibrium. In this section, we characterize the properties of bridge portfolios and examine analytically the way in which a portfolio is decomposed into the bridge and value portfolios. The consequences of portfolio decomposition will provide a cornerstone for investigating an appropriate notion of arbitrage and the existence of equilibrium in constrained asset markets.

**Definition 3.1:** A null-income portfolio $\eta$ in $V^\perp$ is called a *bridge portfolio* if for all $\lambda \geq 0$, it satisfies both $\lambda\eta \in \sum_{i\in I}(\Theta_i \cap V^\perp)$ and $\lambda(-\eta) \in \sum_{i\in I}(\Theta_i \cap V^\perp)$. Let $\mathcal{N}$ denote the set of bridge portfolios.[25] The projection $\eta$ of a portfolio $\theta \in \mathbb{R}^J$ onto $\mathcal{N}$ is called the *bridge component* (or *bridge portfolio*) of $\theta$ and $\theta - \eta$ is called the *value component* (or *value portfolio*) of $\theta$.

As shown below, the bridge component of a portfolio does not create any value, but may matter to risk sharing because the value component alone need not be feasible under the portfolio constraints. It is clear that $\mathcal{N} = \mathcal{L}\left[\sum_{i\in I}(\Theta_i \cap V^\perp)\right]$, and by $(ii)$ of Lemma 2.1, $\mathcal{N} = \mathcal{L}\left[\sum_{i\in I}(\Phi_i \cap V^\perp)\right]$. Let $\mathcal{M}$ denote the orthogonal complement of $\mathcal{N}$ in $V^\perp$. As shown later, $\mathcal{N}$ is closely related to the set $N$. The following theorem shows that bridge portfolios have null value in equilibrium.

**Theorem 3.1 :** Let $q$ be an equilibrium asset price of $\mathcal{E}$. Then $q \cdot \theta = 0$ for all $\theta \in \mathcal{N}$.

[24] This condition is assumed in Siconolfi (1986).

[25] It is easy to see that $\mathcal{N}$ is a subspace of $\mathbb{R}^J$.

PROOF : See the appendix. ■

Theorem 3.1 states that bridge portfolios are always free in equilibrium regardless of the distribution of initial endowments over agents. Since elements in $\mathcal{N}$ are a null-income portfolio which provides a formula of replicating redundant assets, Theorem 3.1 amounts to a constrained version of the law of one price that holds for asset positions which constitute a portfolio in $\mathcal{N}$.[26] In this respect, bridge portfolios are a special type of null-income portfolios because other types of null-income portfolios may have nonzero value under portfolio constraints. The bridge component of optimal portfolios for agents looks trivial in terms of valuation but is complementary in risk sharing to the value component which need not be feasible in constrained asset markets.

As remarked before, bridge portfolios are unnecessary for risk sharing in unconstrained markets. In this case, there are two conceivable ways of eliminating nontrivial bridge portfolios. The first one is to remove all the redundant assets from the asset markets. In this case, the rank of the return matrix without redundant assets has full rank and therefore, the only bridge portfolio is the null portfolio. This approach will be called the *exclusion method*, which is adopted in handling redundant assets by the classical literature of incomplete markets such as Werner (1985) and Balasko and Cass (1989). The second one is to project portfolios in $\mathbb{R}^S$ onto $V$. Since $\mathcal{N} = V^\perp$ in unconstrained markets, the *projection method* allows us to remove nonzero bridge portfolios in $V^\perp$ from $\mathbb{R}^J$ and keep value portfolios in $V$. By definition, the null portfolio is the unique bridge portfolio in $V$. In unconstrained asset markets, both methods are indistinguishable in terms of income transfers.

Null-income portfolios, however, are of distinct nature in constrained markets in several respects. First, they need not be a bridge portfolio because of the 'shadow price' of the portfolio constraints. The exclusion method is not appropriate in handling constrained portfolios because, as shown below, the elimination of redundant assets may lead to sharp reduction in income transfer opportunities. The projection method is also problematic because the projection of feasible portfolios onto $V$ need not be feasible in constrained markets.[27] Nonetheless, the projection method will give an insight into the decomposition of constrained portfolios into the bridge and value components.

[26]The set $\mathcal{N}$ equals $V^\perp$ in unconstrained asset markets. In this case, Theorem 3.1 is a verification of the law of one price in equilibrium.

[27]To take an example, we consider a two-asset, two-state economy such that both assets are risk free. Suppose that the first asset pays one dollar in each state and the second asset pays two dollars in each state. In this case, $V$ is a subspace of $\mathbb{R}^2$ spanned by the vector $(1, 2)$. We assume that $\Theta_i = \{(a, b) \in \mathbb{R}^2 : a \geq -1 \text{ and } b \geq 0\}$. It is easy to see that the projection of $(-1, 0) \in \Theta_i$ onto $V$ is not in $\Theta_i$ and thus, not feasible.

**Example 3.1:** It is illustrated here that the exclusion approach can lead to drastic reduction in income spanning opportunities under short-selling restrictions. We consider two asset structures represented by the payoff matrix

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The asset structure 1 is a sub-structure of the asset structure 2 which does not contain the redundant assets 3 and 4. Clearly, both asset structures have $\mathbb{R}^2$ as the set of income transfers when they are not subject to portfolio constraints. To apply the projection method to the asset structure 2, we decompose $\mathbb{R}^4$ as the direct sum of $\mathbb{R}^4 = V + V^{\perp}$ where $V$ is the subspace of $\mathbb{R}^4$ spanned by the rows of $R_2$ and $V^{\perp}$ its orthogonal complement in $\mathbb{R}^4$. A simple linear algebra shows that the set of portfolios in $V$ generates the same set $\mathbb{R}^2$ of income transfers as $\mathbb{R}^4$.

The two ways of treating null-income portfolios have different consequences when asset markets are subject to short-selling restrictions. Consider the case where the set of feasible portfolios is equal to $\mathbb{R}^2_+$ under the asset structure 1 and to $\mathbb{R}^4_+$ under the asset structure 2. In this case, the sets of income transfers for the asset structures 1 and 2 are $\mathbb{R}^2_+ \cap \{(x, y) \in \mathbb{R}^2 : y \leq x\}$ and $\mathbb{R}^2_+$, respectively. This result illustrates that the removal of redundant assets leads to a drastic change in the set of risk-sharing opportunities under the short-selling restrictions.

Now each $\theta \in \mathbb{R}^4_+$ is uniquely decomposed as $\theta = \hat{\theta} + \tilde{\theta}$ where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^{\perp}$. The point $\hat{\theta}$ is the projection of $\theta$ onto $V$. Since $R_2 \cdot \theta = R_2 \cdot \hat{\theta}$ for all $\theta \in \mathbb{R}^4_+$, the set of income transfers is kept intact under the projection of portfolios in $\mathbb{R}^4_+$ onto $V$. In this respect, the projection method is in contrast to the exclusion method. As mentioned above, however, the projection method may fail to keep feasibility with the income-keeping portfolio $\hat{\theta} \in V$ in general. $\square$

In the rest of the section, we will search for an appropriate way of decomposing constrained portfolios into the bridge and value components by modifying the projection method in a sophisticated way. As shown later in this section, the consequences of portfolio decomposition allow us to construct an artificial economy which is free from bridge portfolios and moreover, is indistinguishable from the original economy in terms of optimal consumptions and income transfers. The properties of the artificial economy will be exploited in verifying the existence of equilibrium of the original economy.

### 3.1. CONE CONSTRAINTS

First, we consider the case that each $\Theta_i$ is a convex cone with vertex.[28] In this case, the decomposition of constrained portfolios into the bridge and value portfolios is finalized with their one-shot projection onto $V + M$. As shown later, however, the decomposition process with convex constraints is much trickier because it involves a series of deliberately-chosen multi-stage projections.

**Proposition 3.1 :** Suppose that $0 \in \Theta_i$ and $\Theta_i$ is a closed convex cone with vertex for all $i \in I$. Then the following results hold true.

($i$) $\sum_{i\in I} \Phi_i = N + \sum_{i\in I} \hat{\Phi}_i$.

($ii$) If $\sum_{i\in I} \Phi_i$ is closed in $\mathbb{R}^J$, then $\sum_{i\in I} \hat{\Phi}_i$ is closed in $\mathbb{R}^J$ and $\sum_{i\in I} c\ell(\hat{\Phi}_i) = \sum_{i\in I} \hat{\Phi}_i$.

($iii$) For each $i \in I$, let $\omega_i$ be a point in $\mathbb{R}^S$ such that $\Phi_i(\omega_i) \neq \emptyset$. If each $\hat{\Theta}_i$ is closed in $\mathbb{R}^J$, then $\mathcal{L}\left[\sum_{i\in I}(\mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M)\right] = \mathcal{L}\left[\sum_{i\in I}(\hat{G}_i \cap M)\right] = \{0\}$.

($iv$) If each $\hat{\Theta}_i$ is closed in $\mathbb{R}^J$, then $N = \mathcal{N}$.

PROOF : See the appendix. ■

The first part of Proposition 3.1 states that aggregate portfolios in $\sum_{i\in I} \Phi_i$ are orthogonally decomposed into portfolios in $N$ and $\hat{\Phi}_i$'s. If $\sum_{i\in I} \Phi_i$ is closed in $\mathbb{R}^J$, it follows from ($i$) and ($ii$) of Proposition 3.1 that $\sum_{i\in I} \Phi_i = N + \sum_{i\in I} c\ell(\hat{\Phi}_i)$. By ($iii$) of Proposition 3.1, the maximal subspace contained in $\sum_{i\in I}(\mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M)$ is the null vector when each $\hat{\Theta}_i$ is closed in $\mathbb{R}^J$ and each $\Phi_i(\omega_i)$ is not empty. The last part of Proposition 3.1 shows that if each $\hat{\Theta}_i$ is closed in $\mathbb{R}^J$, then $N$ coincides with the set $\mathcal{N}$ of bridge portfolios. As illustrated later, however, ($iii$) and ($iv$) of Proposition 3.1 are no longer true in the case that $\Theta_i$ is a non-conic convex set. We notice that if $\Phi_i(\omega_i) \neq \emptyset$ for a point $\omega_i \in \mathbb{R}^S$, by Lemma 2.1, ($i$) and ($ii$) of Proposition 3.1 also hold for the case where $\Phi_i$'s are replaced by $\Phi_i(\omega_i)$'s.

We can apply the consequences of Proposition 3.1 to Balasko, Cass and Siconolfi (1990) where portfolio constraints are expressed as a homogeneous system of linear equations. In particular, Assumption 2.3 is fulfilled in Balasko, Cass and Siconolfi (1990).

[28]For a positive integer $m$, a set $A$ in $\mathbb{R}^m$ is a cone if $\lambda v \in A$ for all $v \in A$ and all $\lambda \geq 0$. The set $A$ is a cone with vertex if $A - v$ is a cone for some $v \in \mathbb{R}^m$.

**Example 3.2 :** We consider the constrained markets of Balasko, Cass and Siconolfi (1990) where each $\Theta_i$ is represented by the set $\{\theta_i \in \mathbb{R}^J : B_i \cdot \theta_i = 0\}$ for some $m_i \times J$ matrix $B_i$ with the nonnegative integer $m_i$. For analytical convenience, we suppose that there exists $w_i \in \Theta_i$ for each $i \in I$ such that $R \cdot w_i > 0$.[29] Let $\hat{w}_i$ denote the projection of $w_i$ onto $V + M$. Then we have $R \cdot \hat{w}_i > 0$. Since $\Theta_i$ is a subspace of $\mathbb{R}^J$, we have $C_i = \Theta_i$ and $\Theta_i \cap V^\perp = \{\theta_i \in \mathbb{R}^J : R \cdot \theta_i = 0$ and $B_i \cdot \theta_i = 0\}$ for all $i \in I$, and $N = \sum_{i \in I}(\Theta_i \cap V^\perp)$.

Notice that the result of Lemma 2.2 is not applicable here because $\{C_i \cap V^\perp\}$ is not positively semi-independent in general. To check that Assumption 2.3 holds here, let $\omega_i$ be a point in $\mathbb{R}^S$ for each $i$. Since $\hat{\Theta}_i$ is a subspace in this example, it is trivially closed and therefore, $\hat{\Phi}_i(\omega_i) = \hat{\Theta}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta + \omega_i \geq 0\}$ is closed for each $i \in I$. In particular, $\hat{\Phi}_i$ is closed. It follows from Lemma 2.1 and Proposition 3.1 that

$$\sum_{i \in I} \Phi_i(\omega_i) = N + \sum_{i \in I} \hat{\Phi}_i(\omega_i). \tag{1}$$

First we show that $\sum_{i \in I} \hat{\Phi}_i(\omega_i)$ is closed. To prove the claim, we choose $\{v^n\}$ in $\sum_{i \in I} \hat{\Phi}_i(\omega_i)$ which converges to a point $v$. For each $i \in I$ and $n$, there exists $v_i^n \in \hat{\Phi}_i(\omega_i)$ such that $v^n = \sum_{i \in I} v_i^n$. Let $\hat{v}^n$ and $\tilde{v}^n$ denote the projection of $v^n$ onto $V$ and $M$, respectively, and $\hat{v}_i^n$ and $\tilde{v}_i^n$ the projection of $v_i^n$ onto $V$ and $M$, respectively. Then $v_i^n = \hat{v}_i^n + \tilde{v}_i^n$ for each $i$, and $\hat{v}^n = \sum_{i \in I} \hat{v}_i^n$ and $\tilde{v}^n = \sum_{i \in I} \tilde{v}_i^n$. Since $\{R \cdot \hat{v}^n\}$ is bounded and $R \cdot \hat{v}_i^n \geq \omega_i$ for each $i \in I$ and $n$, each $\{R \cdot \hat{v}_i^n\}$ is bounded. Noting that $\hat{v}_i^n \in V$ for all $i \in I$ and $n$, this implies that $\{\hat{v}_i^n\}$ is bounded. We claim that each $\{\tilde{v}_i^n\}$ is bounded. Suppose otherwise. Then we have $a^n \equiv \sum_{i \in I} \|\tilde{v}_i^n\| \to \infty$. Since $\{\tilde{v}_i^n / a^n\}$ is bounded, without loss of generality, we can assume that it converges to a point $\eta_i \in M$. Recalling that $\{\hat{v}_i^n\}$ is bounded, we have $v_i^n / a^n \to \eta_i$ for all $i \in I$. This implies that $\eta_i \in \mathcal{K}(\hat{\Phi}_i(\omega_i))$ and therefore, $\eta_i \in \mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M$. Since $\sum_{i \in I} \eta_i = 0$ and $\eta_i \neq 0$ for some $i \in I$, we have $\mathcal{L}\left[\sum_{i \in I}(\mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M)\right] \neq \{0\}$, which contradicts $(iii)$ of Proposition 3.1. Thus, each $\{v_i^n\}$ is bounded and has a subsequence convergent to a point $v_i$. Since $\hat{\Phi}_i(\omega_i)$ is closed, $v_i$ is in $\hat{\Phi}_i(\omega_i)$ for each $i \in I$ and therefore, $v$ is in $\sum_{i \in I} \hat{\Phi}_i(\omega_i)$. Thus, $\sum_{i \in I} \hat{\Phi}_i(\omega_i)$ is closed. On the other hand, $\sum_{i \in I} \hat{\Phi}_i(\omega_i) \subset V + M$ and $(V + M) \cap N = \{0\}$. This implies that $\sum_{i \in I} \hat{\Phi}_i(\omega_i)$ and $N$ are positively semi-independent. Thus, $N + \sum_{i \in I} \hat{\Phi}_i(\omega_i)$ is closed, and by (1), $\sum_{i \in I} \Phi_i(\omega_i)$ is closed. Consequently, Assumption 2.3 is fulfilled in this example. $\square$

Let $\hat{\mathcal{E}}$ denote the economy which is the same as $\mathcal{E}$ except that $\Theta_i$ is replaced by $\hat{\Theta}_i$ for each $i$. We claim that

[29] As shown in Proposition A1 of the appendix, this condition can be assumed without loss of generality.

$i)$ if $\mathcal{E}$ has an equilibrium $(p, q, x, \theta)$, then $(p, q, x, \hat{\theta})$ is an equilibrium of $\hat{\mathcal{E}}$ where $\hat{\theta}_i$ is the projection of $\theta_i$ onto $V + M$, and

$ii)$ if $\hat{\mathcal{E}}$ has an equilibrium $(p, q, x, \theta^*)$ with $q \in V + M$, then there exists $\theta_i \in \Theta_i$ for each $i$ such that $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$.

By Theorem 3.1 and $(iv)$ of Proposition 3.1, $q$ is in $V + M$. We decompose the portfolio $\theta_i \in \Theta_i$ as $\theta_i = \hat{\theta}_i + \tilde{\theta}_i$ where $\hat{\theta}_i \in V + M$ and $\tilde{\theta}_i \in N$. The proof of the statement i) is immediate from the result that $W(q) \cdot \theta_i = W(q) \cdot \hat{\theta}_i$.

To verify $ii)$, we have to find an optimal portfolio for each $i$ which generates the same income transfers as $\theta_i^*$ and clears asset markets. Notice that $\theta_i^* \in \hat{\Theta}_i$ need not be feasible in $\mathcal{E}$, i.e., $\theta_i^* \notin \Theta_i$ in general. By Lemma 2.1, we can apply the consequences of Proposition 3.1 to $\Phi_i(-R \cdot \theta_i^*)$'s. Since $\theta_i^* \in \hat{\Phi}_i(-R \cdot \theta_i^*)$, by $(i)$ of Proposition 3.1, there exists $\theta_i \in \Theta_i$ such that $R \cdot \theta_i \geq R \cdot \theta_i^*$ for all $i \in I$ and $\sum_{i \in I} \theta_i^* = \sum_{i \in I} \theta_i$. The condition $\sum_{i \in I} \theta_i^* = \sum_{i \in I} \theta_i$ implies that $R \cdot \theta_i = R \cdot \theta_i^*$ for all $i \in I$ and $\sum_{i \in I} \theta_i = 0$. Now show that $q \cdot \theta_i = 0$ for all $i \in I$. Let $\hat{\theta}_i$ denote the projection of $\theta_i$ onto $V + M$. Then $\hat{\theta}_i \in \hat{\Theta}_i$ for all $i \in I$. Since $R \cdot \hat{w}_i > 0$, we must have $q \cdot \hat{\theta}_i \geq 0$. Otherwise, by the monotonicity of $u_i$, agent $i$ could get better than at $x_i$ by adding a little bit of $\hat{w}_i$ to $\hat{\theta}_i$ in the economy $\hat{\mathcal{E}}$, which contradicts the optimality of $(x_i, \hat{\theta}_i)$ in the budget set. Since $\sum_{i \in I} \hat{\theta}_i = 0$, $q \cdot \hat{\theta}_i \geq 0$ for all $i \in I$ implies that $q \cdot \hat{\theta}_i = 0$ for all $i \in I$. Recalling that $q \in V + M$, we have $q \cdot \theta_i = 0$ for all $i \in I$. Thus, $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$. $\square$

In the previous example, the set $N$ of bridge portfolios is relatively easier to identify because each portfolio constraint $\Theta_i$ is a subspace of $\mathbb{R}^J$. In this example, the economies $\mathcal{E}$ and $\hat{\mathcal{E}}$ face distinct portfolio constraints but they are indistinguishable in equilibrium in terms of optimal consumption and income transfers. The consequences of Example 3.2 will be extended to general cases where each $\Theta_i$ need not be a cone.

Proposition 3.1 summarizes the results of portfolio decomposition when $\Theta_i$ is a convex cone with vertex for each $i \in I$. The following example, however, shows that Proposition 3.1 is no longer applicable to the case where $\Theta_i$'s are a non-conic convex set.

**Example 3.3 :** We consider an economy with $L = 1, I = 2$, and $S = J = 3$ where the $3 \times 3$

return matrix has the form

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The return vectors $r_1$ and $r_2$ are trivially linearly dependent on $r_3$.[30] We see that $V$ is a line spanned by $(0,0,1)$ and $V^\perp$ a plane spanned by $(1,0,0)$ and $(0,1,0)$, *i.e.*, $V$ is the $z$-axis and $V^\perp$ is the $xy$-plane in the three dimensional Euclidean space. We suppose that

$$\begin{aligned} \Theta_1 &= \left\{(a,b,c) \in \mathbb{R}^3 : a \geq b^2 + c^2,\ b \geq 0,\ c \geq 0\right\}, \\ \Theta_2 &= \left\{(a,b,c) \in \mathbb{R}^3 : -a \geq b^2 + c^2,\ b \leq 0,\ c \geq 0\right\}. \end{aligned}$$

Clearly, $C_1 = \{(a,0,0) : a \geq 0\}$ and $C_2 = \{(a,0,0) : a \leq 0\}$. Then we see that $\Phi_i = \Theta_i$ and $G_i = C_i$ for each $i \in I$.

The set $G_1 + G_2$ coincides with the $x$-axis of $\mathbb{R}^3$ and therefore,

$$N = \mathcal{L}((G_1 \cap V^\perp) + (G_2 \cap V^\perp)) = G_1 + G_2 \subset V^\perp.$$

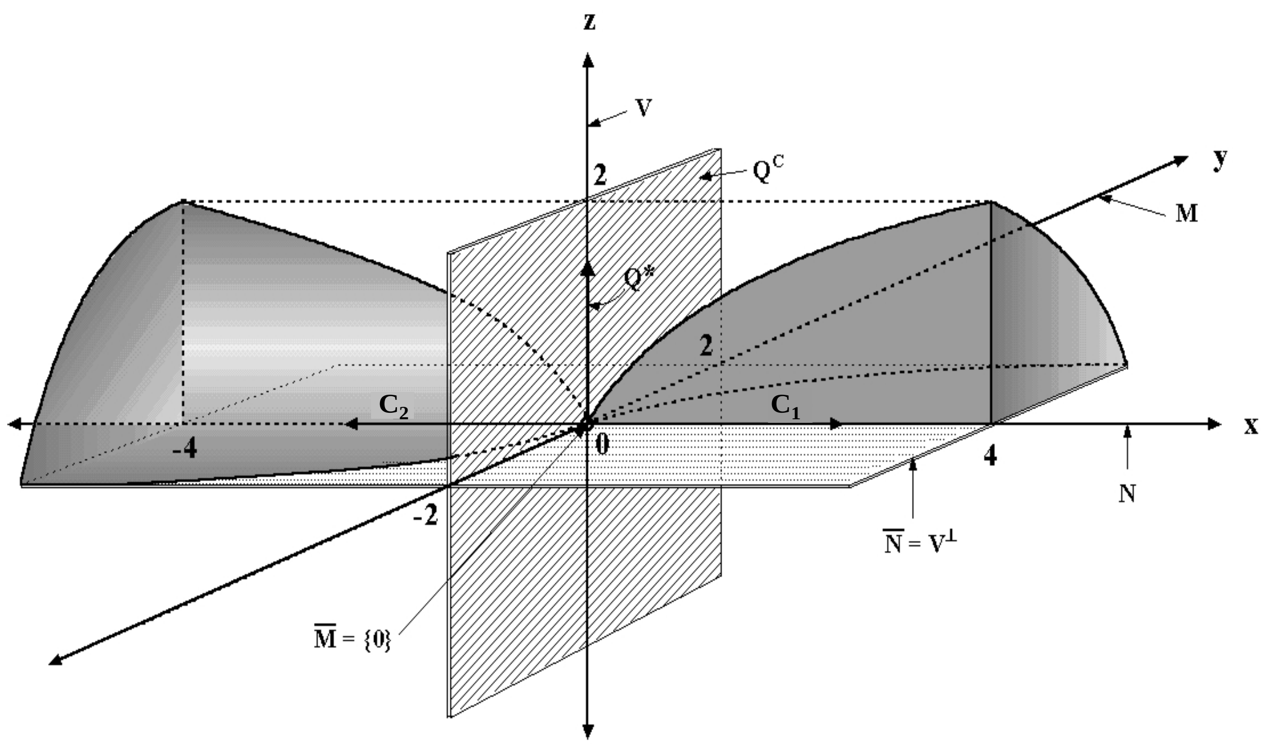
This example is depicted graphically in Figure 1.

Since $N = G_1 + G_2$ is the $x$-axis and $V^\perp$ is the $xy$-plane of $\mathbb{R}^3$, $M$ is the $y$-axis of $\mathbb{R}^3$. It follows that

$$\begin{aligned} \hat{\Theta}_1 = \hat{\Phi}_1 = \mathcal{K}(\hat{\Phi}_1) = \left\{(a,b,c) \in \mathbb{R}^3 : a = 0,\ b \geq 0,\ c \geq 0\right\}, \\ \hat{\Theta}_2 = \hat{\Phi}_2 = \mathcal{K}(\hat{\Phi}_2) = \left\{(a,b,c) \in \mathbb{R}^3 : a = 0,\ b \leq 0,\ c \geq 0\right\}. \end{aligned}$$

Clearly, $M = \mathcal{L}[(\mathcal{K}(\hat{\Phi}_1) \cap M) + (\mathcal{K}(\hat{\Phi}_2) \cap M)]$. Since $M \neq \{0\}$, this implies that $(iii)$ of Proposition 3.1 is not valid any more. As shown later, $\mathcal{N}$ is identified with $V^\perp$ in this example. Since $N \neq V^\perp$, $(iv)$ of Proposition 3.1 is violated as well. □

[30] The choice of null-return assets in $R$ is just a matter of convenience. If $R$ is multiplied by a nonsingular matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, then it is transformed into $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{bmatrix}$, which exhibits linear dependence between nonzero return vectors.




[Figure 1]

### 3.2. GENERAL CONVEX CONSTRAINTS

The previous example illustrates that it is much subtler to implement the portfolio decomposition in the case with non-conic portfolio constraints. The following is a generalization of Proposition 3.1 to the case that the convex sets $\Phi_i$'s need not be a cone.

**Proposition 3.2 :** Under Assumptions 2.2 and 2.3, there exists a pair of subspaces $\overline{M}$ and $\overline{N}$ in $\mathbb{R}^J$ with $N \subset \overline{N}$ that satisfy the following relations

$(i)$ $V^\perp = \overline{M} + \overline{N},\ \overline{M} \cap \overline{N} = \{0\}$,

$(ii)$ $\sum_{i\in I} \Phi_i = \overline{N} + \sum_{i\in I} \overline{\Phi}_i$ and $\sum_{i\in I} \overline{\Phi}_i$ is closed in $\mathbb{R}^J$,

(iii) $\sum_{i\in I} \overline{\Phi}_i = \sum_{i\in I} c\ell(\overline{\Phi}_i)$,

(iv) $\mathcal{L}\left[\sum_{i\in I}(\overline{G}_i \cap \overline{M})\right] = \{0\}$,

where each $\overline{\Phi}_i$ is the orthogonal projection of $\Phi_i$ onto $V + \overline{M}$ and $\overline{G}_i = \mathcal{K}(\overline{\Phi}_i)$, and it holds that

(v) $\sum_{i\in I}(\Phi_i \cap V^{\perp})$ is closed in $\mathbb{R}^J$,

(vi) $\overline{N} = \mathcal{N}$,

PROOF : See the appendix. ■

The first part of Proposition 3.2 states that $V^{\perp}$ is the direct sum of $\overline{M}$ and $\overline{N}$. The first part of (ii) of Proposition 3.2 shows that aggregate portfolios in $\sum_{i\in I} \Phi_i$ are orthogonally decomposed into portfolios in $N$ and $\overline{\Phi}_i$'s. The fourth part of Proposition 3.2 means that the subspace $\mathcal{L}\left[\sum_{i\in I}(\overline{G}_i \cap \overline{M})\right]$ is the null vector. The fifth part of Proposition 3.2 states that $\sum_{i\in I}(\Phi_i \cap V^{\perp})$ are closed in $\mathbb{R}^J$ while the last one shows that $\overline{N}$ coincides with the set $\mathcal{N}$ of bridge portfolios. From now on, $\overline{N}$ and $\overline{M}$ will replace $\mathcal{N}$ and $\mathcal{M}$ to indicate the set of bridge portfolios and its orthogonal complement in $V^{\perp}$, respectively. As illustrated in Example 3.3, Proposition 3.2 is a strict generalization of Proposition 3.1 to the case that $\Phi_i$'s are a convex set. The following consequences are immediate from Proposition 3.2.

**Corollary 3.1:** The following results hold true under Assumptions 2.2 and 2.3.

(i) $\sum_{i\in I} \Phi_i = \overline{N} + \sum_{i\in I} c\ell(\overline{\Phi}_i)$,

(ii) $\mathcal{L}\left[\sum_{i\in I}(c\ell(\overline{\Phi}_i) \cap \overline{M})\right] = \mathcal{L}\left[\sum_{i\in I}(\overline{G}_i \cap \overline{M})\right] = \{0\}$.

(iii) $\sum_{i\in I} c\ell(\overline{\Phi}_i)$ is closed in $\mathbb{R}^J$.

It is worth noting that if $\Phi_i(\omega_i) \neq \emptyset$ for a point $\omega_i \in \mathbb{R}^S$, by Lemma 2.1, the consequences of Proposition 3.2 and Corollary 3.1 also hold for the case where $\Phi_i$'s are replaced by $\Phi_i(\omega_i)$'s. They will play a crucial role in investigating the properties of equilibrium asset prices and the existence of equilibrium. It is illustrated below how Proposition 3.2 is fulfilled in Example 3.3.

**Example 3.4. (Continued) :** To apply the results of Proposition 3.2 to Example 3.3, let $\overline{\Phi}_i$ denote the projection of $\hat{\Phi}_i$ onto $V$, and $\overline{G}_i$ the set $\mathcal{K}(\overline{\Phi}_i)$ for each $i = 1, 2$. Then we see that for each $i = 1, 2$,

$$\overline{\Phi}_i = \overline{G}_i = \left\{(a, b, c) \in \mathbb{R}^3 : a = b = 0,\ c \geq 0\right\}.$$

We set $\overline{N} = N + M$ and $\overline{M} = \{0\}$. Since $\overline{N} = V^{\perp}$, $\overline{N}$ is the $xy$-plane of $\mathbb{R}^3$. The subspaces $\overline{M}$ and $\overline{N}$ are depicted in Figure 1. It follows that

(*i*) $V^{\perp} = \overline{N} + \overline{M},\ \overline{N} \cap \overline{M} = \{0\}$,

(*ii*) $\sum_{i \in I} \Phi_i = \overline{N} + \sum_{i \in I} \overline{\Phi}_i$,

(*iii*) $\left(\overline{G}_i \cap V^{\perp}\right) = \left(\overline{G}_i \cap V^{\perp}\right) = \{0\}$,

(*iv*) $\overline{N} = \mathcal{L}((\Phi_1 \cap V^{\perp}) + (\Phi_2 \cap V^{\perp}))$.

(*v*) $\sum_{i \in I} \overline{\Phi}_i = \left\{(a, b, c) \in \mathbb{R}^3 : a = b = 0,\ c \geq 0\right\}$ is closed in $\mathbb{R}^3$.

In this example, the one-shot projection was not enough to obtain the results of Proposition 3.1 while the two-shot projections lead to the consequences explained in Proposition 3.2. □

Let $\overline{\Theta}_i$ denote the projection of $\Theta_i$ onto $V + \overline{M}$ and $\overline{C}_i$ the set $\mathcal{K}(\overline{\Theta}_i)$ for each $i \in I$. Let $\overline{\mathcal{E}}$ denote the economy which is the same as $\mathcal{E}$ except that $\Theta_i$ is replaced by $c\ell(\overline{\Theta}_i)$ for each $i$.[31] The consequences of Proposition 3.2 play a critical role in verifying the following generalization of the statements made in Example 3.2.

**Theorem 3.2:** The following hold true under Assumptions 2.1-2.3.

*i*) If $\mathcal{E}$ has an equilibrium $(p, q, x, \theta)$ where each $\Theta_i$ possesses $\zeta_i$ with $q \cdot \zeta_i < 0$, then $(p, q, x, \overline{\theta})$ is an equilibrium of $\overline{\mathcal{E}}$ where $\overline{\theta}_i$ is the projection of $\theta_i$ onto $V + \overline{M}$.

*ii*) If $\overline{\mathcal{E}}$ has an equilibrium $(p, q, x, \theta^*)$ with $q \in V + \overline{M}$, then there exists $\theta_i \in \Theta_i$ for each $i$ such that $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$.

PROOF : See the appendix. ■

[31] The reason for taking $c\ell(\overline{\Theta}_i)$ instead of $\overline{\Theta}_i$ is that $\overline{\Theta}_i$ is not closed in general.

The consequences of Theorem 3.2 will be useful in showing the existence of equilibrium for the economy $\mathcal{E}$. It is worth noting that a portfolio $\theta_i \in c\ell(\overline{\Theta}_i)$ may not be feasible in the economy $\mathcal{E}$, i.e., $\theta_i \notin \Theta_i$ in general. That is, $\overline{\mathcal{E}}$ looks different from $\mathcal{E}$ in terms of feasible portfolios. Nonetheless, they are indistinguishable in equilibrium in terms of consumptions and income transfers.

## IV. Projective Arbitrage

In the first part of the section, we briefly review well-known notions of arbitrage and show that the no arbitrage conditions may not be compatible with viability of asset prices in constrained asset markets. Then we introduce the notion of projective arbitrage and characterize equilibrium prices in terms of projective arbitrage. It is shown that projective arbitrage has a comparative advantage over the existing notions of arbitrage in explaining the viability property of asset prices.

### 4.1. ARBITRAGE AND VIABILITY

We discuss well-known notions of arbitrage and check their consistency with the properties of equilibrium asset prices. The following is a typical form of arbitrage for unconstrained asset markets.

**Definition 4.1 :** Suppose that $\Theta_i = \mathbb{R}^J$ for all $i \in I$. Then an asset price $q \in \mathbb{R}^J$ admits *no unconstrained arbitrage* if there is no $v \in \mathbb{R}^J$ which satisfies $W(q) \cdot v > 0$.

Werner (1985) among others investigates the existence of equilibrium with incomplete markets by taking advantage of Definition 4.1.[32] The following is one of the extensions of Definition 4.1 to the constrained asset markets of the economy $\mathcal{E}$.

**Definition 4.2 :** A price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for agent $i$ in the

[32] Magill and Quinzii (1996) are a great reference on the classical notion of arbitrage.

economy $\mathcal{E}$ if there is no $v_i \in C_i$ such that $W(q) \cdot v_i > 0$. A price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for the economy $\mathcal{E}$ if it admits no constrained arbitrage for every agent $i \in I$.

The above notion of arbitrage is extensively used in the literature ( Luttmer (1996), Chen (1995), Pham and Touzi (1999), Cvitanic and Karatzas (1993), and Broadie, Cvitanic and Soner (1998) among others).

Let $Q^N$ and $Q_i^C$ denote the set of prices which do not admit an unconstrained arbitrage and constrained arbitrage for agent $i$, respectively. We set $Q^C = \bigcap_{i \in I} Q_i^C$. The set $Q^C$ denotes the set of prices which do not admit constrained arbitrage for the economy. Noting that $\Theta_i \subset \mathbb{R}^J$ for all $i \in I$, we have $Q^N \subset Q^C$. But the converse is not true in general.[33]

Let $Q^*$ denote the set of equilibrium prices for $\mathcal{E}$. Then $Q^* \subset Q^C$, *i.e.*, the no constrained arbitrage condition holds in equilibrium.[34] As shown below, however, $Q$ can be excessively large to contain equilibrium prices. To examine the relationship between equilibrium and arbitrage-free prices, we define the viability of asset prices as follows.

**Definition 4.3 :** An asset prices $q \in \mathbb{R}^J$ is said to be *viable* if $\xi_i(p, q) \neq \emptyset$ for some $p \in \mathbb{R}_+^{LS}$ and all $i \in I$.

By Harrison and and Kreps (1979), asset prices which admit no unconstrained arbitrage are viable in unconstrained markets. Thus, we are tempted to believe that each $q \in Q^C$ is viable in the constrained markets. As shown below, however, prices in $Q^C$ need not be viable. That is, $Q^C$ may overestimate the set of viable prices.

**Example 4.1 :** We consider the economy of Example 3.3 where for each $i = 1, 2$, $u_i$ satisfies Assumption 2.1. It should be noticed that the following arguments do not rely on a specific choice of either utility functions or the initial endowments of goods. Let $q = (q_1, q_2, q_3)$ denote

[33] If $\Theta_i = C_i$ and $\Theta_i$ is a strict subset of $\{v \in \mathbb{R}^J : R \cdot v > 0\}$ for all $i \in I$, then $Q^C \not\subset Q^N$.

[34] Suppose that there exists an equilibrium $(p, q, x, \theta)$ of $\mathcal{E}$ where $q$ admits a constrained arbitrage for some $i \in I$. Then there exists $v_i \in C_i$ such that $W(q) \cdot v_i > 0$. It follows that

$$\theta_i + v_i \in \Theta_i, \text{ and } W(q) \cdot (\theta_i + v_i) > W(q) \cdot \theta_i.$$

For the price system $(p, q)$, this implies that $(x_i, \theta_i)$ cannot be an optimal choice of agent $i$ in the economy $\mathcal{E}$, which leads to a contradiction.

an asset price. It is clear that $Q_1^C = \{q \in \mathbb{R}^3 : q_1 \geq 0\}$ and $Q_2^C = \{q \in \mathbb{R}^3 : q_1 \leq 0\}$ and therefore, $Q^C = Q_1^C \cap Q_2^C = \{q \in \mathbb{R}^3 : q_1 = 0\}$.

We claim that $Q^* = \{(q_1, q_2, q_3) \in \mathbb{R}^3 : q_1 = q_2 = 0,\ q_3 > 0\}$ or each $q \in Q^C \backslash Q^*$ is not viable. Suppose that $q^* = (q_1^*, q_2^*, q_3^*)$ and $\theta_i = (a_i, b_i, c_i)$ are an equilibrium asset price and an optimal portfolio for each $i = 1, 2$, respectively. We claim that $q^* \in Q^*$. Since $q^* \in Q^C$, $q_1^* = 0$. We have only to show that $q_2^* = 0$ and $q_3^* > 0$.

First, to show that $q_2^* = 0$, consider the case with $q_2^* > 0$. Since $\theta_2 \in \Theta_2$, we have that $-(a_2 - 1) > (b_2)^2 + (c_2)^2$. Thus there exists a number $\varepsilon > 0$ such that $-(a_2 - 1) > (b_2 - \varepsilon)^2 + (c_2)^2$ and therefore,

$$(a_2 - 1, b_2 - \varepsilon, c_2) = (a_2, b_2, c_2) + (1, -\varepsilon, 0) \in \Theta_2.$$

Since $q^* \cdot (a_2 - 1, b_2 - \varepsilon, c_2) = q^* \cdot \theta_2 - \varepsilon q_2^* < q^* \cdot \theta_2$ and $(-1, -\varepsilon, 0) \in V^\perp$, we must have $W(q^*) \cdot (a_2 - 1, b_2 - \varepsilon, c_2) > W(q^*) \cdot \theta_2$ which contradicts the optimality of the choices of agent 2 in equilibrium. In the case with $q_2^* < 0$, we can obtain a contradiction by applying the same argument for agent 1. Thus any $q^* \in Q^C$ with $q_2^* \neq 0$ does not allow an agent to find optimal portfolios, which is impossible. Therefore we have $q_2^* = 0$.

Now, to show $q_3^* > 0$, suppose that $q_3^* \leq 0$. Since $\theta_1 \in \Theta_1$, we have that $a_1 + 1 > (b_1)^2 + (c_1)^2$. Thus there exists a number $\varepsilon > 0$ such that $a_1 + 1 > (b_1)^2 + (c_1 + \varepsilon)^2$ and therefore,

$$(a_1 + 1, b_1, c_1 + \varepsilon) = (a_1, b_1, c_1) + (1, 0, \varepsilon) \in \Theta_1.$$

Since $q^* \cdot (a_1, b_1, c_1 + \varepsilon) = q^* \cdot \theta_1 + \varepsilon q_3^* \leq q^* \cdot \theta_1$ and $R \cdot (a_1 + 1, b_1, c_1 + \varepsilon) = R \cdot \theta_1 + R \cdot (1, 0, \varepsilon) > R \cdot \theta_1$, we must have $W(q^*) \cdot (a_1, b_1, c_1 + \varepsilon) > W(q^*) \cdot \theta_1$ which contradicts the optimality of the choices of agent 1 in equilibrium. Therefore, we conclude that any equilibrium prices must be in $Q^*$ or for each $q \in Q^C \backslash Q^*$, there exists $i \in \{1, 2\}$ which cannot find optimal portfolios. $\square$

The above example shows that the notion of constrained arbitrage fails to characterize the set of viable prices.

### 4.2. PROJECTIVE ARBITRAGE

We provide the notion of projective arbitrage for $\mathcal{E}$ which turns out to give a more precise description of equilibrium conditions for $\mathcal{E}$.

**Definition 4.4 :** A price vector $q \in V + \overline{M}$ admits *no projective arbitrage for agent* $i$ in the economy $\mathcal{E}$ if there is no $\theta_i \in \Theta_i$ such that $\bar{\theta}_i \in \overline{C}_i$ and $W(q) \cdot \theta_i > 0$, where $\bar{\theta}_i$ is the projection of $\theta_i$ onto $V + \overline{M}$. A price vector $q \in V + \overline{M}$ admits *no projective arbitrage for the economy* $\mathcal{E}$ if it admits no projective arbitrage for every agent $i \in I$.

No projective arbitrage prices are in $V + \overline{M}$. This means that bridge portfolios must have null value at no projective arbitrage prices. If $\overline{N} = \{0\}$, then both constrained and projective arbitrages coincide. Their equivalence is also true when $I$ consists of a single agent.[35] As shown below, however, they differ in general. Let $Q_i$ denote the set of no projective arbitrage prices for agent $i$. We set $Q = \bigcap_{i \in I} Q_i$. Then $Q$ denotes the set of prices which admit no projective arbitrage for the economy $\mathcal{E}$. The following results show that equilibrium prices satisfy the no projective arbitrage condition.

**Proposition 4.2 :** Let $q$ be an equilibrium asset price of $\mathcal{E}$. Then the following hold true.

$(i)$ If there exists $\zeta_i \in \Theta_i$ for each $i \in I$ such that $q \cdot \zeta_i < 0$, then $q$ is in $Q$.

$(ii)$ If $\overline{\Theta}_i$ is closed for each $i \in I$, then $q$ is in $Q$.

PROOF : See the appendix. ■

The following result shows the relationship between constrained and projective arbitrages.

**Proposition 4.3 :** Then the following hold under Assumption 2.2.

$(i)$ Suppose that for each $i \in I$, there exists $w_i \in C_i$ such that $R \cdot w_i > 0$. Then for all $i \in I$, we have $Q_i = \left\{ q \in V + \overline{M} : q \cdot \theta_i > 0, \forall \theta_i \in \Theta_i \;\; s.t. \;\; \overline{\theta}_i \in \overline{C}_i \;\&\; R \cdot \theta_i > 0 \right\}$ where $\overline{\theta}_i$ denotes the projection of $\theta_i$ onto $V + \overline{M}$.

$(ii)$ $Q \subset Q^C$.

$(iii)$ Suppose that each $\Theta_i$ is a closed convex cone with vertex and $\hat{C}_i$ is closed in $\mathbb{R}^J$. Then we have $Q = Q^C$.

[35] To show this, let $\Theta$ denote the portfolio constraint imposed on the single agent and $C$ the recession cone of $\Theta$. Then we have $\overline{N} = N = C \cap V^{\perp}$, $\overline{M} = M$, and $\overline{\Theta} = \hat{\Theta} = \Theta \cap (V + M)$. By Corollary 8.3.3 of Rockafellar (1970), we obtain $\overline{C} = C \cap (V + M) = \hat{C}$. It follows that for a point $\theta \in C$, $W(q) \cdot \theta > 0$ if and only if $W(q) \cdot \overline{\theta} > 0$ and $\overline{\theta} \in \overline{C}$ where $\overline{\theta}$ is the projection of $\theta$ onto $V + \overline{M}$. Thus $q \in Q^C$ if and only if $q$ admits no projective arbitrage.

PROOF : See the appendix. ■

The first part of Proposition 4.2 provides a slightly different characterization of the projective arbitrage-free prices, and the second part shows that $Q$ is a subset of $Q^C$. The third part of Proposition 4.2 states that as far as portfolio constraints are expressed as a closed convex cone with vertex, the no constrained arbitrage condition is equivalent to the no projective arbitrage condition. As illustrated below, projective arbitrage may differ from constrained arbitrage in multi-agent markets and moreover, is more relevant to characterizing the viability property of asset prices than constrained arbitrage in general when $\Theta_i$ is a non-conic convex set for some $i \in I$.

**Example 4.2 :** We consider the economy of Examples 3.3 and 4.1. It is illustrated that $Q^*$ is much smaller than $Q^C$. Recalling from Example 3.3 that for each $i = 1, 2$,

$$\overline{\Theta}_i = \overline{C}_i = \left\{(a, b, c) \in \mathbb{R}^3 : a = b = 0,\ c \geq 0\right\},$$

we have $Q = \left\{(q_1, q_2, q_3) \in \mathbb{R}^3 : q_1 = q_2 = 0,\ q_3 > 0\right\}$ and therefore, $Q = Q^*$. As shown in Example 4.1, any $q \in Q^C \backslash Q$ is not viable. □

The above example shows that projective arbitrage exactly characterizes the viability property of asset prices while constrained arbitrage may not.

## V. The Existence of Equilibrium

If agents are endowed with a positive amount of some commodities in each state of the second period, they always survive in unconstrained asset markets.[36] This is not the case, however, with constrained asset markets. As illustrated below, equilibrium may fail to exist in constrained asset markets when each agent has the endowment of commodities in the interior of the consumption set. We provide a survival condition with constrained asset markets which is indispensable for the existence of equilibrium. An important corollary is that the asset-market

[36]Gottardi and Hens (1996) treat exhaustively the survival problem with unconstrained asset markets.

survival condition is unnecessary for the existence of equilibrium in markets where agents are endowed with a positive amount of some commodities in the beginning period.[37]

Another difficulty arises with the Cass trick. The method of Cass (1984) does not directly apply to the case where no agent is allowed to behave in equilibrium as if in complete markets. It should be noted that such an ideal agent is only an instrument to verify the boundary behavior of excess demand functions, as in complete markets. It is demonstrated in the appendix that if asset markets are subject to portfolio constraints, then goods and asset markets are jointly responsible for the boundary behavior of the excess demand correspondence. In particular, survival conditions with asset markets are needed to guarantee not only the continuity of demand correspondences but also the desired boundary behavior of excess demand in constrained markets. We impose the following condition on $\mathcal{E}$.

**Assumption 5.1 :** For each $q \in Q$, there is $\theta_i \in \Theta_i$ for all $i \in I$ such that $q \cdot \theta_i < 0$.

Assumption 5.1 requires that agents be able to have positive income in the first period through asset markets at each $q \in Q$. It is worth noting that the condition is imposed on prices in $Q$ alone but not on prices in the set $c\ell(Q)\backslash Q$. A special remark is in order.

**Remark 5.1 :** There are two types of survival condition involved in verifying the existence of equilibrium in the economy $\mathcal{E}$ in general. One is for goods markets such as ($ii$) of Assumption 2.1, and the other is for asset markets such as Assumption 5.1. A good pair of survival conditions will reveal tension between the two conditions in that if one condition gets stronger, the other condition can get loose. Example 5.1 below illustrates that the combination of ($ii$) of Assumption 2.1 and Assumption 5.1 are qualified as a good pairing. □

We are ready to provide the main theorem of the paper.

**Theorem 5.1 :** Under Assumptions 2.1, 2.2, 2.3, and 5.1, there exists an equilibrium for the economy $\mathcal{E}$.

[37] As shown below, the survival conditions with asset markets introduced in Siconolfi (1986), and Cass, Siconolfi and Villanacci (2001) are unnecessary for the existence of equilibrium.

PROOF : See the appendix. ■

We provide an example where the absence of equilibrium is attributed to the failure of the economy to fulfill Assumption 5.1. It is worth noting that the following example goes beyond the asset market framework of Siconolfi (1986) which assumes that $C_i \cap V^\perp = \{0\}$ for each $i \in I$.

**Example 5.1 :** To illustrate that Assumption 5.1 cannot be dispensed with, we consider a two-state two-agent economy where two Arrow-Debreu securities and one redundant assets are traded and only one good is consumed in each state. The good is also used as a numeraire. We assume that the payoff matrix is given by the $2 \times 3$ matrix

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then we have $V^\perp = \{v \in \mathbb{R}^3 : \lambda(1, -1, 1) \text{ for some } \lambda \in \mathbb{R}\}$. Both agents have the same endowment of goods and distinct preferences.

$$\begin{aligned} u_1(x) &= 2\sqrt{x(1)+1} + \sqrt{x(2)}, \qquad e_1 = (1,1), \\ u_2(x) &= \sqrt{x(1)+1} + 2\sqrt{x(2)}, \qquad e_2 = (1,1). \end{aligned}$$

We assume that for each $i = 1, 2$,

$$\Theta_i = \{(a, b, c) \in \mathbb{R}^3 : a + b \geq 0, c \geq 0\}.$$

Clearly, $C_i = \Theta_i$ for all $i = 1, 2$, and $N = \{0\}$. Thus, the economy satisfies Assumptions 2.1, 2.2, and 2.3. It is worth noting that $C_i \cap V^\perp \neq \{0\}$ for each $i \in I$.

Consider an asset price $q = (1, 1, 0)$. We claim that $q \in Q$, *i.e.*, $q$ is a no projection arbitrage price. Let $\theta = (a, b, c)$ be any portfolio in $C_i$ such that $R \cdot \theta > 0$. Then we have $a + b > 0$. Since $q \cdot \theta = a + b > 0$, it holds that $q \in Q$. Now we show that the current economy fails to fulfill Assumption 5.1. Let $\theta = (a, b, c)$ be a point in $\Theta_i$. In particular, we have $a + b \geq 0$. Since $q \cdot \theta = a + b \geq 0$, Assumption 5.1 does not hold for $q$.

Now we show that the economy has no equilibrium. We set $\hat{u}_1(a, b, c) = 2\sqrt{a - c + 2} + \sqrt{b + c + 1}$ and $\hat{u}_2(a, b, c) = \sqrt{a - c + 2} + 2\sqrt{b + c + 1}$, and $\mathcal{A}_i(q) = \{(a, b, c) \in \mathbb{R}^3 : q_1 a + q_2 b + q_3 c \leq 0,\ (a, b, c) \in \Theta_i\}$ for each $i = 1, 2$. The function $\hat{u}_i$ is a reduced-form utility function defined over feasible portfolios and $\mathcal{A}_i(q)$ is the budget set for agent 1 at the price $q$. Then the utility maximization problem for agent $i = 1, 2$ is reduced to the following relations.

$$\max_{(a,b,c)\in\mathcal{A}_i(q)} \hat{u}_i(a, b, c).$$

Suppose that there exists an equilibrium $\{(q_1, q_2, q_3), (a_1, b_1, c_1), (a_2, b_2, c_2)\}$. The no arbitrage condition implies that $q_1 > 0$ and $q_2 > 0$. Since $a_i + b_i \geq 0$ and $c_i \geq 0$ for each $i = 1, 2$, by the market clearing condition we have $a_i + b_i = 0$ and $c_i = 0$. In particular, the first order conditions for utility maximization at equilibrium choices yield the following relations

$$\begin{aligned}
&\lambda_1(q_1 - q_2) + \frac{1}{\sqrt{2 + a_1}} - \frac{1}{2\sqrt{1 - a_1}} = 0, \quad \lambda_1 > 0,\\
&\lambda_2(q_1 - q_2) + \frac{1}{2\sqrt{2 - a_1}} - \frac{1}{\sqrt{1 + a_1}} = 0, \quad \lambda_2 > 0,\\
&(q_1 - q_2)a_1 = 0,
\end{aligned}$$

where the last equation is derived from the budget constraint of each agent, and $\lambda_1$ and $\lambda_2$ are the Lagrangian multipliers for the budget constraint of agent 1 and 2, respectively.

The relation $(q_1-q_2)a_1 = 0$ implies that $q_1 = q_2$ or $a_1 = 0$. If $q_1 = q_2$, then the first and second equations are not compatible. Suppose that $a_1 = 0$. Then $\lambda_1(q_1 - q_2) > 0$ and $\lambda_2(q_1 - q_2) < 0$, which is impossible. Therefore, we conclude that the economy has no equilibrium.

**Remark 5.2 :** The results of Theorem 5.1 can be applied to show the existence of equilibrium in the economy where agents are allowed to consume and have a positive endowment of goods in the beginning period. Let $\mathcal{E}'$ be the economy with the same asset market characteristics as $\mathcal{E}$ where consumption is allowed to arise in the first period. More precisely, the consumption space of $\mathcal{E}'$ is augmented by adding $\mathbb{R}^L$ to $\mathbb{R}^{SL}$. It is assumed that agent $i$ has the consumption set $X_i' := \mathbb{R}^L_+ \times X_i$ and the endowment of goods $(e_i(0), e_i)$ where $e_i(0) \in \mathbb{R}^L_+ \setminus \{0\}$ for each $i \in I$. Let $\nu_i$ denote a utility function in $\mathbb{R}^L_+ \times X_i$ for agent $i$ which is continuous and strictly increasing. Then for a given price pair $(p, q) \in \mathbb{R}^{L(S+1)}_+ \times \mathbb{R}^J$, agent $i$ is supposed to choose $(\overline{y}_i, \overline{\theta}_i)$ which maximizes $\nu_i$ in the budget set

$$\begin{aligned}
&p(0) \cdot (y_i(0) - e_i(0)) + q \cdot \theta_i \leq 0,\\
&p(s) \cdot (y_i(s) - e_i(s)) \leq r(s) \cdot \theta_i, \ \forall\, s \in S,\\
&\theta_i \in \Theta_i, \ y_i \in X_i'.
\end{aligned}$$

The economy $\mathcal{E}'$ is summarized as the profile $\langle (X_i', \Theta_i, \nu_i, (e_i(0), e_i))_{i\in I}, R\rangle$. We assume that it satisfies the following conditions.

**Assumption 2.1$'$ :** The following hold true.

$(i)$ Each $v_i : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$ is continuous, strictly increasing, and quasiconcave.

$(ii)$ $e_i(s) > 0$ for each $i$ and $s = 0, 1, \ldots, S$, and $\sum_{i\in I}(e_i(0), e_i) \gg 0$.

Assumption 2.1$'$ is the $\mathcal{E}'$ version of Assumption 2.1 for the economy $\mathcal{E}$. Now we show that $\mathcal{E}'$ can be transformed into the economy of the same type as $\mathcal{E}$ where no consumption arises in the beginning period. To do this, we add state $0$ to the second period and introduce asset $0$ which pays one unit of money in state $0$ and nothing in the other states of the second period. Thus, the new asset structure is described by the $(S+1)\times(J+1)$ return matrix $\tilde{R}$ :

$$\tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}.$$

Let $q_0$ and $\eta_i(0)$ denote a price and amount of asset $0$ held by agent $i$, respectively. We assume that no short-selling restriction is imposed on asset $0$. For each $y_i(0)$ and $p(0)$, we set $\eta_i(0) = p(0)\cdot(y_i(0) - e_i(0))$ and $\tilde{q} = (q_0, q)$. Then the budget set for agent $i$ is transformed as

$$\begin{aligned}
&(q_0, q)\cdot(\eta(0)_i, \theta_i) \le 0,\\
&p(0)\cdot(y_i(0) - e_i(0)) \le \eta_i(0) = (1, 0)\cdot(\eta_i(0), \theta_i),\\
&p(s)\cdot(y_i(s) - e_i(s)) \le r(s)\cdot\theta_i = (0, r(s))\cdot(\eta_i(0), \theta_i),\ \forall\, s\in S,\\
&(\eta_i(0), \theta_i) \in \tilde{\Theta}_i = \mathbb{R}\times\Theta_i,\ y_i \in X_i'
\end{aligned}$$

The second inequality can be considered as the budget set which faces agent $i$ in the state $0$ of the second period. Let $\mathcal{E}'' = \langle (X_i', \tilde{\Theta}_i, \nu_i, (e_i(0), e_i))_{i\in I}, \tilde{R}\rangle$ denote the transformed economy where no consumption arises in the initial period and agent $i$ has the portfolio constraint set $\tilde{\Theta}_i$. Let Assumptions 2.2$''$, 2.3$''$ and 5.1$''$ denote the $\mathcal{E}''$ version of Assumptions 2.2, 2.3 and 5.1, respectively. It is straightforward to see that if $\mathcal{E}'$ satisfies Assumptions 2.2 and 2.3, then $\mathcal{E}''$ satisfies Assumptions 2.2$''$ and 2.3$''$.

Now we show that Assumption 5.1$''$ is always fulfilled in $\mathcal{E}''$. Let $\tilde{q} = (q_0, q)$ be a price which admits no projective arbitrage for $\mathcal{E}''$. Then we must have $q_0 > 0$ because $\nu_i$ is strictly increasing and asset 0 delivers positive income transfer. Since $(-1, 0) \in \mathbb{R} \times \Theta_i$ and $(q_0, q) \cdot (-1, 0) = -q_0 < 0$ for all $q \in \mathbb{R}^J$, $\mathcal{E}''$ satisfies Assumption 5.1$''$. Thus, if $\mathcal{E}''$ satisfies Assumptions 2.1$'$, 2.2$''$ and 2.3$''$, by Theorem 5.1 it has an equilibrium. Therefore, there exists an equilibrium of the economy $\mathcal{E}'$ if it satisfies Assumptions 2.1$'$, 2.2 and 2.3. This fact implies that the asset-market survival condition is unnecessary for the existence of equilibrium of the economy $\mathcal{E}'$. □

The existence of equilibrium for the economy $\mathcal{E}'$ is immediate from Theorem 5.1 and Remark 5.2.

**Corollary 5.1 :** Under Assumptions 2.1$'$, 2.2 and 2.3, there exists an equilibrium of the economy $\mathcal{E}'$.

In particular, Corollary 5.1 can be applied to Siconolfi (1986) which examines the existence of equilibrium of $\mathcal{E}'$. Siconolfi (1986) assumes that (GMS) $(e_i(0), e_i) \gg 0$ for all $i \in I$, and

**(AMS)** for every $j \in J$, there exist $\theta^+ \in \Theta_{i_j^+}$ and $\theta^- \in \Theta_{i_j^-}$ for some $i_j^+$ and $i_j^-$ in $I$ such that $pr_k(\theta^+) > 0$ if $k = j$ and $pr_k(\theta^+) = 0$ if $k \neq j$, and $pr_k(\theta^-) < 0$ if $k = j$ and $pr_k(\theta^-) = 0$ if $k \neq j$, where $pr_k(v)$ for some $v \in \mathbb{R}^J$ denotes the $k$th coordinate of $v$.

It is easy to check that (AMS) implies that (AMS$'$) $0 \in \text{int}(\sum_{i \in I} \Theta_i')$. As remarked above, (AMS$'$) is unnecessary for the existence of equilibrium of $\mathcal{E}'$ as far as $\mathcal{E}'$ satisfies (GMS).

## VI. Concluding Remarks

We have shown the existence of equilibrium in the asset markets where portfolio constraints are expressed as a convex set. The consequence of the paper is a substantial extension of the literature of equilibrium theory on constrained asset markets. A main difficulty with constrained asset markets is the presence of an unbounded amount of bridge portfolios in $\overline{N}$ which matter to risk sharing in general and have information on no arbitrage prices. The presence of nonzero bridge portfolios invalidates the closedness condition on either the individual portfolio constraint set or the marketed set of income transfers as a sufficient condition for the existence of optimal portfolios or equilibrium. To address the puzzling problem with bridge portfolios, we relies on the new condition that the aggregate set of feasible portfolios which yield nonnegative income in all future states for each agent be closed.

To analyze the effect of constrained null-income portfolios on equilibrium prices, we have developed the portfolio decomposition technique which leads to Propositions 3.1 and 3.2. These propositions provide a basis for identifying the set $\overline{N}$ of bridge portfolios which are free in equilibrium, and enable us to get the notion of projective arbitrage for the economy $\mathcal{E}$. As shown in Example 4.2, projective arbitrage gives a precise description of equilibrium prices but constrained arbitrage may not. Thus, projective arbitrage seems to be more relevant to studying arbitrage pricing theory in the case where asset markets are subject to non-conic convex portfolio constraints.

Portfolio constraints also raise a survival problem with asset markets. Specifically, Assumption 5.1 is introduced as a survival condition with constrained markets. As illustrated in Example 5.1, the asset-market survival condition is quite tight when it is combined with $(ii)$ of Assumption 2.1 which imposes a survival condition on goods markets.

This paper can be extended in some ways. A challenging issue is to investigate the existence of equilibrium with constrained markets in a multi-period economy by fully accounting for the effect of bridge portfolios on equilibrium.[38] It will be also interesting to study the implications of projective arbitrage to arbitrage pricing in constrained markets where $\Theta_i$'s are a convex subset of an infinite-dimensional portfolio space. Another interesting theme is to extend the consequence of the paper to the case that portfolio constraints are endogenously determined.

[38]Equilibrium may not exist even in unconstrained multi-period markets because the prices of long-lived assets are involved in spanning state-contingent incomes. See Magill and Quinzii (1996).

## APPENDIX

PROOF OF LEMMA 2.2 : Let $\omega_i$ be a point in $\mathbb{R}^S$ for each $i \in I$. If $\Phi_i(\omega_i)$ is empty for some $i \in I$, then $\sum_{i\in I} \Phi_i(\omega_i)$ is empty and thus, trivially closed. Thus, without loss of generality, we can assume that $\Phi_i(\omega_i) \neq \emptyset$ for each $i \in I$. Let $\{\theta^n\}$ be a sequence in $\sum_{i\in I} \Phi_i(\omega_i)$ which converges to $\theta$. For each $n$, we pick $\theta_i^n$ for each $n$ and $i \in I$ such that $\theta^n = \sum_{i\in I} \theta_i^n$ and $\theta_i^n \in \Phi_i$.

We claim that $\{\theta_i^n\}$ is bounded for all $i \in I$. Otherwise, $\sum_{i\in I} \|\theta_i^n\| \to \infty$. We set $a^n = 1/\sum_{i\in I} \|\theta_i^n\|$ for all $n$. By multiplying both sides of $\sum_{i\in I} \theta_i^n = \theta^n$ by $a^n$, we obtain $\sum_{i\in I} a^n\theta_i^n = a^n\theta^n$. Since $\{a^n\theta_i^n\}$ is bounded for each $i \in I$, it has a subsequence which converges to a point $\dot{\theta}_i$. It is clear that $\dot{\theta}_i \in C_i$ for each $i \in I$, $\sum_{i\in I} \dot{\theta}_i = 0$, and $\sum_{i\in I} \|\dot{\theta}_i\| = 1$. On the other hand, we know that $R \cdot \theta_i^n + \omega_i \geq 0$ for each $n$. By multiplying both sides of $R \cdot \theta_i^n + \omega_i \geq 0$ by $a^n$, we obtain $R \cdot (a^n\theta_i^n) + a^n\omega_i \geq 0$ for all $n$ and $i \in I$. By passing to the limit, we have $R \cdot \dot{\theta}_i \geq 0$ for all $i \in I$. Recalling that $\sum_{i\in I} \dot{\theta}_i = 0$, this implies that $R \cdot \dot{\theta}_i = 0$ and therefore, $\dot{\theta}_i \in V^\perp$ for all $i \in I$. It follows that $\dot{\theta}_i \in C_i \cap V^\perp$ for all $i \in I$ and $\sum_{i\in I} \dot{\theta}_i = 0$ where $\dot{\theta}_i \neq 0$ for some $i \in I$. This contradicts the positive semi-independence of $\{C_i \cap V^\perp, i \in I\}$. Thus, $\{\theta_i^n\}$ is bounded for all $i \in I$.

Since $\{\theta_i^n\}$ is bounded for all $i \in I$, it has a bounded subsequence which converges to a point $\theta_i$. Since $\Phi_i$ is closed for each $i \in I$, we have $\theta_i \in \Phi_i(\omega_i)$ for all $i \in I$ and therefore, $\theta = \sum_{i\in I} \theta_i \in \sum_{i\in I} \Phi_i(\omega_i)$. Thus, we conclude that $\sum_{i\in I} \Phi_i(\omega_i)$ is closed. ■

To facilitate subsequent analysis, we will take advantage of the following condition.

**PV :** For each $i \in I$, there exists $w_i \in C_i$ which satisfies $R \cdot w_i \gg 0$.

This condition turns out to be very convenient in characterizing equilibrium prices as well as arbitrage-free prices but can be assumed without loss of generality. To show that the condition PV is innocuous, we introduce an augmented asset structure by adding a fictitious asset 0 with the return $r_0 = (r_{s0})_{s\in S}$ to the original asset structure $J$. We assume that asset 0 is riskless, *i.e.*, $r_{s0} = 1$ for all $s \in S$. We set $J^a = \{0\} \cup J$. Let $R^a = [r_0, r_1, \ldots, r_J]$ denote the $S \times (J+1)$ return matrix and $q_0$ denote a price of the riskless asset. We define the set $\Theta_i^a = \mathbb{R}_+ \times \Theta_i$ for each $i \in I$. Let $C_i^a$ denote the recession cone of $\Theta_i^a$ for each $i \in I$. Then we have $C_i^a = \mathbb{R}_+ \times C_i$. Let $\mathcal{E}^a$ denote the economy which is the same as $\mathcal{E}$ except that $J$ and $\Theta_i$ are replaced by $J^a$ and $\Theta_i^a$ for each $i \in I$, respectively. Then the set $\Theta_i^a$ can be considered the portfolio constraint on agent $i$ in the economy $\mathcal{E}^a$. The following lemma shows that $\mathcal{E}^a$ satisfies the $\mathcal{E}^a$ version of the condition PV, and if $\mathcal{E}^a$ has an equilibrium, then $\mathcal{E}$ has an equilibrium which is indistinguishable from that of $\mathcal{E}^a$ in terms of income transfers and consumption.

**Proposition A1 :** The following hold true.

($i$) For each $i \in I$, there exists a nonzero portfolio $w_i^a \in C_i^a$ such that $R^a \cdot w_i^a \gg 0$.

($ii$) If $(p, (q_0, q), x, (\theta_{i0}, \theta_i)_{i\in I})$ is an equilibrium of $\mathcal{E}^a$, then $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$.

PROOF : ($i$) For each $i \in I$, let $w_i^a$ denote the vector in $\mathbb{R}^{J+1}$ where the first element is 1 and the other elements are zero. Clearly, $w_i^a \in C_i^a$ and $R^a \cdot w_i^a \gg 0$ for all $i \in I$. $\square$

($ii$) Suppose that $(p, (q_0, q), x, (\theta_{i0}, \theta_i)_{i\in I})$ is an equilibrium of $\mathcal{E}^a$. Since $\theta_{i0}^a \geq 0$ for all $i \in I$ and $\sum_{i\in I} \theta_{i0} = 0$, we must have $\theta_{i0} = 0$ for all $i \in I$. Thus, $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$. ■

The original economy $\mathcal{E}$ can be transformed into $\mathcal{E}^a$ which satisfies ($i$) of Proposition A1. By ($ii$) of Proposition A1, $\mathcal{E}$ will have equilibrium if $\mathcal{E}^a$ does. Thus, without loss of generality, we can assume that $\mathcal{E}$ satisfies the condition PV in investigating the existence of equilibrium of $\mathcal{E}$.

PROOF OF THEOREM 3.1 : Let $(p, q, x, \theta)$ be an equilibrium of $\mathcal{E}$. Suppose that there exists $v \in \mathcal{N}$ such that $q \cdot v \neq 0$. Since $-v \in \mathcal{N}$, without loss of generality we may assume that $q \cdot v < 0$. For each $\lambda > 0$ and $i \in I$, we pick $v_i(\lambda) \in \Theta_i \cap V^\perp$ such that $\lambda^2 v = \sum_{i\in I} v_i(\lambda)$. Since $\lim_{\lambda\to\infty} \lambda q \cdot v = \lim_{\lambda\to\infty} \sum_{i\in I} q \cdot [v_i(\lambda)/\lambda] = -\infty$ and $\lambda^2 v \in \mathcal{N}$ for all $\lambda > 0$, there exists $i \in I$ such that $\lim_{\lambda\to\infty} q \cdot [v_i(\lambda)/\lambda] = -\infty$. Without loss of generality, we will assume that $i = 1$.

On the other hand, by the condition PV, we have $w_1 \in C_1$ with $R \cdot w_1 > 0$. Then there exists $\delta > 0$ in $\mathbb{R}^{SL}$ such that $u_1(x_1 + \delta) > u_1(x_1)$ and $p \,\square\, (x_1 + \delta - e_1) < R \cdot (\theta_1 + w_1)$. By the continuity of $u_1$, there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$,

$$u_1\left((1 - 1/\lambda)(x_1 + \delta)\right) > u_1(x_1).$$

Clearly, for all $\lambda > 0$ we have $p \,\square\, [(1 - 1/\lambda)(x_1 + \delta) - e_1] < R \cdot [(1 - 1/\lambda)(\theta_1 + w_1)]$. Since $\lim_{\lambda\to\infty} q \cdot [v_1(\lambda)/\lambda] = -\infty$, there exists $\lambda_2 > 0$ such that $q \cdot [(1 - 1/\lambda)(\theta_1 + w_1) + v_1(\lambda)/\lambda] < 0$ for all $\lambda \geq \lambda_2$. Recalling that $v_1(\lambda) \in V^\perp$, we have $R \cdot [(1 - 1/\lambda)(\theta_1 + w_1) + v_1(\lambda)/\lambda] = R \cdot [(1 - 1/\lambda)(\theta_1 + w_1)]$. It follows that for all $\lambda > \max\{\lambda_1, \lambda_2\}$,

($a$) $[(1-1/\lambda)(x_1+\delta), (1-1/\lambda)(\theta_1+w_1)+v_1(\lambda)/\lambda] \in \mathcal{B}_1(p,q)$, and

($b$) $u_1[(1-1/\lambda)(x_1+\delta)] > u_1(x_1)$.

This contradicts the optimality of $(x_1, \theta_1)$ in $\mathcal{B}_1(p,q)$. Thus, we conclude that $q \cdot v = 0$ for all $v \in \mathcal{N}$. ■

PROOF OF PROPOSITION 3.1 : ($i$) First show that $\sum_{i\in I}\Phi_i \subset N + \sum_{i\in I}\hat{\Phi}_i$. Let $\theta_i$ be a point in $\Phi_i$ for each $i \in I$. We have the decomposition $\theta_i = \hat{\theta}_i + \eta_i$ where $\hat{\theta}_i \in V + M$ and $\eta_i \in N$. Then we see

$$\sum_{i\in I}\theta_i = \sum_{i\in I}\eta_i + \sum_{i\in I}\hat{\theta}_i \in N + \sum_{i\in I}\hat{\Phi}_i.$$

Conversely, let $\theta$ be a point in $N + \sum_{i\in I}\hat{\Phi}_i$. Then there exist $\eta \in N$ and $\hat{\theta}_i \in \hat{\Phi}_i$ for each $i \in I$ such that $\theta = \eta + \sum_{i\in I}\hat{\theta}_i$. We choose $\eta_i \in N$ such that $\hat{\theta}_i + \eta_i \in \Phi_i$. On the other hand, we have $C_i \cap V^\perp \subset \Theta_i \cap V^\perp \subset \Phi_i$ for all $i \in I$. This implies that $N \subset \mathcal{L}(\sum_{i\in I}\Phi_i)$ and therefore, $N + \sum_{i\in I}\Phi_i \subset \sum_{i\in I}\Phi_i$. It follows that

$$\theta = \eta + \sum_{i\in I}\hat{\theta}_i = \left(\eta - \sum_{i\in I}\eta_i\right) + \sum_{i\in I}(\hat{\theta}_i + \eta_i) \in N + \sum_{i\in I}\Phi_i \subset \sum_{i\in I}\Phi_i.$$

It implies that $N + \sum_{i\in I}\hat{\Phi}_i \subset \sum_{i\in I}\Phi_i$ and therefore, $N + \sum_{i\in I}\hat{\Phi}_i = \sum_{i\in I}\Phi_i$. □ □

($ii$) Suppose that $\sum_{i\in I}\Phi_i$ is closed. Let $\{v^n\}$ be a sequence in $\sum_{i\in I}\hat{\Phi}_i$ which converges to a point $v \in V + M$. Since $0 \in N$, the result of ($i$) implies that $\{v^n\}$ is in $\sum_{i\in I}\Phi_i$. Since $\sum_{i\in I}\Phi_i$ is closed, this implies that $v$ is in $\sum_{i\in I}\Phi_i$ and therefore, in $N + \sum_{i\in I}\hat{\Phi}_i$. Since $(V+M)\cap N = \{0\}$, it follows that $v$ is in $\sum_{i\in I}\hat{\Phi}_i$. Thus, the set $\sum_{i\in I}\hat{\Phi}_i$ is closed.

Clearly, $\sum_{i\in I}\hat{\Phi}_i \subset \sum_{i\in I} c\ell(\hat{\Phi}_i)$. Let $\theta$ be a point in $\sum_{i\in I} c\ell(\hat{\Phi}_i)$. Then there exists $\theta_i \in c\ell(\hat{\Phi}_i)$ for each $i \in I$ such that $\theta = \sum_{i\in I}\theta_i$. We pick $\{\theta_i^n\}$ in $\hat{\Phi}_i$ which converges to $\theta_i$. Since each $\sum_{i\in I}\theta_i^n$ is in the closed set $\sum_{i\in I}\hat{\Phi}_i$, its limit $\theta = \sum_{i\in I}\theta_i$ is in $\sum_{i\in I}\hat{\Phi}_i$ as well. Thus, we conclude that $\sum_{i\in I} c\ell(\hat{\Phi}_i) \subset \sum_{i\in I}\hat{\Phi}_i$ and therefore, $\sum_{i\in I}\hat{\Phi}_i = \sum_{i\in I} c\ell(\hat{\Phi}_i)$. □

($iii$) For each $i \in I$, let $\psi_i$ denote the vertex of $\Theta_i$. Since $C_i$ is the recession cone of $\Theta_i$, we have $\Theta_i = C_i + \psi_i$. The point $\psi_i$ has the decomposition $\psi_i = \hat{\psi}_i + \tilde{\psi}_i$ where $\hat{\psi}_i \in V + M$ and $\tilde{\psi}_i \in N$. Clearly, $\hat{\Theta}_i = \hat{C}_i + \hat{\psi}_i$ and $\hat{\Phi}_i = (\hat{C}_i + \hat{\psi}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$ for each $i \in I$. Since $\hat{\Theta}_i$ is closed, $\hat{C}_i$ is closed. For each $i \in I$, let $\omega_i$ be a point in $\mathbb{R}^S$ with $\Phi_i(\omega_i) \neq \emptyset$. It follows by Corollary 8.3.3 of Rockafellar (1970) that for each $i \in I$, $G_i = \mathcal{K}[(C_i + \psi_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}] = C_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$ and $\mathcal{K}(\hat{\Phi}_i(\omega_i)) = \mathcal{K}[(\hat{C}_i + \hat{\psi}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq -\omega_i\}] =$

$\hat{C}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$. Since $\hat{G}_i = \hat{C}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta \geq 0\}$, it holds that $\mathcal{K}(\hat{\Phi}_i(\omega_i)) = \hat{G}_i$ for each $i \in I$.

To show that $\mathcal{L}\left[\sum_{i \in I}(\mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M)\right] = \{0\}$, we choose $v \in \mathcal{L}\left[\sum_{i \in I}(\mathcal{K}(\hat{\Phi}_i) \cap M)\right]$. By the previous result, $v$ is in $\mathcal{L}\left[\sum_{i \in I}(\hat{G}_i \cap M)\right]$. For each $i \in I$, we pick $v_i$ and $v_i'$ in $\hat{G}_i \cap M$ such that $v = \sum_{i \in I} v_i$ and $-v = \sum_{i \in I} v_i'$. Since $v_i \in \hat{G}_i$ and $v_i' \in \hat{G}_i$, there exists $\eta_i$ and $\eta_i'$ in $N$ for all $i \in I$ such that $v_i + \eta_i \in G_i$ and $v_i' + \eta_i' \in G_i$. Before going further, we verify the following result.

**CLAIM :** For each $i \in I$, let $A_i$ be a nonempty convex set in $\mathbb{R}^J$ and $\mathfrak{L}$ the lineality space of $\sum_{i \in I} \mathcal{K}(A_i)$. Then we have

$$\mathfrak{L} = \sum_{i \in I} \left(\mathcal{K}(A_i) \cap \mathfrak{L}\right).$$

PROOF OF THE CLAIM: It is clear that $\sum_{i \in I}(\mathcal{K}(A_i) \cap \mathfrak{L}) \subset \mathfrak{L}$. We have only to show the converse. Let $v$ be a point in $\mathfrak{L}$. Pick $v_i \in \mathcal{K}(A_i)$ such that $v = \sum_{i \in I} v_i$. We introduce a set

$$H = \{x \in \mathbb{R}^J : x = \alpha_0(-v) + \sum_{i \in I} \alpha_i v_i \text{ for some } \alpha = (\alpha_0, \ldots, \alpha_I) \in \mathbb{R}_+^{I+1}\}.$$

We claim that $H$ is a subspace of $\mathbb{R}^J$. Let $x$ and $y$ be points in $H$. Then there exist $\alpha$ and $\beta$ in $\mathbb{R}_+^{I+1}$ such that

$$x = \alpha_0(-v) + \sum_{i \in I} \alpha_i v_i, \quad y = \beta_0(-v) + \sum_{i \in I} \beta_i v_i.$$

For any real numbers $a$ and $b$, we set

$$c = \max\{|a\alpha_i + b\beta_i| : i = 0, \ldots, I\}.$$

Then it is clear that $c + a\alpha_i + b\beta_i \geq 0$ for all $i = 0, \ldots, I$. Since $-v + \sum_{i \in I} v_i = 0$, it follows that and

$$ax + by = (c + a\alpha_0 + b\beta_0)(-v) + \sum_{i \in I}(c + a\alpha_i + b\beta_i)v_i.$$

This implies that $ax + by \in H$ for any real numbers $a$ and $b$ and therefore, $H$ is a subspace of $\mathbb{R}^J$.

Since $-v \in \mathfrak{L}$ and $v_i \in \mathcal{K}(A_i)$ for all $i \in I$, the set $H$ is in a convex cone $\sum_{i \in I} \mathcal{K}(A_i) + \mathfrak{L}$. On the other hand, the cone $\sum_{i \in I} \mathcal{K}(A_i) + \mathfrak{L}$ has the lineality space $\mathfrak{L}$. These results imply $H \subset \mathfrak{L}$. In particular, we see each $v_i \in \mathfrak{L}$ and therefore, $v_i \in \mathcal{K}(A_i) \cap \mathfrak{L}$ and $v \in \sum_{i \in I}(\mathcal{K}(A_i) \cap \mathfrak{L})$. ||

It follows from the above claim that

$$N = \sum_{i \in I} \Big[ (G_i \cap V^\perp) \cap N \Big] = \sum_{i \in I} (G_i \cap N).$$

Since $-\sum_{i \in I}(\eta_i + \eta_i') \in N$, we can choose $\tilde{\eta}_i \in G_i \cap N$ such that $\sum_{i \in I} \tilde{\eta}_i = -\sum_{i \in I}(\eta_i + \eta_i')$. It follows that for all $i \in I$,

$$v_i + \eta_i \in G_i \cap V^\perp, \quad v_i' + \eta_i' + \tilde{\eta}_i \in G_i \cap V^\perp,$$

and

$$\sum_{i \in I}(v_i + v_i' + \eta_i + \eta_i' + \tilde{\eta}_i) = \sum_{i \in I}(v_i + v_i') = v - v = 0.$$

The last relation implies that

$$-\sum_{i \in I}(v_i + \eta_i) = \sum_{i \in I}(v_i' + \eta_i' + \tilde{\eta}_i) \in \sum_{i \in I}(G_i \cap V^\perp).$$

Thus, we see that $\sum_{i \in I}(v_i + \eta_i) \in \sum_{i \in I}(G_i \cap V^\perp)$ and $-\sum_{i \in I}(v_i + \eta_i) \in \sum_{i \in I}(G_i \cap V^\perp)$. Noting that $N$ is the lineality space of $\sum_{i \in I}(G_i \cap V^\perp)$, we have $\sum_{i \in I}(v_i + \eta_i) \in N$, and therefore, $\sum_{i \in I} v_i \in N$. Since $v_i \in M$ for all $i \in I$, This implies that $\sum_{i \in I} v_i \in M \cap N$ and therefore, $v = \sum_{i \in I} v_i = 0$. Thus, we conclude that $\mathcal{L}\Big[\sum_{i \in I}(\mathcal{K}(\hat{\Phi}_i(\omega_i)) \cap M)\Big] = \mathcal{L}\Big[\sum_{i \in I}(\hat{G}_i \cap M)\Big] = \{0\}$. $\square$

($iv$) Since $C_i \cap V^\perp \subset \Theta_i \cap V^\perp$ for each $i$, by definition, $N = \mathcal{L}[\sum_{i \in I}(C_i \cap V^\perp)] \subset \mathcal{N} = \mathcal{L}[\sum_{i \in I}(\Theta_i \cap V^\perp)]$. To show the converse, we choose a point $v \in \mathcal{L}[\sum_{i \in I}(\Theta_i \cap V^\perp)]$. Then for each $n$, there exists $\theta_i(n) \in \Theta_i \cap V^\perp$ such that $nv = \sum_{i \in I} \theta_i(n)$. Since $\sum_{i \in I}(\Theta_i \cap V^\perp) \subset V^\perp$, we have the orthogonal decompositions $v = \hat{v} + \tilde{v}$ and $\theta_i(n) = \hat{\theta}_i(n) + \tilde{\theta}_i(n)$ where $\hat{v}$ and $\hat{\theta}_i(n)$ are in $M$, and $\tilde{v}$ and $\tilde{\theta}_i(n)$ are in $N$. It follows that $\hat{\theta}_i(n) \in \hat{\Phi}_i$ for each $i$ and $n\hat{v} = \sum_{i \in I} \hat{\theta}_i(n)$. Similarly there exists $\theta_i(-n) \in \Theta_i \cap V^\perp$ such that $-nv = \sum_{i \in I} \theta_i(-n)$. Let $\hat{\theta}_i(-n)$ denote the projection of $\theta_i(-n)$ onto $M$ for each $i$. It follows that $\hat{\theta}_i(-n) \in \hat{\Phi}_i$ for each $i$ and $-n\hat{v} = \sum_{i \in I} \hat{\theta}_i(-n)$. Then we have $\sum_{i \in I}(\hat{\theta}_i(n) + \hat{\theta}_i(-n)) = 0$.

We claim that $\hat{v} = 0$. Otherwise, $\sum_{i \in I}(\|\hat{\theta}_i(n)\| + \|\hat{\theta}_i(-n)\|) \to \infty$. For each $n$, we set $a_n = 1/\sum_{i \in I}(\|\hat{\theta}_i(n)\| + \|\hat{\theta}_i(-n)\|)$. Since $\{a_n\hat{\theta}_i(n)\}$ and $\{a_n\hat{\theta}_i(-n)\}$ is bounded for each $i$, they have a subsequence convergent to $v_i^+$ and $v_i^-$ in $M$, respectively. It follows that $v_i^+$ and $v_i^-$ are in $\mathcal{K}(\hat{\Phi}_i \cap M)$ for each $i$, $\sum_{i \in I}(v_i^+ + v_i^-) = 0$, and $\sum_{i \in I}(\|v_i^+\| + \|v_i^-\|) = 1$. Since $\mathcal{K}(\hat{\Phi}_i \cap M) \subset \mathcal{K}(\hat{\Phi}_i) \cap M$, both $v_i^+$ and $v_i^-$ are in $\mathcal{K}(\hat{\Phi}_i) \cap M$. These results imply that $\mathcal{L}[\sum_{i \in I}(\mathcal{K}(\hat{\Phi}_i) \cap M)] \neq \{0\}$, which contradicts ($iii$). Thus, we conclude that $v = \tilde{v} \in N$ and therefore, $\mathcal{L}[\sum_{i \in I}(\Theta_i \cap V^\perp)] \subset N$.

■

PROOF OF PROPOSITION 3.2 : Since $(i)$, $(ii)$ and $(iii)$ are interrelated to each other, they will be verified at the same time. To do this, we conduct a series of orthogonal projections with $\Phi_i$ in the following way. For notational consistence, we set $M^0 = M$, $N^0 = N$ and $\Phi_i^0 = \Phi_i$ and $G_i^0 = G_i$ for all $i$. The space $\mathbb{R}^J$ is expressed as the direct sum

$$\mathbb{R}^J = V \oplus N^0 \oplus M^0.$$

In the first stage of decomposition, let $\Phi_i^1$ denote the orthogonal projection of $\Phi_i$ onto $V + M^0$, $G_i^1$ the asymptotic cone $\mathcal{K}(\Phi_i^1)$, $N^1$ the subspace $\mathcal{L}[\sum_{i\in I}(G_i^1 \cap M^0)]$, and $M^1$ the orthogonal complement of $N^1$ in $M^0$. By the same arguments made in the proof of $(i)$ and $(ii)$ of Proposition 3.1, it holds that $\sum_{i\in I}\Phi_i = N^0 + \sum_{i\in I}\Phi_i^1 = N^0 + \sum_{i\in I} c\ell(\Phi_i^1)$, and $\sum_{i\in I}\Phi_i^1$ and $\sum_{i\in I} c\ell(\Phi_i^1)$ are closed.

In the 2nd stage, let $\Phi_i^2$ be the orthogonal projection of $c\ell(\Phi_i^1)$ onto $V + M^1$, $G_i^2$ the asymptotic cone $\mathcal{K}(\Phi_i^2)$, $N^2$ the subspace $\mathcal{L}[\sum_{i\in I}(G_i^2 \cap M^1)]$, and $M^2$ the orthogonal complement of $N^2$ in $M^1$. Since $\sum_{i\in I} c\ell(\Phi_i^1)$ is closed, it follows from the same arguments made in the proof of $(i)$ and $(ii)$ of Proposition 3.1 that $\sum_{i\in I} c\ell(\Phi_i^1) = N^1 + \sum_{i\in I}\Phi_i^2$, $\sum_{i\in I}\Phi_i^2$ is closed, and $\sum_{i\in I}\Phi_i^2 = \sum_{i\in I} c\ell(\Phi_i^2)$. In particular, we have

$$\begin{aligned}\sum_{i\in I}\Phi_i &= N^0 + \sum_{i\in I} c\ell(\Phi_i^1)\\ &= N^0 + N^1 + \sum_{i\in I} c\ell(\Phi_i^2).\end{aligned}$$

In the $t$-th stage ($t \geq 1$), let $\Phi_i^t$ be the orthogonal projection of $c\ell(\Phi_i^{t-1})$ onto $V + M^{t-1}$, $G_i^t$ the asymptotic cone $\mathcal{K}(\Phi_i^t)$, $N^t$ the subspace $\mathcal{L}[\sum_{i\in I}(G_i^t \cap M^{t-1})]$, and $M^t$ the orthogonal complement of $N^t$ in $M^{t-1}$. Since $\sum_{i\in I} c\ell(\Phi_i^{t-1})$ is closed, again by the same arguments made in the proof of $(i)$ and $(ii)$ of Proposition 3.1, we see that $\sum_{i\in I} c\ell(\Phi_i^{t-1}) = N^{t-1} + \sum_{i\in I}\Phi_i^t$, $\sum_{i\in I}\Phi_i^t$ is closed, and $\sum_{i\in I}\Phi_i^t = \sum_{i\in I} c\ell(\Phi_i^t)$. It follows that

$$\begin{aligned}\sum_{i\in I}\Phi_i &= N^0 + \sum_{i\in I} c\ell(\Phi_i^1)\\ &\;\;\vdots\\ &= N^0 + N^1 + \ldots + N^{t-2} + \sum_{i\in I} c\ell(\Phi_i^{t-1})\\ &= N^0 + N^1 + \ldots + N^{t-1} + \sum_{i\in I} c\ell(\Phi_i^t).\end{aligned}$$

Moreover, $\mathbb{R}^J$ is orthogonally decomposed as

$$\mathbb{R}^J = V \oplus N^0 \oplus \cdots \oplus N^{t-1} \oplus M^{t-1}.$$

Suppose $N^t \neq \{0\}$. Then we go to the $(t+1)$-th round of decomposition. The process will stop at some integer $T$ with $N^T = \{0\}$. In fact, $T$ must satisfy $1 \leq T \leq J-$ dim$(V)$, because the dimension of $M^t$ strictly decreases as the decomposition process goes on. We set $N_t = N^0 \oplus \cdots \oplus N^t$ for each $t \geq 0$, $\overline{N} = N_{T-1}$, $\overline{M} = M^{T-1}$, and $\overline{\Phi}_i = \Phi_i^T$ and $\overline{G}_i = G_i^T$ for each $i \in I$. It follows that

$$\overline{G}_i = \mathcal{K}(\overline{\Phi}_i),\ V^\perp = \overline{M} + \overline{N},\ \overline{M} \cap \overline{N} = \{0\},\ \mathbb{R}^J = V \oplus \overline{M} \oplus \overline{N},$$

and moreover, $\sum_{i\in I} \Phi_i = \overline{N} + \sum_{i\in I} \overline{\Phi}_i$, $\sum_{i\in I} \overline{\Phi}_i$ is closed, and $\sum_{i\in I} \overline{\Phi}_i = \sum_{i\in I} c\ell(\overline{\Phi}_i)$. □

$(iv)$ The result is immediate from the fact that $N^T = \{0\}$. □

$(v)$ Let $v$ be a point in $c\ell\left(\sum_{i\in I}(\Phi_i \cap V^\perp)\right)$. Since $\sum_{i\in I} \Phi_i$ is closed, $v$ is in $\sum_{i\in I} \Phi_i$. Then there exists $w_i \in \Phi_i$ for each $i \in I$ such that $v = \sum_{i\in I} w_i$. Recalling that $R \cdot v = 0$ and $R \cdot w_i \geq 0$, we have $R \cdot w_i = 0$ and thus, $w_i \in V^\perp$ for all $i \in I$. Consequently, we have $v = \sum_{i\in I} w_i = \sum_{i\in I}(\Phi_i \cap V^\perp)$. Therefore, we conclude that $\sum_{i\in I}(\Phi_i \cap V^\perp)$ is closed. □

$(vi)$ First, we show that $\mathcal{L}[\sum_{i\in I}(\Phi_i \cap V^\perp)] \subset \overline{N}$. To do this, we choose a point $v \in \mathcal{L}[\sum_{i\in I}(\Theta_i \cap V^\perp)]$. Then for each $n$, there exists $\theta_i(n) \in \Theta_i \cap V^\perp$ such that $nv = \sum_{i\in I} \theta_i(n)$. Since $\sum_{i\in I}(\Theta_i \cap V^\perp) \subset V^\perp$, we have the orthogonal decompositions $v = \overline{v} + \tilde{v}$ and $\theta_i(n) = \overline{\theta}_i(n) + \tilde{\theta}_i(n)$ where $\overline{v}$ and $\overline{\theta}_i(n)$ are in $\overline{M}$, and $\tilde{v}$ and $\tilde{\theta}_i(n)$ are in $\overline{N}$. It follows that $\overline{\theta}_i(n) \in \overline{\Phi}_i$ for each $i$ and $n\overline{v} = \sum_{i\in I} \overline{\theta}_i(n)$. Similarly there exists $\theta_i(-n) \in \Theta_i \cap V^\perp$ such that $-nv = \sum_{i\in I} \theta_i(-n)$. Let $\overline{\theta}_i(-n)$ denote the projection of $\theta_i(-n)$ onto $\overline{M}$ for each $i$. It follows that $\overline{\theta}_i(-n) \in \overline{\Phi}_i$ for each $i$ and $-n\overline{v} = \sum_{i\in I} \overline{\theta}_i(-n)$. Then we have $\sum_{i\in I}(\overline{\theta}_i(n) + \overline{\theta}_i(-n)) = 0$.

We claim that $\overline{v} = 0$. Otherwise, $\sum_{i\in I}(\|\overline{\theta}_i(n)\| + \|\overline{\theta}_i(-n)\|) \to \infty$. For each $n$, we set $a_n = 1/\sum_{i\in I}(\|\overline{\theta}_i(n)\| + \|\overline{\theta}_i(-n)\|)$. Since $\{a_n\overline{\theta}_i(n)\}$ and $\{a_n\overline{\theta}_i(-n)\}$ is bounded for each $i$, they have a subsequence convergent to $v_i^+$ and $v_i^-$ in $\overline{M}$, respectively. It follows that $v_i^+$ and $v_i^-$ are in $\mathcal{K}(\overline{\Phi}_i \cap \overline{M})$ for each $i$, $\sum_{i\in I}(v_i^+ + v_i^-) = 0$, and $\sum_{i\in I}(\|v_i^+\| + \|v_i^-\|) = 1$. Since $\mathcal{K}(\overline{\Phi}_i \cap \overline{M}) \subset \mathcal{K}(\overline{\Phi}_i) \cap \overline{M} = \overline{G}_i \cap \overline{M}$. Thus, $v_i^+$ and $v_i^-$ are in $\overline{G}_i \cap \overline{M}$ for each $i$. These results imply that $\mathcal{L}[\sum_{i\in I}(\overline{G}_i \cap \overline{M})] \neq \{0\}$, which contradicts $(iv)$. Thus, we conclude that $v = \tilde{v} \in \overline{N}$ and therefore, $\mathcal{L}[\sum_{i\in I}(\Theta_i \cap V^\perp)] \subset \overline{N}$.

To show the converse, let $v \in \overline{N}$. By the first result of $(i)$ of Proposition 3.2, $v$ is in $\sum_{i\in I} \Phi_i$. Thus, there exists $w_i \in \Phi_i$ for each $i \in I$ such that $v = \sum_{i\in I} w_i$. In particular, each $w_i \in \Phi_i$

satisfies $R \cdot w_i \geq 0$. Since $R \cdot v = 0$, this implies that $R \cdot w_i = 0$ or $w_i \in V^\perp$. Thus, $w_i \in \Phi_i \cap V^\perp = \Theta_i \cap V^\perp$ for each $i \in I$ and therefore, $v \in \sum_{i \in I}(\Theta_i \cap V^\perp)$. By applying the same arguments to $\lambda v$ for each $\lambda \in \mathbb{R}$, we can show that $\lambda v \in \sum_{i \in I}(\Theta_i \cap V^\perp)$ for all $\lambda \in \mathbb{R}$. This implies that $v \in \mathcal{L}[\sum_{i \in I}(\Theta_i \cap V^\perp)]$ and therefore, $\overline{N} \subset \mathcal{L}[\sum_{i \in I}(\Theta_i \cap V^\perp)]$. ■

PROOF OF THEOREM 3.2 : Let $\mathcal{B}_i'(p,q)$ and $\xi_i'(p,q)$ indicate the budget constraint and demand correspondence for the economy $\overline{\mathcal{E}}$

$i)$ By Theorem 3.1, $q$ is in $V + \overline{M}$ and by the monotonicity of $u_i$, $p \gg 0$. We decompose the portfolio $\theta_i \in \Theta_i$ as $\theta_i = \overline{\theta}_i + \tilde{\theta}_i$ where $\overline{\theta}_i \in V + \overline{M}$ and $\tilde{\theta}_i \in \overline{N}$. It is clear that $\overline{\theta}_i \in \overline{\Theta}_i$, $\sum_{i \in I} \overline{\theta}_i = 0$, and $W(q) \cdot \theta_i = W(q) \cdot \overline{\theta}_i$ for each $i \in I$. We claim that $(x_i, \overline{\theta}_i) \in \xi_i'(p,q)$ for each $i \in I$. Suppose otherwise. Then there exists $v_i \in c\ell(\overline{\Theta}_i)$ and $y_i \in X_i$ for some $i \in I$ such that $(y_i, v_i) \in \mathcal{B}_i'(p,q)$ and $u_i(y_i) > u_i(x_i)$. Since $p \gg 0$, there exists $\alpha < 1$ such that $u_i(\alpha y_i) > u_i(x_i)$ and $p \,\square\, (\alpha y_i - e_i) \ll R \cdot (\alpha v_i)$. Recalling that $v_i \in c\ell(\overline{\Theta}_i)$, we can pick $\{v_i^n\}$ in $\overline{\Theta}_i$ such that $v_i^n \to v_i$. Then for sufficiently large $n$, we have $p \,\square\, (\alpha y_i - e_i) \ll R \cdot (\alpha v_i^n)$. We choose $\eta_i^n \in \overline{N}$ for each $n$ such that $v_i^n + \eta_i^n \in \Theta_i$. Since $q \in V + \overline{M}$, we have $q \cdot \eta_i^n = 0$ for each $n$ and thus, $q \cdot (v_i^n + \eta_i^n) \to q \cdot v_i = 0$. Let $\zeta_i$ be a point in $\Theta_i$ with $q \cdot \zeta_i < 0$. Then for sufficiently large $n$, there exists $\beta \in (0,1)$ such that $\beta\alpha(v_i^n + \eta_i^n) + (1-\beta)\zeta_i \in \Theta_i$, $q \cdot [\beta\alpha(v_i^n + \eta_i^n) + (1-\beta)\zeta_i] < 0$, and $p \,\square\, (\alpha y_i - e_i) \ll R \cdot [\beta\alpha(v_i^n + \eta_i^n) + (1-\beta)\zeta_i]$. In short, $u_i(\alpha y_i) > u_i(x_i)$ and for sufficiently large $n$, $(\alpha y_i, \beta\alpha(v_i^n + \eta_i^n) + (1-\beta)\zeta_i) \in \mathcal{B}_i(p,q)$ which contradicts the fact that $(x_i, \theta_i) \in \xi_i(p,q)$. Thus, $(p, q, x, \overline{\theta})$ is an equilibrium of the economy $\overline{\mathcal{E}}$. □

$ii)$ The results of the theorem will be verified by applying Corollary 3.1 to find an optimal portfolio for each $i$ which generates the same income transfers as $\theta_i^*$ and clears asset markets. By Proposition A1, without loss of generality, we can assume the condition PV for the economy $\mathcal{E}$. Then we can pick $w_i \in C_i$ which satisfies $R \cdot w_i \gg 0$. Let $\overline{w}_i$ denote the projection of $w_i$ onto $V + \overline{M}$. Then we have $R \cdot \overline{w}_i \gg 0$ and $\overline{w}_i \in \overline{C}_i$. Thus, for any $v_i \in \overline{\Theta}_i$, we have $v_i + \overline{w}_i \in c\ell(\overline{\Theta}_i)$ and $R \cdot (v_i + \overline{w}_i) \gg R \cdot v_i$. This implies that there exists $y_i \in ri(\overline{\Theta}_i)$ sufficiently close to $v_i + \overline{w}_i$ such that $R \cdot y_i \gg R \cdot v_i$. Thus, we see that $ri(\overline{\Theta}_i) \cap ri(\{v' \in \mathbb{R}^J : R \cdot v' - R \cdot v_i \geq 0\}) \neq \emptyset$. It follows from Theorem 6.5 of Rockafellar that

$$c\ell(\overline{\Phi}_i(-R \cdot v_i)) = c\ell(\overline{\Theta}_i) \cap \{v' \in \mathbb{R}^J : R \cdot v' - R \cdot v_i \geq 0\}. \tag{2}$$

Since $\theta_i^* \in c\ell(\overline{\Theta}_i)$, $\theta_i^*$ is in $c\ell(\overline{\Theta}_i) \cap \{v' \in \mathbb{R}^J : R \cdot v' - R \cdot \theta_i^* \geq 0\}$. Thus, by (2) we have $\theta_i^* \in c\ell(\overline{\Phi}(-R \cdot \theta_i^*))$. By Lemma 2.1, we can apply the consequences of Corollary 3.1 to $\Phi_i(-R \cdot \theta_i^*)$'s.

Since $\theta_i^* \in c\ell(\overline{\Phi}(-R \cdot \theta_i^*))$ and $\sum_{i \in I} \theta_i^* = 0$, we see that

$$0 \in \overline{N} + \sum_{i \in I} c\ell(\overline{\Phi}_i(-R \cdot \theta_i^*)).$$

By $(i)$ of Corollary 3.1, this implies that $0 \in \sum_{i \in I} \Phi_i(-R \cdot \theta_i^*)$. Thus, there exists $\theta_i \in \Theta_i$ such that $R \cdot \theta_i \geq R \cdot \theta_i^*$ for all $i \in I$ and $\sum_{i \in I} \theta_i = 0$. These results also imply that $R \cdot \theta_i = R \cdot \theta_i^*$ for all $i \in I$.

Now we show that $q \cdot \theta_i = 0$ for all $i \in I$. Let $\overline{\theta}_i$ denote the projection of $\theta_i$ onto $V + \overline{M}$ for each $i$. Then $\overline{\theta}_i \in \overline{\Theta}_i$ for all $i \in I$. We claim that $q \cdot \overline{\theta}_i \geq 0$ for all $i \in I$. Suppose otherwise. Then there exists $i \in I$ such that $q \cdot \overline{\theta}_i < 0$.

Let $c$ be a positive number such that $q \cdot \overline{\theta}_i + q \cdot (c\overline{w}_i) < 0$. Since $R \cdot \overline{w}_i > 0$, we pick $s^* \in S$ with $r(s^*) \cdot (c\overline{w}_i) > 0$. Then there exists a consumption $\tau \in \mathbb{R}^L_{++}$ such that $p(s^*) \cdot \tau < r(s^*) \cdot (c\overline{w}_i)$. Choose $\varepsilon \in \mathbb{R}^{SL}_+$ such that $\varepsilon(s) = \tau$ if $s = s^*$ and $\varepsilon(s) = 0$ if $s \in S \backslash \{s^*\}$. Clearly, $u_i(x_i + \varepsilon) > u_i(x_i)$ and $p \,\square\, \varepsilon < R \cdot (c\overline{w}_i)$. Then there exists $\bar{\alpha} \in (0, 1)$ which satisfies $u_i(\alpha x_i + \varepsilon) > u_i(x_i)$ and $q \cdot (\alpha\overline{\theta}_i + c\overline{w}_i) < 0$ for all $\alpha \in [\bar{\alpha}, 1)$. Since $e_i(s) > 0$ for all $s \in S$ and the strict monotonicity of $u_i$ implies $p \gg 0$, we can choose $\alpha \in (\bar{\alpha}, 1)$ such that

$$\begin{aligned} 0 &< -q \cdot (\alpha\overline{\theta}_i + c\overline{w}_i), \\ p \,\square\, (\alpha x_i + \varepsilon - e_i) &\ll R \cdot (\alpha\overline{\theta}_i + c\overline{w}_i). \end{aligned}$$

Recalling that $\overline{w}_i \in \mathcal{K}(\overline{\Theta}_i)$, we have $\alpha\overline{\theta}_i + c\overline{w}_i \in c\ell(\overline{\Theta}_i)$. Consequently, it holds that $u_i(\alpha x_i + \varepsilon) > u_i(x_i)$ and $(\alpha x_i + \varepsilon, \alpha\overline{\theta}_i + c\overline{w}_i) \in \mathcal{B}'_i(p, q)$, which contradicts the optimality of $(x_i, \theta_i^*)$ in $\mathcal{B}'_i(p, q)$.

Since $\sum_{i \in I} \overline{\theta}_i = 0$, $q \cdot \overline{\theta}_i \geq 0$ for all $i \in I$ implies that $q \cdot \overline{\theta}_i = 0$ for all $i \in I$. Recalling that $q \in V + \overline{M}$, we have $q \cdot \theta_i = 0$ for all $i \in I$. Thus, $(p, q, x, \theta)$ is an equilibrium of $\mathcal{E}$. ■

PROOF OF PROPOSITION 4.2 : Let $(p, q, x, \theta)$ be an equilibrium of $\mathcal{E}$. Suppose that $q$ admits a projective arbitrage $v_i \in \Theta_i$ for some $i \in I$. For each $i \in I$, let $\overline{\theta}_i$ and $\overline{v}_i$ denote the projection of $\theta_i$ and $v_i$ onto $V + \overline{M}$, respectively. It is clear that $\overline{\theta}_i \in \overline{\Theta}_i$, $\overline{v}_i \in \overline{C}_i$, and $W(q) \cdot \overline{v}_i > 0$ for each $i \in I$.

$(i)$ By $i)$ of Theorem 3.2, $(p, q, x, \overline{\theta})$ is an equilibrium of $\overline{\mathcal{E}}$. But the result that $\overline{\theta}_i + \overline{v}_i \in c\ell(\overline{\Theta}_i)$ and $W(q) \cdot (\overline{\theta}_i + \overline{v}_i) > W(q) \cdot \overline{\theta}_i$ contradicts the optimality of $(x_i, \overline{\theta}_i)$ in the budget for the economy $\overline{\mathcal{E}}$. Consequently, $q$ is in $Q$. □

$(ii)$ Since $\overline{\Theta}_i$ is closed and convex, $\overline{C}_i$ coincides with the recession cone of $\overline{\Theta}_i$. Thus, we have $\lambda\overline{v}_i \in \overline{\Theta}_i$ for all $\lambda > 0$. We pick $\eta_i(\lambda) \in \overline{N}$ such that $\lambda\overline{v}_i + \eta_i(\lambda) \in \Theta_i$. Since $q \in V + \overline{M}$, it follows that for all $\lambda > 0$, $(1-1/\lambda)\theta_i + (1/\lambda)(\overline{v}_i + \eta_i(\lambda)) \in \Theta_i$ and

$$\begin{aligned} W(q)\cdot[(1-1/\lambda)\theta_i + (1/\lambda)(\overline{v}_i + \eta_i(\lambda))] &= W(q)\cdot[(1-1/\lambda)\theta_i + \overline{v}_i] \\ &> W(q)\cdot[(1-1/\lambda)\theta_i]. \end{aligned}$$

On the other hand, $R \cdot \overline{v}_i > 0$. Let $s'$ be a state in $S$ with $r(s') \cdot \overline{v}_i > 0$. Then we can pick a consumption $\tau \in \mathbb{R}^L_{++}$ such that $p(s') \cdot \tau < r(s') \cdot \overline{v}_i$. Choose $\varepsilon \in \mathbb{R}^{SL}_{+}$ such that $\varepsilon(s) = \tau$ if $s = s'$ and $\varepsilon(s) = 0$ if $s \neq s'$. Clearly, $u_i(x_i + \varepsilon) > u_i(x_i)$. Thus, for sufficiently large $\lambda > 0$, we have $u_i((1-1/\lambda)x_i + \varepsilon) > u_i(x_i)$. On the other hand, it holds that for all $\lambda > 1$,

$$\begin{aligned} p \,\square\, [(1-1/\lambda)x_i - e_i] &\ll R\cdot[(1-1/\lambda)\theta_i] \\ &< R\cdot[(1-1/\lambda)\theta_i + (1/\lambda)(\overline{v}_i + \eta_i(\lambda))]. \end{aligned}$$

It follows that for sufficiently large $\lambda > 0$, $((1-1/\lambda)\theta_i + (1/\lambda)(\overline{v}_i + \eta_i(\lambda)), (1-1/\lambda)x_i + \varepsilon) \in \mathcal{B}_i(p,q)$ and $u_i((1-1/\lambda)x_i + \varepsilon) > u_i(x_i)$, which contradicts the fact that $(x_i, \theta_i) \in \xi_i(p,q)$. ■

PROOF OF PROPOSITION 4.3 : ($i$) For every $i \in I$, we define the set

$$Q'_i = \left\{ q \in V + \overline{M} : q \cdot \theta_i > 0, \forall\, \theta_i \in \Theta_i \text{ such that } \overline{\theta}_i \in \overline{C}_i \text{ and } R\cdot\theta_i > 0 \right\}.$$

If $q \in Q_i$, then it is clear that $q \in Q'_i$ and therefore, $Q_i \subset Q'_i$. Suppose that there exists $q \in Q'_i \backslash Q_i$. Since $q \notin Q_i$, there exists $\theta_i \in \Theta_i$ such that $\bar{\theta}_i \in \overline{C}_i$ and $W(q)\cdot\theta_i > 0$ where $\bar{\theta}_i$ is the projection of $\theta_i$ onto $V + \overline{M}$. There are two possibilities, ($a$) $q \cdot \theta_i \leq 0$ and $R \cdot \theta_i > 0$ or ($b$) $q \cdot \theta_i < 0$ and $R \cdot \theta_i = 0$. The case ($a$) leads to an immediate contradiction to the fact that $q \in Q'_i$. Consider the case ($b$). Let $\overline{w}_i$ denote the projection of $w_i$ onto $V + \overline{M}$. Then we have $R \cdot \overline{w}_i > 0$. Since $C_i \subset \Theta_i$ and $\overline{C}_i = \mathcal{K}(\overline{\Theta}_i)$, $\bar{w}_i$ is in $\overline{C}_i$. Thus, we can choose a small number $\alpha > 0$ such that $\alpha\overline{w}_i + (1-\alpha)\overline{\theta}_i \in \overline{C}_i$, $\;q\cdot(\alpha\overline{w}_i + (1-\alpha)\overline{\theta}_i) = q\cdot(\alpha w_i + (1-\alpha)\theta_i) < 0$ and $R\cdot(\alpha\overline{w}_i + (1-\alpha)\overline{\theta}_i) = R\cdot(\alpha w_i + (1-\alpha)\theta_i) > 0$, which contradicts the fact that $q \in Q'_i$ as well. Thus, we conclude that for each $i \in I$, $Q_i = Q'_i$. □

($ii$) Let $q$ be a price in $Q$. Suppose that $q \notin Q^C$. Then there exists $i \in I$ such that $W(q)\cdot v_i > 0$ for some $v_i \in C_i$. Let $\bar{v}_i$ be the projection of $v_i$ onto $V + \overline{M}$. Since $C_i \subset \Theta_i$ and $\overline{C}_i = \mathcal{K}(\overline{\Theta}_i)$, it holds that $\bar{v}_i$ is in $\overline{C}_i$ and $W(q)\cdot\bar{v}_i = W(q)\cdot v_i > 0$, , which contradicts the fact that $q \in Q$. □

($iii$) By ($ii$), we have $Q \subset Q^C$. Thus, we have only to show that $Q^C \subset Q$. Suppose that there exists $q \in Q^C \backslash Q$. Then there exists $v_i \in \Theta_i$ for some $i \in I$ such that $\overline{v}_i \in \overline{C}_i$ and $W(q)\cdot\overline{v}_i > 0$. Since $\Theta_i$ is a closed convex cone with vertex for all $i \in I$, by Proposition 3.1 and 3.2, we have

$N = \overline{N}$ and $M = \overline{M}$. By $(iii)$ of Proposition 3.1, we have $\overline{C}_i = c\ell(\hat{C}_i)$. Since $\hat{C}_i$ is closed, $\overline{v}_i$ is in $\hat{C}_i$. Let $\tilde{v}_i$ be a point in $N$ such that $\hat{v}_i + \tilde{v}_i \in C_i$. Since $W(q) \cdot \tilde{v}_i = 0$, we have $W(q) \cdot (\hat{v}_i + \tilde{v}_i) > 0$ which contradicts the fact that $q \in Q^C$. ∎

Now we investigate the properties of demand correspondences for $\mathcal{E}$. Let $K$ be a closed rectangle in $\mathbb{R}^{SL} \times \mathbb{R}^J$ with center at the origin. We set $P = (\mathbb{R}^L_+ \setminus \{0\})^S$. The set $P$ denotes the set of nonnegative prices which excludes the zero price in each contingency of the second period. For each $(p, q) \in P \times c\ell(Q)$, we define the budget set $\mathcal{B}_i(p, q; K) = \mathcal{B}_i(p, q) \cap K$ for every $i \in I$, which is a compact truncation of the budget set $\mathcal{B}_i(p, q)$. We are ready to define demand correspondences with respect to the truncated budget set:

$$\xi_i(p, q; K) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : (x_i, \theta_i) \in \underset{(x,\theta) \in \mathcal{B}_i(p,q;K)}{\arg\max} \; u_i(x) \right\}.$$

As implicitly shown in Example 5.1, the correspondence $\xi_i(p, q; K)$ may fail to be continuous at asset prices which do not allow income transfer from the beginning period to the next period. To circumvent this problem, we introduce the following artifact. For a point $(p, q) \in P \times c\ell(Q)$ which satisfies either $(p, q) \notin \mathbb{R}^{SL}_{++} \times Q$ or $\min q \cdot \Theta_i = 0$, we define the set

$$\varphi_i(p, q; K) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i \;\middle|\; \begin{array}{l} \exists \{(p^n, q^n)\} \text{ in } \mathbb{R}^{SL}_{++} \times Q \text{ such that } (p^n, q^n) \to (p, q), \\ \min q^n \cdot \Theta_i < 0 \text{ for each } n, \text{ and} \\ (x^n_i, \theta^n_i) \to (x_i, \theta_i) \text{ for some } (x^n_i, \theta^n_i) \in \xi_i(p^n, q^n; K) \end{array} \right\}$$

We claim that $\varphi_i(p, q; K)$ is closed. Let $\{(x^n_i, \theta^n_i)\}$ be a sequence in $\varphi_i(p, q; K)$ which converges to some point $(x_i, \theta_i)$ in $X_i \times \Theta_i$. Then there exists $(p^{n,m}, q^{n,m}) \to (p, q)$ such that $(p^{n,m}, q^{n,m}) \in \mathbb{R}^{SL}_{++} \times Q$ for each $m$, and $(x^{n,m}_i, \theta^{n,m}_i) \to (x^n_i, \theta^n_i)$ which satisfies $(x^{n,m}_i, \theta^{n,m}_i) \in \xi_i(p^{n,m}, q^{n,m}; K)$ for each $m$. Recalling that $(x^n_i, \theta^n_i) \to (x_i, \theta_i)$, by the diagonal sequence theorem there exists $\{n_k\}$ such that $(x^{n,n_k}_i, \theta^{n,n_k}_i) \to (x_i, \theta_i)$. Thus we have $(x_i, \theta_i) \in \varphi_i(p, q; K)$.[39]

We define the correspondence $\xi'_i$ on $P \times c\ell(Q)$ by.

$$\xi'_i(p, q; K) = \begin{cases} \xi_i(p, q; K), & \text{if } (p, q) \in \mathbb{R}^{SL}_{++} \times Q \text{ and } \min q \cdot \Theta_i < 0, \\ \varphi_i(p, q; K), & \text{otherwise.} \end{cases}$$

Let $\hat{\xi}_i(p, q; K)$ and $\hat{\varphi}_i(p, q; K)$ denote the convex hull of $\xi'_i(p, q; K)$ and $\varphi_i(p, q; K)$, respectively. Since $\varphi_i(p, q; K)$ is compact, so is $\hat{\varphi}_i(p, q; K)$.

[39] The reader is referred to Kantorvich and Akilov (1982) for the diagonal sequence theorem.

**Lemma A1 :** Each $\hat{\xi}_i(p,q;K)$ is upper hemicontinuous with nonempty compact convex values at each $(p,q) \in P \times c\ell(Q)$.

PROOF : Since $K$ is compact and $u_i$ is continuous, $\hat{\xi}_i(p,q;K)$ is nonempty and compact. If $\xi'_i$ is upper hemicontinuous, so is the convex hull $\hat{\xi}_i$.[40] Thus we have only to show the upper hemicontinuity of $\xi'_i$. Choose a sequence $\{(p^n, q^n, x_i^n, \theta_i^n)\}$ which converges to $(p, q, x_i, \theta_i)$ such that $(x_i^n, \theta_i^n) \in \xi'_i(p^n, q^n; K)$.

Suppose that $(p,q) \in [P \times c\ell(Q)] \backslash (\mathbb{R}^{SL}_{++} \times Q)$ or $\min q \cdot \Theta_i = 0$. Then there are two possibilities;

$(a)$ $(p^n, q^n) \in \mathbb{R}^{SL}_{++} \times Q$ and $\min q^n \cdot \Theta_i < 0$ for infinitely many $n$'s or

$(b)$ $(p^n, q^n) \in [P \times c\ell(Q)] \backslash (\mathbb{R}^{SL}_{++} \times Q)$ or $\min q^n \cdot \Theta_i = 0$ for infinitely many $n$'s.

If (a) holds, then $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K)$ for infinitely many $n$. By definition, $(x_i, \theta_i) \in \varphi_i(p,q;K)$ and therefore, $(x_i, \theta_i) \in \xi'_i(p,q;K)$. If (b) holds, then $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K)$ for infinitely many $n$'s. Without loss of generality, we may assume that $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K)$ for all $n$. Then there exists $\{(p^{n,m}, q^{n,m})\}$ in $\mathbb{R}^{SL}_{++} \times Q$ for each $n$ such that $\min q^{n,m} \cdot \Theta_i < 0$ for all $m$, $(p^{n,m}, q^{n,m}) \to (p^n, q^n)$ and $(x_i^{n,m}, \theta_i^{n,m}) \to (x_i^n, \theta_i^n)$ with $(x_i^{n,m}, \theta_i^{n,m}) \in \xi_i(p^{n,m}, q^{n,m}; K)$. Recalling that $(x_i^n, \theta_i^n) \to (x_i, \theta_i)$, by the diagonal sequence theorem there exists $\{n_k\}$ such that $(x_i^{n,n_k}, \theta_i^{n,n_k}) \to (x_i, \theta_i)$. Thus we have $(x_i, \theta_i) \in \varphi_i(p,q;K)$ and therefore, $(x_i, \theta_i) \in \xi'_i(p,q;K)$.

We turn to the case that $(p,q) \in \mathbb{R}^{SL}_{++} \times Q$ and $\min q \cdot \Theta_i < 0$. Suppose that $(x_i, \theta_i) \in \mathcal{B}_i(p,q;K) \backslash \xi_i(p,q;K)$. Then there is $(x'_i, \theta'_i) \in \mathcal{B}_i(p,q;K)$ such that $u_i(x'_i) > u_i(x_i)$. It follows from the continuity of preferences that for some $\alpha \in (0,1)$, $u_i(\alpha x'_i) > u_i(x_i)$. Since $e_i > 0$ and $p(s) \gg 0$ for all $s \in S$, we see that $q \cdot (\alpha\theta'_i) \leq 0$ and $p \,\square\, (\alpha x'_i - e_i) \ll R \cdot (\alpha\theta'_i)$.

Since $\min q \cdot \Theta_i < 0$, we can choose $\zeta_i \in \Theta_i$ with $q \cdot \zeta_i < 0$. Then there exists a positive number $\beta < 1$ which satisfies $q \cdot (\beta\alpha\theta'_i + (1-\beta)\zeta_i) < 0$, $p \,\square\, (\beta\alpha x'_i - e_i) \ll R \cdot (\beta\alpha\theta'_i + (1-\beta)\zeta_i)$ and $u_i(\beta\alpha x'_i) > u_i(x_i)$. Then for sufficiently large $n$, we have

$$u_i(\beta\alpha x'_i) > u_i(x_i^n) \text{ and } (\beta\alpha x'_i, \beta\alpha\theta'_i + (1-\beta)\zeta_i) \in \mathcal{B}_i(p^n, q^n; K).$$

This contradicts the optimality of $(x_i^n, \theta_i^n)$ in $\mathcal{B}_i(p^n, q^n; K)$. ■

For each $i \in I$, we set $\Gamma_i = \{v_i \in \overline{C}_i : R \cdot v_i > 0 \text{ and } \exists \eta_i \in \overline{N} \;\; s.t. \;\; v_i + \eta_i \in \Theta_i\}$. Each $\Gamma_i$ is nonempty under the condition PV. We define the sets $\Gamma = \sum_{i \in I}(\Gamma_i \cup \{0\})$ and $\bar{\Gamma} = \sum_{i \in I} c\ell(\Gamma_i)$. The sets $Q$ and $\Gamma$ are characterized as following.

[40] For details on this point, see Hildenbrand (1974).

**Lemma A2 :** The sets $Q$ and $\Gamma$ have the following property.

($i$) $Q = \{q \in V + \overline{M} : q \cdot v > 0, \forall\, v \in \Gamma \backslash \{0\}\}$.

($ii$) The set $\bar{\Gamma}$ is a closed convex pointed cone.[41]

PROOF : ($i$) Let $q \in \{q' \in V + \overline{M} : q' \cdot v > 0$ for all nonzero $v \in \Gamma\}$. Since $\Gamma_i \subset \Gamma$ for each $i \in I$, we see that $q \cdot v_i > 0$ for all $v_i \in \Gamma_i$. It implies that $q \in Q_i$ for all $i \in I$ and therefore, $q \in Q$. To prove the converse, let $q \in Q$. Then $q \in Q_i$ for all $i \in I$. For a nonzero point $v \in \Gamma$, there exists $v_i \in \Gamma_i \cup \{0\}$ for each $i \in I$ such that $v = \sum_{i \in I} v_i$. Then we have $q \cdot v_i > 0$ for all $i \in I$ with $v_i \neq 0$ and therefore, $q \cdot v > 0$. Thus we have $q \in \{q' \in V + \overline{M} : q' \cdot v > 0, \forall\, v \in \Gamma \backslash \{0\}\}$. $\square$

($ii$) Clearly, $\bar{\Gamma}$ is convex. For closedness, it suffices to show that $\{c\ell(\Gamma_i) : i \in I\}$ are positively semi-independent, since $c\ell(\Gamma_i)$ is itself a closed convex cone. Pick $\theta_i \in c\ell(\Gamma_i)$ for every $i \in I$ such that $\sum_{i \in I} \theta_i = 0$. Then $R \cdot (\sum_{i \in I} \theta_i) = 0$. Since $\theta_i \in c\ell(\Gamma_i)$ implies $R \cdot \theta_i \geq 0$, we see that $R \cdot \theta_i = 0$, *i.e.*, $\theta_i \in \overline{M}$. We know that $\Gamma_i \subset \overline{G}_i$, and therefore $c\ell(\Gamma_i) \subset \overline{G}_i$ for each $i \in I$. Thus, we have $\theta_i \in \overline{G}_i \cap \overline{M}$ for every $i \in I$. By ($iv$) of Proposition 3.2, we see that each $\theta_i$ is equal to 0. Hence $\{c\ell(\Gamma_i) : i \in I\}$ are positively semi-independent. To show that $\bar{\Gamma}$ is pointed, pick $\overline{\theta} \in \bar{\Gamma} \cap (-\bar{\Gamma})$. Then there exist $\theta_i \in c\ell(\Gamma_i)$ and $\theta'_i \in c\ell(\Gamma_i)$ such that $\overline{\theta} = \sum_{i \in I} \theta_i$ and $-\overline{\theta} = \sum_{i \in I} \theta'_i$, respectively. Since $R \cdot \theta_i \geq 0$ and $R \cdot \theta'_i \geq 0$, we have $R \cdot \overline{\theta} \geq 0$ and $R \cdot (-\overline{\theta}) \geq 0$, which implies $R \cdot \overline{\theta} = 0$. It follows that $\theta_i \in \overline{G}_i \cap \overline{M}$ and $\theta'_i \in \overline{G}_i \cap \overline{M}$ for all $i \in I$ and therefore $\overline{\theta} \in \sum_{i \in I}(\overline{G}_i \cap \overline{M})$ and $-\overline{\theta} \in \sum_{i \in I}(\overline{G}_i \cap \overline{M})$. By ($iv$) of Proposition 3.2, we have $\overline{\theta} = 0$. ■

We show that there exists at least one agent that has 'large' purchasing power around the boundary of $P \times c\ell(Q)$. This leads to the explosion of aggregate demand on the relevant boundary of $P \times c\ell(Q)$ in frictional markets. Let $\{K_n\}$ be an increasing sequence of rectangles in $\mathbb{R}^{SL} \times \mathbb{R}^J$ which satisfies $\bigcup_n K_n = \mathbb{R}^{SL} \times \mathbb{R}^J$.

**Proposition A2 :** Let $\{(p^n, q^n)\}$ be a sequence of prices in $P \times c\ell(Q)$ convergent to a point $(p, q) \in [P \times c\ell(Q)] \backslash (\mathbb{R}^{SL}_{++} \times Q)$. Then for a sequence of allocations $\{(x^n, \theta^n)\}$ such that $(x^n_i, \theta^n_i) \in \hat{\xi}_i(p^n, q^n; K_n)$ for all $n$ and all $i \in I$, we have $\sum_{i \in I} \|x^n_i\| \to \infty$.

PROOF : Since $\hat{\xi}_i(p^n, q^n; K_n)$ is the convex hull of $\xi'_i(p^n, q^n; K_n)$ for each $n$, we have only to verify the current proposition for $\xi'_i$. Thus without loss of generality we may assume that

[41] A cone $C$ is pointed if $C \cap (-C) = \{0\}$.

$(x_i^n, \theta_i^n) \in \xi_i'(p^n, q^n; K_n)$ for all $n$. To the contrary, suppose that $\{\sum_{i \in I} \|x_i^n\|\}$ is bounded. Then, $\{x_i^n\}$ is bounded for every $i \in I$. Decompose $\theta_i^n = \hat{\theta}_i^n + \tilde{\theta}_i^n$ where $\hat{\theta}_i^n \in V$ and $\tilde{\theta}_i^n \in V^\perp$. Then we see that for every $n$,

$$p^n \,\square\, (x_i^n - e_i) = R \cdot \theta_i^n = R \cdot \hat{\theta}_i{}^n.$$

Suppose that $\{\hat{\theta}_i^n\}$ is unbounded. Since $\{\hat{\theta}_i^n / \|\hat{\theta}_i^n\|\}$ is bounded, it has a subsequence converging to a nonzero vector $v_i \in V$. Recalling that $\{x_i^n\}$ and $\{p^n\}$ are bounded, we have $R \cdot v_i = 0$. This implies $v_i = 0$, which is impossible. Since the sequence $\{\hat{\theta}_i^n\}$ is bounded, without loss of generality we can take $\{(x_i^n, \hat{\theta}_i^n)\}$ as a convergent subsequence. Let $(x_i, \hat{\theta}_i)$ denote its limit point in $X_i \times V$.

We can consider two possibilities for $\{(p^n, q^n)\}$; either $(a)$ $(p^n, q^n) \in \mathbb{R}_{++}^{SL} \times Q$ and $\min q^n \cdot \Theta_i < 0$ for infinitely many $n$'s or $(b)$ $(p^n, q^n) \in [P \times c\ell(Q)] \backslash (\mathbb{R}_{++}^{SL} \times Q)$ or $\min q^n \cdot \Theta_i = 0$ for infinitely many $n$'s. Suppose that $(a)$ holds. Then $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K^n)$ for infinitely many $n$. Suppose that $(b)$ holds. Then $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K^n)$ for infinitely many $n$. By definition, there exists a sequence $\{(p^{n,m}, q^{n,m})\}$ in $\mathbb{R}_{++}^{SL} \times Q$ such that $\min q^{n,m} \cdot \Theta_i < 0$, $(p^{n,m}, q^{n,m}) \to (p^n, q^n)$ and $(x_i^{n,m}, \theta_i^{n,m}) \to (x_i^n, \theta_i^n)$ with $(x_i^{n,m}, \theta_i^{n,m}) \in \xi_i(p^{n,m}, q^{n,m}; K^n)$ for each $m$. Since $x_i^n \to x_i$ and $(p^n, q^n) \to (p, q)$, by the diagonal sequence theorem there exists $\{n_k\}$ such that $(p^{n,n_k}, q^{n,n_k}) \to (p, q)$ and $x_i^{n,n_k} \to x_i$. In particular, $(x_i^{n,n_k}, \theta_i^{n,n_k})$ is in $\xi_i(p^{n,n_k}, q^{n,n_k}; K^n)$.

Thus without loss of generality, we may assume that $(p^n, q^n) \in \mathbb{R}_{++}^{SL} \times Q$, $\min q^n \cdot \Theta_i < 0$, and $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K^n)$ for all $n$ and all $i \in I$. Since $(p, q) \in [P \times c\ell(Q)] \backslash (\mathbb{R}_{++}^{SL} \times Q)$, either (CASE 1) $p^n(s) \to p(s) \in \partial \mathbb{R}_+^L \backslash \{0\}$ for some $s \in S$ or (CASE 2) $q^n \to q \in c\ell(Q) \backslash Q$ and $p^n \to p \gg 0$.

(CASE 1) Suppose that $p^n(s) \to p(s) \in \partial \mathbb{R}_+^L \backslash \{0\}$ for some $s \in S$. Then we can set $\delta$ in a way that $\delta(s) > 0$ and $\delta(s') = 0$ if $s' \neq s$. Since $\sum_{i \in I} e_i \gg 0$, there exists $i \in I$ such that $p(s) \cdot e_i(s) > 0$. Then we have $p(s) \cdot [\alpha(x_i(s) + \delta(s))] < p(s) \cdot e_i(s) + r(s) \cdot (\alpha \hat{\theta}_i)$ and $u_i[\alpha(x_i + \delta)] > u_i(x_i)$ for some $\alpha \in (0, 1)$. Thus, for sufficiently large $n$, we see that $p^n(s) \cdot [\alpha(x_i^n(s) + \delta(s))] < p^n(s) \cdot e_i(s) + r(s) \cdot (\alpha \theta_i^n)$ and $u_i[\alpha(x_i^n + \delta)] > u_i(x_i^n)$. Since $(x_i^n, \theta_i^n) \in \mathcal{B}_i(p^n, q^n; K_n)$, it holds that for each $s' \neq s$ and each $n$, $p^n(s') \cdot [\alpha x_i^n(s')] \leq p^n(s') \cdot e_i(s') + r(s') \cdot (\alpha \theta_i^n)$. It follows that $u_i[\alpha(x_i^n + \delta)] > u_i(x_i^n)$ and $(\alpha(x_i^n + \delta), \alpha \theta_i^n) \in \mathcal{B}_i(p^n, q^n; K_n)$ for sufficiently large $n$, which contradicts the fact that $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K_n)$. ‖

(CASE 2) Suppose that $q^n \to q \in c\ell(Q) \backslash Q$ and $p^n \to p \gg 0$. First of all, we claim that there exists $\gamma \in \Gamma \backslash \{0\}$ such that $q \cdot \gamma = 0$. Since $\Gamma \subset \bar{\Gamma}$ and $q \in c\ell(Q)$, $q \cdot \gamma' \geq 0$ for all $\gamma' \in \Gamma$. Now suppose that $q \cdot \gamma' > 0$ for all nonzero $\gamma' \in \Gamma$. Then by $(i)$ of Lemma A2, we must have $q \in Q$, which is impossible. This proves the claim. $\|$

For each $i \in I$, we pick $\gamma_i \in \Gamma_i \cup \{0\}$ such that $\gamma = \sum_{i \in I} \gamma_i$. Since $q \cdot \gamma_i \geq 0$, $q \cdot \gamma = 0$ implies that $q \cdot \gamma_i = 0$ for each $i \in I$. Recalling that $\gamma \neq 0$, we can pick $i \in I$ such that $\gamma_i \neq 0$ and therefore, $R \cdot \gamma_i > 0$. Since $\min q^n \cdot \Theta_i < 0$, there exists $\zeta_i^n \in \Theta_i$ for each $n$ which satisfies $q^n \cdot \zeta_i^n < 0$. On the other hand, $R \cdot \gamma_i > 0$. Let $s^*$ be a state in $S$ with $r(s^*) \cdot \gamma_i > 0$. Then we can pick a consumption $\tau \in \mathbb{R}^L_{++}$ such that $p(s^*) \cdot \tau < r(s^*) \cdot \gamma_i$. Choose $\varepsilon \in \mathbb{R}^{SL}_{+}$ such that $\varepsilon(s) = \tau$ if $s = s^*$ and $\varepsilon(s) = 0$ if $s \in S \backslash \{s^*\}$. Clearly, $u_i(x_i + \varepsilon) > u_i(x_i)$. Then there exists $\bar{\alpha} \in (0, 1)$ which satisfies $u_i(\alpha x_i + \varepsilon) > u_i(x_i)$ for all $\alpha \in [\bar{\alpha}, 1)$. Recalling that $q^n \cdot \theta_i^n \leq 0$ and $q^n \cdot \zeta_i^n < 0$, we have $q^n \cdot (\alpha \theta_i^n + (1 - \alpha)\zeta_i^n) < 0$ for sufficiently large $n$. Let $\{\gamma_i^n\}$ be a sequence in $ri(\overline{C}_i)$ with $\gamma_i^n \to \gamma_i$. Since $ri(\overline{C}_i)$ is a subset of the recession cone of $\overline{\Theta}_i$, for each $\lambda > 0$ and $n$, we have $\lambda \gamma_i^n \in \overline{\Theta}_i$. We choose $\eta_i^n(\lambda) \in \overline{N}$ such that $\lambda \gamma_i^n + \eta_i^n(\lambda) \in \Theta_i$. Recalling that $q \in V + \overline{M}$, we have $W(q) \cdot \eta_i^n(\lambda) = 0$ for all $\lambda > 0$ and $n$. Since $p \gg 0$, $q \cdot \gamma_i = 0$ and $R \cdot \gamma_i > 0$, we can choose $\alpha < 1$ such that for sufficiently large $n$ and $\lambda > 0$, $((1 - 1/\lambda)(\alpha \theta_i^n + (1 - \alpha)\zeta_i^n) + (1/\lambda)(\lambda \gamma_i^n + \eta_i^n(\lambda)) \in \Theta_i$, and

$$\begin{aligned} 0 &< -q^n \cdot [(1 - 1/\lambda)(\alpha \theta_i^n + (1 - \alpha)\zeta_i^n) + \gamma_i^n], \\ p^n \,\square\, (\alpha x_i + \varepsilon - e_i) &\ll R \cdot [(1 - 1/\lambda)(\alpha \theta_i^n + (1 - \alpha)\zeta_i^n) + \gamma_i^n]. \end{aligned}$$

This implies that for sufficiently large $n$, $u_i(\alpha x_i + \varepsilon) > u_i(x_i^n)$ and $(\alpha x_i + \varepsilon, (1 - 1/\lambda)(\alpha \theta_i^n + (1 - \alpha)\zeta_i^n) + (1/\lambda)(\lambda \gamma_i^n + \eta_i^n(\lambda)) \in \mathcal{B}_i(p^n, q^n; K_n)$. This contradicts the fact that $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K_n)$. ■

Proposition A2 states that if the sequence of optimal choices $\{(x_i^n, \theta_i^n)\}$ is bounded for each $i$, then $\{(p^n, q^n)\}$ in $P \times c\ell(Q)$ converges to the point $(p, q) \in \mathbb{R}^{SL}_{++} \times Q$, *i.e.*, $p \gg 0$ and $q \in Q$. This result is used to show that equilibrium prices of $\mathcal{E}$ are in $\mathbb{R}^{SL}_{++} \times Q$.

## Proof of Theorem 5.1

The proof of Theorem 5.1 needs several preliminary steps. By Proposition A1, an equilibrium will exist in the economy $\mathcal{E}$ if an equilibrium exists in the economy $\mathcal{E}^a$. To work on $\mathcal{E}^a$, as alluded before, we adopt the same notational scheme in $\mathcal{E}^a$ as in $\mathcal{E}$ by attaching the superscript $a$ to the notation of $\mathcal{E}$. The following conditions are a $\mathcal{E}^a$ version of Assumptions 2.2, 2.3 and 5.1 for the economy $\mathcal{E}$.

**Assumption 2.2a :** The set $\Theta_i^a$ is a closed, convex set in $\mathbb{R}^{J+1}$ with $0 \in \Theta_i^a$.

**Assumption 2.3a :** The set $\sum_{i\in I} \Phi_i^a$ is closed in $\mathbb{R}^{J+1}$.

**Assumption 5.1a :** For each $q^a \in Q^a$, there is $\theta_i^a \in \Theta_i^a$ for all $i \in I$ such that $q^a \cdot \theta_i^a < 0$.

**Proposition A3 :** If $\mathcal{E}$ satisfies Assumptions 2.2, 2.3 and 5.1, then $\mathcal{E}^a$ satisfies Assumptions 2.2a, 2.3a and 5.1a.

PROOF : Assumptions 2.2a trivially holds when Assumptions 2.2 holds. For each $v^a \in \mathbb{R}^{J+1}$, we will write $v^a = (v_0, v)$ where $v_0 \in \mathbb{R}$ and $v \in \mathbb{R}^J$. We denote the $\mathcal{E}^a$ version of Proposition 3.2 and Corollary 3.1 by Proposition 3.2$^a$ and Corollary 3.1$^a$, respectively. Let $V^a$ denote the subspace spanned by the row vectors of $R^a$ and $V^{\perp,a}$ its orthogonal complement in $\mathbb{R}^{J+1}$, and $\overline{M}^a$ and $\overline{N}^a$ the subspaces of $V^{\perp,a}$ obtained in Proposition 3.2$^a$.

First we claim that $\overline{N}^a = \{0\} \times \overline{N}$ and $V^a + \overline{M}^a = \mathbb{R} \times (V + \overline{M})$. Let $\theta^a$ be a point in $\overline{N}^a = \mathcal{L}[\sum_{i\in I}(\Theta_i^a \cap V^{\perp,a})]$. Then there exist $v_i^a$ and $z_i^a$ in $\Theta_i^a \cap V^{\perp,a}$ for each $i$ such that $\theta^a = \sum_{i\in I} v_i^a$ and $-\theta^a = \sum_{i\in I} z_i^a$. In particular, we have $\theta_0^a = \sum_{i\in I} v_{i0}^a$ and $-\theta_0^a = \sum_{i\in I} z_{i0}^a$. This implies that $\sum_{i\in I}(v_{i0}^a + z_{i0}^a) = 0$. Since $v_{i0}^a \geq 0$ and $z_{i0}^a \geq 0$, this in turn implies that $v_{i0}^a = z_{i0}^a = 0$ for all $i \in I$. Thus, we have $\theta^a = (0, \theta)$, $v_i^a = (0, v_i)$, $R \cdot v_i = 0$, and $z_i^a = (0, z_i)$, $R \cdot z_i = 0$ for all $i \in I$. Since $\theta = \sum_{i\in I} v_i \in \sum_{i\in I}(\Theta_i \cap V^\perp)$ and $-\theta = \sum_{i\in I} z_i \in \sum_{i\in I}(\Theta_i \cap V^\perp)$, this implies that $\theta \in \overline{N}$ and therefore, $\overline{N}^a \subset \{0\} \times \overline{N}$. For a point $\theta \in \overline{N}$, clearly we have $(0, \theta) \in \overline{N}^a$ and therefore, $\{0\} \times \overline{N} \subset \overline{N}^a$. Thus, we conclude that $\overline{N}^a = \{0\} \times \overline{N}$ and $V^a + \overline{M}^a = \mathbb{R} \times (V + \overline{M})$.

Now we show that Assumption 2.3 implies Assumption 2.3a. Suppose that $\sum_{i\in I} \Phi_i$ is closed in $\mathbb{R}^J$. Let $\{\theta^{a,n}\}$ be a sequence in $\sum_{i\in I} \Phi_i^a$ which converges to $\theta^a = (\theta_0, \theta)$. For each $i \in I$ and $n$, we choose $\theta_i^{a,n} = (\theta_{i0}^n, \theta_i^n)$ in $\Phi_i^a$ such that $\theta^{a,n} = \sum_{i\in I} \theta_i^{a,n}$. Then we have $\theta_0^n = \sum_{i\in I} \theta_{i0}^n$ and $R \cdot \theta_i^n + \theta_{i0}^n r_0 \geq 0$. Since $\theta_0^n \to \theta_0$ and $\theta_{i0}^n \in \mathbb{R}_+$ for each $n$ and $i \in I$, each $\{\theta_{i0}^n\}$ is bounded and therefore, has a subsequence convergent to a point $\theta_{i0} \in \mathbb{R}_+$.

Now for each $n$ and $i \in I$, we let $\overline{\theta}^n$ and $\overline{\theta}_i^n$ denote the projection of $\theta^n$ and $\theta_i^n$ onto $V + \overline{M}$ for each $n$ and $i \in I$. Then it is clear that $\overline{\theta}_i^n \in \overline{\Phi}_i$ for each $i \in I$ and $\overline{\theta}^n = \sum_{i\in I} \overline{\theta}_i^n$. We claim that each $\{\overline{\theta}_i^n\}$ is bounded. Suppose otherwise. Then $\sum_{i\in I} \|\overline{\theta}_i^n\| \to \infty$. For each $n$, we set $a^n = 1/\sum_{i\in I} \|\overline{\theta}_i^n\|$. Clearly, $\sum_{i\in I} \|a^n \overline{\theta}_i^n\| = 1$ for all $n$ and therefore, each $\{a^n \overline{\theta}_i^n\}$ is bounded. Thus, it has a subsequence convergent to a point $v_i \in \overline{G}_i$. Since $\{\overline{\theta}^n\}$ is bounded, by passing to the limit the relation $a^n \overline{\theta}^n = \sum_{i\in I} a^n \overline{\theta}_i^n$, we obtain $\sum_{i\in I} v_i = 0$. On the other hand, we have

$R \cdot (a^n \theta_i^n) \geq -(a^n \theta_{i0}^n r_0)$ for each $n$. By passing it to the limit, we obtain $R \cdot v_i \geq 0$. Since $\sum_{i \in I} v_i = 0$, it implies that $R \cdot v_i = 0$. Combining $v_i \in V + \overline{M}$ with $v_i \in V^{\perp}$, we have $v_i \in \overline{M}$. Consequently, we have $v_i \in \overline{G}_i \cap \overline{M}$ and therefore, $v_i \in \sum_{i \in I}(\overline{G}_i \cap \overline{M})$. On the other hand, $v_i = \sum_{j \neq i} -v_j \in \sum_{j \in I} -(\overline{G}_j \cap \overline{M})$. Thus, we have $v_i \in \mathcal{L}\left[\sum_{i \in I}(\overline{G}_i \cap \overline{M})\right]$ for each $i \in I$. It follows from $(iv)$ of Proposition 3.2 that $v_i = 0$ for all $i \in I$, which contradicts the fact that $\sum_{i \in I} \|v_i\| = 1$.

Thus, each $\{\overline{\theta}_i^n\}$ has a subsequence convergent to a point $\overline{\theta}_i \in c\ell(\overline{\Theta}_i)$. Since $R \cdot \overline{\theta}_i + \theta_{i0} r_0 \geq 0$, it implies that $\overline{\theta}_i \in c\ell(\overline{\Phi}_i(\theta_{i0} r_0))$. Then we see that $\overline{\theta} = \sum_{i \in I} \overline{\theta}_i \in \sum_{i \in I} c\ell(\overline{\Phi}_i(\theta_{i0} r_0))$ and therefore, $\theta = (\theta - \overline{\theta}) + \overline{\theta} \in \overline{N} + \sum_{i \in I} c\ell(\overline{\Phi}_i(\theta_{i0} r_0))$. By $(i)$ of Corollary 3.1, we have $\theta \in \sum_{i \in I} \Phi_i(\theta_{i0} r_0)$. Then there exists $\theta_i' \in \Phi_i(\theta_{i0} r_0)$ such that $\theta = \sum_{i \in I} \theta_i'$. It follows that $(\theta_{i0}, \theta_i') \in \Theta_i^a$ and $(\theta_{i0}, \theta_i') \in \{\theta_i^a \in \mathbb{R}^{J+1} : R^a \cdot \theta_i^a \geq 0\}$ for each $i \in I$, and therefore, $\theta^a = (\theta_0, \theta) = \sum_{i \in I}(\theta_{i0}, \theta_i') \in \sum_{i \in I} \Phi_i^a$. Thus, we conclude that $\sum_{i \in I} \Phi_i^a$ is closed.

We show that if Assumption 5.1 holds for the economy $\mathcal{E}$, then Assumption 5.1a holds for the economy $\mathcal{E}^a$. Let $Q^a$ denote the set of no projective arbitrage prices for $\mathcal{E}^a$. We claim that $Q^a \subset \mathbb{R}_{++} \times Q$. Let $q^a = (q_0, q)$ be a point in $Q^a$ and $W^a(q^a)$ the matrix $W(q)$ augmented with the asset 0. Since $r_0 > 0$, by the monotonicity of $u_i$ it holds trivially that $q_0 > 0$. Suppose that $q \notin Q$. Then there exists $\theta_i \in \Theta_i$ for some $i \in I$ such that $\overline{\theta}_i \in \overline{C}_i$ and $W(q) \cdot \theta_i > 0$ where $\overline{\theta}_i$ is the projection of $\theta_i$ onto $V + \overline{M}$. Recalling that $V^a + \overline{M}^a = \mathbb{R} \times (V + \overline{M})$, we have $\overline{C}_i^a = \mathbb{R}_+ \times \overline{C}_i$. Thus, $(0, \overline{\theta}_i) \in \overline{C}_i^a$. It follows that $(0, \theta_i) \in \Theta_i^a$, $W^a(q^a) \cdot (0, \theta_i) = W(q) \cdot \theta_i > 0$, and $(0, \overline{\theta}_i)$ is the projection of $(0, \theta_i)$ onto $V^a + \overline{M}^a$. This implies that $q^a$ admits a projective arbitrage $(0, \theta_i)$ for $\mathcal{E}^a$, which is impossible. Thus, we have $Q^a \subset \mathbb{R}_{++} \times Q$.

By Assumption 5.1, for all $i \in I$, there exists $\theta_i \in \Theta_i$ which satisfies $q \cdot \theta_i < 0$ for all $q \in Q$. Let $q^a = (q_0, q)$ be a point in $\mathbb{R}_{++} \times Q$ for some $q_0 > 0$. Since $\theta_i^a = (0, \theta_i)$ is in $\Theta_i^a$ and $q^a \cdot \theta_i^a = q \cdot \theta_i < 0$ for each $i \in I$, Assumption 5.1a holds for $\mathcal{E}^a$. ■

To apply fixed point theorems, we need to find a compact convex set of nonzero prices that generates $c\ell(Q)$ as its conic expansion. If $\bar{\Gamma}$ has the empty interior in $V + \overline{M}$, however, $c\ell(Q)$ is not a pointed cone and therefore, it is impossible to find a compact, convex price simplex which corresponds to $c\ell(Q)$. To circumvent the dilemma, we will generalize the approach of Debreu (1962) to the current setting. We set

$$Q^{\circ} = \left\{q \in V + \overline{M} : q \cdot v > 0 \text{ for all } v \in \bar{\Gamma} \backslash \{0\}\right\}.$$

By $(ii)$ of Lemma A2, $Q^{\circ}$ is an open set in $V + \overline{M}$ which is not a subspace. Lemma 8 of Debreu

(1962) enables us to find a nondecreasing sequence of pointed convex cones in $Q^\circ \cup \{0\}$ whose union contains $Q^\circ$.

**Lemma A3 :** There exists a nondecreasing sequence $\{Q^n\}$ of pointed closed convex cones which satisfies in $Q^n \subset Q^\circ \cup \{0\}$ for all $n$ and $Q^\circ \cup \{0\} = \bigcup_n Q^n$.

PROOF : By Lemma 8 of Debreu (1962), there exists a nondecreasing sequence $\{Q^n\}$ of closed, convex cones in $Q^\circ \cup \{0\}$ such that $\bigcup_n Q^n$ contains the relative interior of $Q^\circ \cup \{0\}$. Since $Q^\circ$ is open in $V + \overline{M}$, we see that $Q^\circ \subset \bigcup_n Q^n \subset Q^\circ \cup \{0\}$ and therefore, $\bigcup_n Q^n = Q^\circ \cup \{0\}$. We claim that each $Q^n$ is pointed. Suppose that $Q^m$ is not pointed for some $m$. Let $L_m$ denote the nonzero lineality space of $Q^m$. Since $L_m \subset Q^{n'}$ for all $n' \geq m$, we have $L_m \subset \bigcup_n Q^n$ and therefore, $L_m \subset Q^\circ \cup \{0\}$. Let $q$ be a nonzero point in $L_m$. Then $-q$ is also in $L_m$. Since $q \in Q^\circ$, we have $q \cdot v > 0$ for all nonzero $v \in \bar{\Gamma}$. This implies that $-q \notin Q^\circ$, which contradicts the fact that $-q \in L_m \backslash \{0\} \subset Q^\circ$. ■

Since $Q^\circ$ is open in $V + \overline{M}$, we may assume that each $Q^n$ has the nonempty interior in $V + \overline{M}$. For each $n$, we define the set

$$\Gamma^n = \{\gamma \in V + \overline{M} : q \cdot \gamma \geq 0, \, \forall q \in Q^n\}.$$

Each $\Gamma^n$ is a pointed closed convex cone. Lemma A3 leads to the following result.

**Lemma A4 :** The sequence $\{\Gamma^n\}$ is nonincreasing and satisfies $\bigcap_n \Gamma^n = \bar{\Gamma}$.

PROOF : Since $\{Q^n\}$ is nondecreasing, $\{\Gamma^n\}$ is nonincreasing. By Lemma A3, $\bar{\Gamma}$ is in $\Gamma^n$ for each $n$ and therefore, $\bar{\Gamma} \subset \bigcap_n \Gamma^n$. Suppose that there exists $\gamma \in (\bigcap_n \Gamma^n) \backslash \bar{\Gamma}$. Then $\gamma \in \Gamma^n$ for all $n$ and $\gamma \notin \bar{\Gamma}$. We recall that $Q^\circ$ is open in $V + \overline{M}$ and $\bar{\Gamma} = \{\gamma \in V + \overline{M} : q \cdot \gamma \geq 0, \, \forall q \in Q^\circ\}$. Thus $\gamma \notin \bar{\Gamma}$ implies that $q \cdot \gamma < 0$ for some $q \in Q^\circ$. By Lemma A3, there exists $m$ which satisfies $q \in Q^m$. Since $\gamma \in \Gamma^m$, we have $q \cdot \gamma \geq 0$, which is contradictory. We conclude that $\bigcap_n \Gamma^n = \bar{\Gamma}$. ■

Since $Q^n$ is pointed, $\Gamma^n$ has the nonempty interior in $V + \overline{M}$. Pick a portfolio $\theta_i^0$ in the relative interior of $c\ell(\Gamma_i)$ for each $i \in I$. Then a portfolio $\theta^0 = \sum_{i \in I} \theta_i^0$ is in the relative interior of $\bar{\Gamma}$. Clearly $q \cdot \theta^0 > 0$ for all $q$ in $c\ell(Q) \backslash \{0\}$. By Lemma A4, $\theta^0$ is in the relative interior of $\Gamma^n$ in $V + \overline{M}$ for all $n$. Then we see that $q \cdot \theta^0 > 0$ for all nonzero $q \in Q^n$. We define the sets of normalized prices.

$$\bar{\Delta} = \left\{(p,q) \in \mathbb{R}^{SL}_{+} \times c\ell(Q) : \|q\| = 1, \sum_{\ell \in L} p_\ell(s) = 1 \text{ for all } s \in S\right\},$$
$$\Delta = \left\{(p,q) \in \mathbb{R}^{SL}_{++} \times Q : \|q\| = 1, \sum_{\ell \in L} p_\ell(s) = 1 \text{ for all } s \in S\right\},$$

$$\Delta^n = \left\{(p,q) \in \mathbb{R}^{SL}_{+} \times Q^n : \|q\| = 1, \sum_{\ell \in L} p_\ell(s) = 1 \text{ for all } s \in S\right\},$$
$$\tilde{\Delta}^n = \left\{(p,q) \in \mathbb{R}^{SL}_{+} \times Q^n : q \cdot \theta^0 = 1, \sum_{\ell \in L} p_\ell(s) = 1 \text{ for all } s \in S\right\}.$$

where $\|\cdot\|$ denotes the Euclidean norm. Clearly, $\Delta^n$ is closed and $\Delta^n \subset \bar{\Delta}$ for all $n$.

PROOF OF THEOREM 5.1 : By Theorem 3.2, it is enough to show the existence of equilibrium in the projection economy $\overline{\mathcal{E}}$. To do this, we need to check Assumptions 2.1-2.3 and 5.1 in terms of $\overline{\mathcal{E}}$. It should be noticed that the portfolio constraint for agent $i \in I$ in the economy $\overline{\mathcal{E}}$ is not $\overline{\Theta}_i$ but $c\ell(\overline{\Theta}_i)$. Let Assumptions 2.1p-2.3p and 5.1p, and the condition PVp denote the $\overline{\mathcal{E}}$ of Assumptions 2.1-2.3 and 5.1, and the condition PV, respectively. Obviously, Assumptions 2.1p, 2.2p and 5.1p hold for $\overline{\mathcal{E}}$ if Assumptions 2.1 and 5.1 hold for $\mathcal{E}$. By (*iii*) of Corollary 3.1, $\sum_{i\in I} c\ell(\overline{\Phi}_i)$ is closed. By the same arguments made in the proof of (*ii*) of Theorem 3.2, we have $c\ell(\overline{\Phi}_i) = c\ell(\overline{\Theta}_i) \cap \{v \in \mathbb{R}^J : R \cdot v \geq 0\}$. Thus, Assumption 2.3p holds for $\overline{\mathcal{E}}$. For each $i \in I$, let $w_i$ be a point in $C_i$ which satisfies the condition PV. Then the projection $\overline{w}_i$ of $w_i$ onto $V + \overline{M}$ satisfies $\overline{w}_i \in \overline{C}_i$ and $R \cdot \overline{w}_i > 0$. Thus, the condition PVp holds for $\overline{\mathcal{E}}$ as well.

Allowing for some notational abuse in the rest of the proof, we keep the same notation for the budget and demand correspondences $\mathcal{B}_i$, $\xi_i$ and $\hat{\xi}_i$ in the economy $\overline{\mathcal{E}}$ as in the original economy $\mathcal{E}$.

(STEP 1) First, we apply the fixed point theorem to the truncated demand correspondences of the economy $\overline{\mathcal{E}}$. Let $K$ be a rectangle in $\mathbb{R}^{SL} \times (V + \overline{M})$ sufficiently large such that it contains $\big(\sum_{i\in I} e_i, 0\big)$. Let $\Delta'$ denote a nonempty, convex and compact set in $P \times c\ell(Q)$.

We define the correspondence $\Psi(\,\cdot\,;K) = \varphi(\,\cdot\,;K)\times\pi(\,\cdot\,;K) : \Delta'\times K \to 2^{\Delta'\times K}$ by $\Psi(p,q,z,w;K) = \varphi(p,q;K)\times\pi(z,w;K)$ for each $((p,q),(z,w)) \in \Delta'\times K$, where $\varphi(z,w;K) = \{(p,q) \in \Delta' : q\cdot w \geq q'\cdot w,\ p \,\square\, z \geq p' \,\square\, z,\ \forall\,(p',q') \in \Delta'\}$ and $\pi(p,q;K) = \sum_{i\in I}\hat{\xi}_i(p,q;K) - \big(\sum_{i\in I} e_i, 0\big)$. By Lemma A1, we can show that $\Psi(\,\cdot\,;K)$ is upper hemicontinuous with nonempty compact convex values. By Kakutani's fixed point theorem, there is a fixed point $(\overline{p},\overline{q},\overline{z},\overline{w}) \in \Psi(\overline{m};K)$.

(STEP 2) Let $\{K_n\}$ be an increasing sequence of rectangles in $\mathbb{R}^{SL} \times \mathbb{R}^J$ such that $\bigcup_n K_n = \mathbb{R}^{SL} \times \mathbb{R}^J$. By applying the result of STEP 1 to the case where $\Delta'$ and $K$ are replaced by $\tilde{\Delta}^n$ and $K_n$, there exists a fixed point $(p^n, q^n, z^n, w^n)$ for $\Psi(\,\cdot\,; K_n)$. Since $(z^n, w^n) \in \pi(p^n, q^n; K_n)$, there is an allocation $(x^n, \theta^n)$ such that $(x_i^n, \theta_i^n) \in \hat{\xi}_i(p^n, q^n; K_n)$ for each $i$, $\sum_{i\in I}(x_i^n - e_i) = z^n$ and $\sum_{i\in I} \theta_i^n = w^n$. The fact that $(p^n, q^n) \in \varphi(z^n, w^n; K_n)$ for all $n$ implies that $p^n \,\square\, \sum_{i\in I}(x_i^n - e_i) \geq p \,\square\, \sum_{i\in I}(x_i^n - e_i)$ for all $p \in \left(\mathbb{R}_+^L \setminus \{0\}\right)^S$ and $q \cdot w^n \leq 0$ for each $q \in Q^n$ which satisfies $q \cdot \theta^0 = 1$. In particular, the second result gives $-w^n \in \Gamma^n$ for each $n$. Since $0 \leq p^n \,\square\, x_i^n \leq p^n \,\square\, e_i + R \cdot \theta_i^n$ for all $i \in I$ and $n$ and $\{p^n \,\square\, e_i\}$ is bounded, there exists $\omega_i \in \mathbb{R}^S$ for each $i \in I$ such that $R \cdot \theta_i^n + \omega_i \geq 0$ and therefore, $\theta_i^n \in \overline{\Phi}_i(\omega_i)$ for all $i \in I$ and $n$.

(STEP 3) We show that $\{x_i^n\}$ and $\{\theta_i^n\}$ are bounded for all $i \in I$. Suppose that $\sum_{i\in I} \|x_i^n\| + \sum_{i\in I} \|\theta_i^n\| + \|w^n\| \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm. We set $a^n = 1/(\sum_{i\in I} \|x_i^n\| + \sum_{i\in I} \|\theta_i^n\| + \|w^n\|)$. Then $a^n \to 0$. Since $\{a^n w^n\}$, and $\{a_n x_i^n\}$ and $\{a_n \theta_i^n\}$ are bounded for each $i \in I$, they have a subsequence convergent to a point $\dot{w}$, $\dot{x}_i$, and $\dot{\theta}_i$, respectively. Recalling that $\theta_i^n \in \overline{\Phi}_i(\omega_i)$, we have $\dot{\theta}_i \in \overline{G}_i$ for each $i \in I$. On the other hand, $\{p^n\}$ is bounded so that it has a subsequence convergent to a point $p^*$ in $\mathbb{R}^{LS}$.

We claim that $-\dot{w} \in \bar{\Gamma}$. Suppose not. Then by Lemma A4, there exists $\bar{n}$ such that $-\dot{w} \notin \Gamma^{\bar{n}}$ and $\Gamma^n \subset \Gamma^{\bar{n}}$ for all $n \geq \bar{n}$. Since $\Gamma^{\bar{n}}$ is closed, there exists an open neighborhood $B$ of $-\dot{w}$ in $V + \overline{M}$ such that $\Gamma^{\bar{n}} \cap B = \emptyset$. We also have $\Gamma^n \cap B = \emptyset$ for all $n \geq \bar{n}$. Since $-a^n w^n \in \Gamma^n$, it implies that $-a^n w^n \notin B$ for all $n \geq \bar{n}$. It contradicts the fact that $-\dot{w}$ is an accumulation point of $\{-a^n w^n\}$.

Let $p$ be a point in $\left(\mathbb{R}_+^L \setminus \{0\}\right)^S$. Since $p \,\square\, \sum_{i\in I}(x_i^n - e_i) \leq p^n \,\square\, \sum_{i\in I}(x_i^n - e_i) = R \cdot w^n$ for each $n$, we see that $p \,\square\, \sum_{i\in I}(a^n x_i^n - a^n e_i) \leq R \cdot (a^n w^n)$. Passing to the limit, we have $p \,\square\, \sum_{i\in I} \dot{x}_i \leq R \cdot \dot{w}$. On the other hand, $-\dot{w} \in \bar{\Gamma}$ means that $R \cdot \dot{w} \leq 0$. It follows that $p \,\square\, \sum_{i\in I} \dot{x}_i \leq 0$ for all $p \in \left(\mathbb{R}_+^L \setminus \{0\}\right)^S$, and therefore, $\sum_{i\in I} \dot{x}_i \leq 0$. Since $\dot{x}_i \geq 0$ for all $i \in I$, we see that $\dot{x}_i = 0$ for all $i \in I$.

Now by passing to the limit the relation $p^n \,\square\, (a^n x_i^n - a^n e_i) = R \cdot (a^n \theta_i^n)$, we see that $0 = p^* \,\square\, \dot{x}_i = R \cdot \dot{\theta}_i$ or $\dot{\theta}_i \in \overline{M}$ for each $i \in I$. Recalling that $\dot{\theta}_i \in \overline{G}_i$, we must have $\dot{\theta}_i \in \overline{G}_i \cap \overline{M}$ for each $i$. Again by multiplying the relation $\sum_{i\in I} \theta_i^n = w^n$ by $a^n$ and passing to the limit, we obtain $\sum_{i\in I} \dot{\theta}_i = \dot{w}$ and thus, $R \cdot \dot{w} = 0$. Since $-\dot{w} \in \bar{\Gamma}$, there exists $\dot{w}_i \in c\ell(\Gamma_i) \subset \overline{G}_i$ for each $i$ such that $\sum_{i\in I} \dot{w}_i = -\dot{w}$. The fact that $R \cdot \dot{w} = 0$ and $R \cdot \dot{w}_i \geq 0$ for each $i \in I$ yields $R \cdot \dot{w}_i = 0$ for all $i \in I$. Consequently, we have $\dot{w}_i \in \overline{G}_i \cap \overline{M}$ for all $i \in I$.

It follows that $\sum_{i\in I}(\dot{\theta}_i + \dot{w}_i) = 0$ and $\dot{\theta}_i + \dot{w}_i \in \overline{G}_i \cap \overline{M}$ for all $i \in I$. By $(iv)$ of Proposition

3.2, we must have $\dot{\theta}_i = \dot{w}_i = 0$ for all $i \in I$. Thus, we see that $\dot{w} = 0$, and $\dot{x}_i = 0$ and $\dot{\theta}_i = 0$ for each $i \in I$. But this result leads to the following contradiction

$$1 = a^n \frac{1}{a^n} = \sum_{i\in I} \|a^n x_i^n\| + \sum_{i\in I} \|a^n \theta_i^n\| + \|a^n w^n\| \to \sum_{i\in I} \|\dot{x}_i\| + \sum_{i\in I} \|\dot{\theta}_i\| + \|\dot{w}\| = 0.$$

Therefore, we conclude that $\{x_i^n\}$ and $\{\theta_i^n\}$ are bounded for all $i \in I$.

(STEP 4) Now we set

$$\dot{q}^n = \frac{q^n}{\|q^n\|}.$$

Clearly, $\{\dot{q}^n\}$ is bounded and $(p^n, \dot{q}^n) \in \Delta^n \subset \bar{\Delta}$ for all $n$. Without loss of generality, we can assume that $((p^n, \dot{q}^n), z^n, x^n, \theta^n) \to ((p^*, q^*), z^*, x^*, \theta) \in \bar{\Delta} \times \mathbb{R}^{SL} \times \mathbb{R}_+^{SLI} \times \left(\prod_{i\in I} c\ell(\overline{\Theta}_i)\right)$. We notice that $\theta_i$ need not be in $\overline{\Theta}_i$ because $\overline{\Theta}_i$ is not closed in general.

Since each $\{\theta_i^n\}$ is bounded, so is $\{w^n\}$. Thus it has a subsequence convergent to a point $w^*$. Recalling from Step 2 that $-w^n \in \Gamma^n$ for all $n$, we have $-w^* \in \bar{\Gamma}$. Thus $R \cdot w^* \leq 0$ and therefore, $z^* \leq 0$. Since $q^n \cdot w^n = 0$ for all $n$, we have $q^* \cdot w^* = 0$.

(STEP 5) Now we check the asset market clearing condition. We choose $w_i \in c\ell(\Gamma_i) \subset \overline{G}_i$ for each $i$ such that $-w^* = \sum_{i\in I} w_i$. Thus, we have $\theta_i + w_i \in c\ell(\overline{\Theta}_i)$ for all $i \in I$ and $\sum_{i\in I}(\theta_i + w_i) = 0$. For each $i \in I$, we set $\theta_i^* = \theta_i + w_i$. For a sufficiently large rectangle $K$ in $\mathbb{R}^{SL} \times (V + \overline{M})$, each $(x_i^*, \theta_i^*)$ is in the interior of $K$. By Lemma A1, we see $(x_i^*, \theta_i^*) \in \hat{\xi}_i(p^*, q^*; K)$. Since $(p^n, \dot{q}^n) \in \bar{\Delta}$ for all $n$, Proposition A2 allows us to have $(p^*, q^*) \in \Delta$ and therefore, $(x_i^*, \theta_i^*) \in \xi_i(p^*, q^*; K)$.[42] Since $K$ is not binding at $(x_i^*, \theta_i^*)$, it is in $\xi_i(p^*, q^*)$. On the other hand, the Walras' law implies that $p^* \,\Box\, z^* = 0$. Since $z^* \leq 0$ and $p^* \gg 0$, we see that $z^* = 0$ or $\sum_{i\in I}(x_i^* - e_i) = 0$. Therefore, the profile $(p^*, q^*, x^*, \theta^*) \in \Delta \times \mathbb{R}_+^{SLI} \times \left(\prod_{i\in I} c\ell(\overline{\Theta}_i)\right)$ is an equilibrium of the economy $\overline{\mathcal{E}}$. By ii) of Theorem 3.2, we conclude that there exists an equilibrium of the economy $\mathcal{E}$. ■

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[42]Recall that $R \cdot w_i \geq 0$ and $q^* \cdot w_i \geq 0$ for each $i \in I$. Since $q^* \cdot (-w^*) = 0$, we have $W(q^*) \cdot w_i \geq 0$ for all $i \in I$. Since $q \in Q$, this implies that $W(q^*) \cdot w_i = 0$.

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