A Co-Moment Criterion for Firms for the Two Canonical Models of Production under Uncertainty

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Abstract. This paper shows that there is a common criterion for firms for the two canonical equilibrium models of production under uncertainty, the state-of-nature model and the probability model. When security markets are sufficiently rich to efficiently allocate firms' profit streams among the consumer-investors, the theoretical criterion in the state-of nature model is market-value maximization, while for the probability model it is maximization of a firm's contribution to social welfare. Both criteria however require information not readily available to firms in practice. We show that if risks are not too large then both criteria can be transformed into a common co-moment criterion which a Nash-competitive firm should maximize, taking as given the production decisions of other firms and the co-moment prices, which can be easily deduced from security prices. The co-moment criterion provides a unifying framework for the two equilibrium models of production under uncertainty, and has the merit of being based on information which is readily available.

1. Introduction

This paper studies the optimal choice of investment by firms in a two-period finance economy under uncertainty. The analysis is made in the most favorable setting where there are sufficiently many financial securities to assure that the output (profit) of firms is efficiently distributed among the agents in the economy. In this setting the theory seems to be clear and well-established: firms should maximize their market value.

We will argue that the situation is not as clear and simple as it might at first appear. First "market-value maximization" is only valid from a normative point of view in a specific model of production under uncertainty, in which the uncertainty is modeled by states of nature with fixed probabilities of occurrence and firms influence how much they produce in each state through their choice of investment and production plan. This is what we call the *state-of-nature* (SN) model.

There is an alternative way of modeling production under uncertainty which fits better with the observed structure of traded financial contracts, which are based on outcomes rather than states of nature. The alternative model takes as a primitive the possible profit outcomes for the firms and assumes that the firms' choices of investment determine the probability distribution over their profit outcomes: we call this the *probability* (\mathcal{P}) model. In the \mathcal{P} model market-value maximization does not lead to a socially optimal outcome and the criterion which emerges from a normative analysis is that each firm should maximize its contribution to expected social welfare. Thus depending on the way uncertainty in production is modeled, the criterion that a firm should adopt to be led to a socially optimal choice of investment, is different.

A second difficulty is that firms may not readily have access to the information required to implement their criterion. In the state-of-nature model the firms need to know the present-value prices or equivalently the stochastic discount factor. In the probability model firms need to know the social welfare function, which seems to require knowing the utility functions of the agents. However we show in Magill-Quinzii (2007) that knowing the stochastic discount factor is sufficient to obtain a good approximation of the social welfare function. Thus the information required by firms to implement their criterion is essentially the same in both models—firms need to know the stochastic discount factor of the economy.

The two standard ways of deriving present-value prices from security prices suggested by theory do not provide a realistic way of obtaining such information. The most direct approach is to assume that there are markets for Arrow securities—i.e. contracts contingent on the realization of a state of nature or an outcome—whose prices would directly reveal the present-value prices. Such contracts are however not observed in practice. In the absence of such markets, the present-value prices can still "in principle" be deduced from the observed security prices if the securities satisfy the full spanning condition: this requires inverting the matrix of payoffs of the securities across the states (or outcomes). In any practical implementation the dimensions of this matrix would be enormous—so that finding the inverse is not feasible: to our knowledge this approach has never been attempted in any empirical study.

The approach that is widely adopted in the macroeconomic, and also in the finance literature, to make it straightforward to derive the stochastic discount factor is to postulate a specific functional form for the agents' preferences. For if the utility function of the representative agent at equilibrium is known—and such a utility function always exists with the full spanning of the securities—then this utility gives the social welfare function and its derivatives give the stochastic discount factors. Macroeconomists typically assume that agents have identical power or log utilities, while in finance it is frequently assumed that agents have mean-variance preferences. Unfortunately either of these assumptions implies homogeneity of the agents' attitudes toward risk, which in turn implies properties for the portfolio behavior of agents—collinear risk holdings and full diversification—which are not observed in practice. They also lead to difficulties in explaining security prices, at the aggregate level for macroeconomics (the equity premium puzzle) and at the firm level in finance (security returns are not satisfactorily explained by covariance with the market).

Our objective is to find a way of expressing the criterion that a firm should maximize, which can be readily estimated, without making restrictive homogeneity assumptions on the functional form of the agents' preferences. The assumption that we make is that the utility function of the 'representative agent' at equilibrium (which aggregates the preferences of the agents) can be developed in a Taylor series expansion around the mean equilibrium aggregate output. The spirit of the analysis is close to that of Judd and Guu (2001) who use Taylor series expansions, retaining only a few of the terms, to obtain an approximation of an equilibrium when risks are "small".

Replacing the stochastic discount factor by its Taylor series expansion leads to a co-moment pricing formula for the securities, where the present value of a security's payoff is expressed as the discounted expected payoff plus a weighted sum of the co-moments of the security's payoff with aggregate output. The coefficients, which depend on the agents' utility functions, can be considered as the "prices" of the associated co-moments. The idea of deriving a co-moment expression for asset prices by using a Taylor series expansion of an agent's utility function seems to have been first introduced by Rubinstein (1973); the equilibrium co-moment pricing formula, stopped at the third co-moment, was derived by Kraus and Litzenberger (1976, 1981).

However simply replacing the stochastic discount factor (in the state-of-nature model) or the

social welfare function (in the probability model) by their Taylor series expansions around aggregate output does not lead to a "natural" objective function for a firm to maximize in order to be led to its optimal choice of investment. This comes from the fact that there are terms which the firm must take as given—the aggregate output in the state-of-nature model or the mean aggregate output in the probability model—which in part depend on its own action. In the state-of-nature model this is just the manifestation of the familiar problem with the competitive assumption that firms must take prices—here state prices i.e. discount factors—as given: however the discount factors depend on aggregate output which in turn depends on the firm's own production.

To avoid this difficulty we derive a function for each firm which has the same first-order conditions as the original criterion but in which the only terms which are taken as given are the actions of all other firms and the coefficients of the co-moment pricing formula. Maximizing this function, which we call the *co-moment criterion* of the firm, corresponds to "competitive/Nash" behavior for the firm, where the competitive part consists in taking the co-moment prices as given, while the Nash part consists in taking the production decisions of other firms as given.

Since the co-moment criterion is derived from a Taylor series expansion it consists of an infinite sum of co-moment terms: since the series converges, the sum can be approximated by a finite number of terms. The smaller the aggregate and individual risks, the smaller the number of terms needed to obtain a good approximation. Thus to obtain an approximate criterion, it suffices to estimate a finite number of co-moment prices from the data on the security prices and returns, which does not seem to be an infeasible empirical task. If the intuition behind the asset pricing literature which extends the CAPM model to a mean-variance-skewness model is a guide, then an approximation obtained by neglecting the terms of order more than three may already provide a reasonable approximation if the risks are suitably restricted.¹

The striking feature of the co-moment criterion is that it is the same function for the two models: in the state-of-nature model firms maximize the criterion by choosing how much to produce in each state (subject to the technology constraints), while in the probability model firms maximize the same function by choosing their probability distribution over the outcomes. Thus the co-moment criterion provides a way of unifying the two canonical models of production under uncertainty.

The plan of the paper is as follows. Section 2 presents the two canonical models of production

¹It was recognized in the seventies that taking into account preference for positive skewness could help explain the observed lack of diversification of most investors' portfolios—for diversification typically destroys skewness: see Simkowicz-Beedles (1973) and, for a more recent empirical study of investors' lack of diversification and its relation to skewness, see Mitton-Vorkink (2007). As far as security prices are concerned, Harvey-Siddique (2000) have shown that security returns, which are not well explained by the mean-variance CAPM, are better explained when the co-skewness of the returns with the market portfolio is taken into account.

under uncertainty, the state-of-nature model and the probability model, and derives the criterion for each firm which leads to a Pareto optimal allocation. Section 3 derives the general co-moment formula for security prices and relates it to the recent literature on mean-variance-skewness asset pricing. Section 4 derives the co-moment formula for the state-of-nature model and relates it to the criterion suggested by Stiglitz (1972) for studying investment in the CAPM model. Section 5 derives the co-moment formula for the probability model. Section 6 concludes with a conjecture on how the analysis is likely to generalize to the case where some stakeholders in firms (managers, employees or large shareholders) cannot fully diversify their positions and hence are inevitably exposed to the firms' idiosyncratic risks.

2. State-of-Nature and Probability Models of Production

Consider a two-period (t = 0, 1) finance economy with I consumer-investors and K firms. Each firm $k \in K$ makes an investment $a_k \in \mathbb{R}_+$ at date 0, which leads to a random output at date 1: the two models differ by the way the date 1 random output is characterized.

State-of-Nature (SN) Model. Uncertainty is modeled by S states of nature with exogenously fixed probabilities $\rho = (\rho_s)_{s \in S}$. A production plan $(a_k, y^k) \in \mathbb{R}_+ \times \mathbb{R}^S_+$, with $y^k = (y_1^k, \ldots, y_S^k)$, is feasible for firm k if $T_k(-a_k, y^k) \leq 0$, where $T_k : \mathbb{R}_+ \times \mathbb{R}^S_+ \to \mathbb{R}$ is a differentiable, increasing and strictly quasi-convex transformation function. Letting $y_0^K = -a_k$, to avoid boundary solutions we assume that

$$\lim_{y_0^k \to 0} \frac{\partial T_k}{\partial y_0^k} > 0, \qquad \lim_{y_s^k \to 0} \frac{\partial T_k}{\partial y_s^k} = 0, \quad \forall s \in S$$

Probability (\mathcal{P}) Model. Each firm $k \in K$ has a fixed set of possible outcomes $\{y_1^k, \ldots, y_{S_k}^k\}$ at date 1, ranked in increasing order, and y^k denotes the random variable with fixed support $\{y_1^k, \ldots, y_{S_k}^k\}$. An outcome for the economy is a realization of the date 1 output for each firm, $y_s = (y_{s_1}^1, \ldots, y_{s_K}^K)$, indexed by the element $s = (s_1, \ldots, s_K)$ of the set $S = S_1 \times \ldots \times S_K$ which is called the outcome space.² The firms' choices of investment $a = (a_1, \ldots, a_K)$ at date 0 determine the joint probability $\rho(a) = (\rho_s(a))_{s \in S}$ of the firms' random outcomes at date 1. Let $Y = \sum_{k \in K} y^k$ denote the random aggregate output and let

$$G(\eta, a) = \sum_{\{s \in S | Y_s \ge \eta\}} \rho_s(a)$$

²To simplify notation we use the same symbol to denote a set and the number of elements in the set: thus S_k denotes both the number of possible outcomes for firm k and the index set of these outcomes.

denote the upper cumulative distribution function for Y: we assume that $\int_0^{\eta} G(t, a) dt$ is increasing and concave in a for all $\eta \leq Y_{\text{max}}$. This implies that investment is productive, in the sense that an increase in any a_k leads to a second-order stochastic dominant shift in the distribution of the aggregate output. Concavity in a_k implies that there are stochastic decreasing returns to scale for firm k's investment. The assumption of joint concavity is more demanding but is needed to obtain the equivalent of the First Welfare Theorem for this economy (see Magill-Quinzii (2007)).

REMARK: When analyzing the investment decision of firm k it will be convenient to use the notation $s = (s_k, s_{-k})$, $a = (a_k, a_{-k})$ and $y = (y^k, y^{-k})$ where '-k' stands for the firms other than k.

Consumption Sector. The consumption sector is the same in both models. The *I* consumerinvestors are the initial owners of the *K* firms and they trade on security markets to share the production risks: to simplify the analysis we assume that these production risks are the only risks to which agents are exposed. Thus the initial endowment of agent *i* consists of an amount ω_0^i of income at date 0 and ownership shares $(\delta_k^i)_{k \in K}$ of the firms, with $\delta_k^i \geq 0$, $\sum_{i \in I} \delta_k^i = 1$ for all $k \in K$. Agents have no endowment income at date 1. As a result each agent's consumption stream at date 1 will depend on the realization of the firms' outputs. Let $x^i = (x_0^i, x_1^i) = (x_0^i, (x_s^i)_{s \in S})$ denote agent *i*'s random consumption stream, where *S* denotes the set of states of nature in the SN model, or the firms' outcomes (the outcome space) in the \mathcal{P} model. Each agent is assumed to have expected utility preferences of the form

$$u^{i}(x^{i}) = u_{0}^{i}(x_{0}^{i}) + \sum_{s \in S} \rho_{s} u_{1}^{i}(x_{s}^{i})$$
(1)

where u_0^i , u_1^i are smooth, increasing, strictly concave functions.

Security Markets. There are J securities which consist of the equities of the K firms, bonds and derivative securities and, in the SN model, possibly additional securities contingent on the occurrence of the states of nature. The firms' production plans, consisting of the vectors (a_k^*, y^{*k}) satisfying $T_k(-a_k^*, y^{*k}) \leq 0$ in the SN model, or the investment a_k^* leading to the probability distribution $\rho(a^*)$ in the \mathcal{P} model, are assumed to be known by the consumer-investors. Thus agents can correctly anticipate the payoffs of the securities and the probability distribution of the payoffs.

To define an exchange equilibrium with fixed production plans using common notation for the two models, we let

•
$$\rho^* = \rho$$
 in the SN model
• $\rho^* = \rho(a^*)$ in the P model

• $y^* = (y^{1*}, \ldots, y^{K*})$, the firms' choices of production feasible with investments $a^* = (a_1^*, \ldots, a_K^*)$ in the SN model

- $y^* = (y^{1*}, \dots, y^{K*}) = (y^1, \dots, y^K)$ the fixed outcomes of the firms in the \mathcal{P} model
- $V^* = [V^j_s(y^*_s)]_{j \in J}$ the payoff matrix in the \mathcal{SN} model
- $V^* = [V^j(y^*_s)]_{\substack{j \in J\\s \in S}}$ the payoff matrix in the \mathcal{P} model

where in the SN model the payoff $V_s^j : \mathbb{R}^K \to \mathbb{R}$ of security j in state s can depend both on the state of nature and on the firms' outputs, while in the \mathcal{P} model the payoff $V^j : \mathbb{R}^K \to \mathbb{R}$ of security j can depend only on the realized outputs of the firms. For both models we assume that the first K securities are the equity of the firms, and the remaining securities are in zero net supply. Let q_j denote the price of security j and let $q = (q_j)_{j \in J}$ denote the vector of security prices. $z^i = (z_j^i)_{j \in J}$ denotes the portfolio of securities purchased or sold by agent i and $z = (z^i)_{i \in I}$ denotes the vector of portfolios of the agents. Finally $x = (x^i)_{i \in I}$ denote the vector of consumption streams of the I agents.

Equilibrium with Fixed Production Plans. We can now define an exchange equilibrium on the security markets for fixed and known production plans for the firms.

Definition 1 (x^*, z^*, q^*) is an exchange equilibrium with fixed production plans (a^*, y^*) if

(i) for each $i \in I$, x^{i*} maximizes $u_0^i(x_0^i) + \sum_{s \in S} \rho_s^* u_s^i(x_s^i)$ subject to

 $x_{0}^{i} = \omega_{0}^{i} + (\hat{q}^{*} - a^{*})\delta^{i} - q^{*}z^{i}, \qquad x_{1}^{i} = V^{*}z^{i}, \qquad z^{i} \in \mathbb{R}^{J}$

(ii) $\sum_{i \in I} z_j^{i*} = 1, \ j = 1, \dots K, \qquad \sum_{i \in I} z_j^{i*} = 0, \ j > K$

where $\hat{q}^* = (q_j^*)_{j=1}^K$ denotes the vector of equity prices.

REMARK: We assume that the investment a_k^* of firm k is financed by the initial shareholders: this is without loss of generality since the Modigliani-Miller theorem on the irrelevance of the choice of financing policy holds for this economy. The market clearing condition (ii) on the security market combined with the agents' budget equations in (i) implies the feasibility of the equilibrium allocation since

$$\sum_{i \in I} x_0^{i*} = \sum_{i \in I} \omega_0^i - \sum_{k \in K} a_k^*, \qquad \sum_{i \in I} x_1^{i*} = V^* \sum_{i \in I} z^{i*} = \sum_{k \in K} y^{k*}$$

To avoid the introduction of multipliers for the non-negativity constraints on agents' consumption streams, we assume either that the marginal utility of consumption tends to infinity when consumption tends to zero (e.g. power or log utility functions) or, if the marginal utility is defined at zero, that consumption is not restricted to be non negative. Of course the first case is the most realistic, but polynomial utilities are convenient for constructing simple examples.

In order that an allocation (x^*, a^*, y^*) be Pareto optimal, two conditions must be satisfied: the distribution $x^* = (x^{i*})_{i \in I}$ of the available resources $(\sum_{i \in I} \omega_0^i - \sum_{k \in K} a_k^*, \sum_{k \in K} y^{k*})$ among consumers must be optimal, and firms must choose their production plans (a^*, y^*) optimally. We assume that the financial markets assure an optimal distribution of resources (income streams) among the agents and focus attention on the second condition.

Assumption EE (Efficiency of Exchange). If (x^*, z^*, q^*) is an exchange equilibrium with fixed production plans (a^*, y^*) , then there exists $\pi^* \in \mathbb{R}^S_{++}$ such that

$$\pi_s^* = \frac{\rho_s^* u_1^{i\,\prime}(x_s^{i*})}{u_0^{i\,\prime}(x_0^{i*})} \qquad s \in S, \ i \in I$$
(2)

Sufficient conditions for optimality of the exchange equilibrium are well known. Assumption EE will hold if either rank $V^* = S$ (condition R), or if for all $i \in I$, u_1^i is in the LRT family with the same marginal coefficient of risk tolerance for all agents, and the equity of firms and a riskless bond are traded (condition LRT).

The rank condition R is satisfied for the SN model if there are complete markets with respect to the states of nature, and for the P model if there is full spanning of the firms' outcomes, a condition which we argue in Magill-Quinzii (2007) is less restrictive than complete markets with respect to states of nature. The alternative condition LRT is stringent on preferences since it requires that agents have a similar attitude toward risk, but is less restrictive on the spanning achieved by securities. Note however that the equity of the firms must be traded so that agents can get rid of their initial risks, and that agents must have access to a riskless bond to attenuate or leverage the risk of equity.

Since all agents have the same probability estimate ρ^* , the equality of the agents' present values of income in each state/outcome implies the equality of their stochastic discount factors (SDF). Let μ^* denote the common SDF defined by

$$\mu_s^* = \frac{u_1^{i\prime}(x_s^{i*})}{u_0^{i\prime}(x_0^{i*})}, \qquad s \in S, \ i \in I$$
(3)

The equality of the agents' stochastic discount factors imply that for each $s \in S$ the allocation

2. State-of-Nature and Probability Models of Production

 $x_s^* = (x_s^{i*})_{i \in I}$ is the solution of the social welfare maximum problem

$$\max_{x_s \in \mathbb{R}^I} \left\{ \sum_{i \in I} \frac{1}{u_0^{i'}(x_0^{i*})} u_1^i(x_s^i) \; \middle| \sum_{i \in I} x_s^i = Y_s^* \right\}$$

where Y_s^* denotes the aggregate output in state or outcome s. Let $\Phi^*(\eta)$ denote the associated convolution function of the date 1 utilities defined by

$$\Phi^{*}(\eta) = \max_{\xi \in \mathbb{R}^{I}} \left\{ \sum_{i \in I} \alpha_{i}^{*} u_{1}^{i}(\xi_{i}) \left| \sum_{i \in I} \xi_{i} = \eta \right. \right\} \quad \text{with} \quad \alpha_{i}^{*} = \frac{1}{u_{0}^{iI}(x_{0}^{i*})}$$
(4)

It is readily shown that the function Φ^* has the two properties (see e.g. Magill-Quinzii (1996, p. 163))

$$\Phi^*(Y_s^*) = \sum_{i \in I} \alpha_i^* u_1^i(x_s^{i*}), \qquad \Phi^{*\prime}(Y_s^*) = \mu_s^*$$
(5)

 Φ^* is the utility function of the "representative agent" at the equilibrium. From the first-order conditions for the optimal choice of a portfolio in (i) of Definition 1, it follows that the equilibrium prices of the securities satisfy

$$q^* = \pi^* V^* = E^*(\mu^* V^*) = E^*(\Phi^{*\prime}(Y^*) V^*)$$
(6)

where E^* denotes the expectation operator with respect to the probability ρ^* , and where the last equality comes from the property (5) of the convolution function.

Optimal Choice of Production Plans. We can now characterize the production plans (a^*, y^*) which lead to Pareto optimality of the combined consumption-production allocation.

Proposition 1. Let (a^*, y^*) be a choice of production plans for the firms and (x^*, z^*, q^*) an associated exchange equilibrium satisfying Assumption EE, with present-value vector $\pi^* = (\pi_s^*)_{s \in S}$ and $SDF \ \mu^* = (\mu_s^*)_{s \in S}$. The consumption-production plan (x^*, a^*, y^*) is a Pareto optimal allocation for the production economy if and only if

(i) in the SN model, for each $k \in K$, (a_k^*, y^{k*}) maximizes

$$M(a_k, y^k) = \sum_{s \in S} \pi_s^* y_s^k - a_k = E(\mu^* y^k) - a_k$$
(7)

subject to $T_k(-a_k, y^k) \leq 0;$

(ii) in the \mathcal{P} model, for each $k \in K$, a_k^* maximizes

$$V(a_k) = \sum_{s \in S} \rho(a_k, a_{-k}^*) \Phi^*(Y_s) - a_k$$
(8)

Proof: (i) is the standard result of the Arrow-Debreu theory that, with convex production sets, Pareto optimality is equivalent to firms maximizing profit at prices π^* collinear to the agents' gradients at the consumption allocation x^* . (ii) is derived in Magill-Quinzii (2007) from the firstorder conditions for Pareto optimality: in the \mathcal{P} model firms must maximize their contribution to social welfare measured in units of date 0 consumption.³

The form (7) and (8) of the firms' criteria are the most convenient for establishing existence of an equilibrium of the production economy: the equilibrium of the SN model is a standard Arrow-Debreu equilibrium which is known to exist under the convexity assumptions on preferences and technology given above. An equilibrium of the P model is shown to exist in Magill-Quinzii (2007) if in addition to the assumptions made above, the market values of the firms are non-negative, or taxes are used to subsidize firms with negative market values.

The criterion (7) of the state-of-nature model also has the apparent merit of simplicity: it is linear in the firm's production plan and, if firms know the present-value prices π^* , the only additional information that each firm needs in order to make an optimal choice of production plan is its own technology T_k . The prices π^* do all the co-ordination of information required for efficiency. However the problem is that in practice there are no "direct" markets for Arrow securities on states of nature, so that firms would need to indirectly deduce the state prices π^* from the observed prices of the securities.

The criterion (8) of the probability model is considerably more complex than the market-value criterion (7): it is a non-linear function of the firm's production plan and requires that firms know the function Φ^* —which in turn requires that they know agents' preferences. Markets, it would seem, have lost their role of providing firms with the requisite information to guide production decisions. In Magill-Quinzii (2007) we argue that if the prices π^* , and hence the SDF μ^* , can be recovered from the financial market equilibrium, then the stochastic discount factor can be "integrated" to obtain an approximation of the function Φ^* , since $\Phi^{*'}(Y_s) = \mu_s^*$. In this way Φ^* can be recovered from the state prices π^* .

Thus to implement either of the criteria (7) or (8), firms need to know the present-value prices π^* . "In principle" π^* can be deduced from the observed security prices q^* of the underlying exchange equilibrium (x^*, z^*, q^*) on the financial markets. But even if the spanning assumption were approximately satisfied, deducing π^* from the security prices would be a daunting task, since it

³The derivation of the criterion V^k in (8) was motivated by the observation in Magill-Quinzii (2006) that if firms maximize the analogue of the market-value criterion $\sum_{s \in S} \rho_s(a_k, , a_{-k}^*) \mu_s^* y_s^k - a_k$ under the assumptions of the \mathcal{P} model, then the resulting allocation is (generically) not Pareto optimal.

would involve inverting a matrix of very large dimension: the equilibrium pricing formula $q^* = \pi^* V^*$ combined with the assumption that V^* is invertible implies $\pi^* = q^* V^{*-1}$ from which, if ρ^* is known, μ^* can be deduced by (2) and (3). In practice, to the best of our knowledge, no empirical paper on asset pricing has ever attempted such a calculation, suggesting that it is extremely unlikely that firms would even attempt to calculate π^* from security prices in this way.

Empirical papers in finance which focus on explaining the prices of individual firms' securities typically base their analysis on the CAPM formula, which can be viewed as the first step in a "moment approach" to asset pricing. The CAPM formula prices the discounted mean of a security return and its covariance (beta) with the market. These are the first two terms which appear if the SDF $\Phi^{*\prime}(Y_s)$ in the pricing formula (6) is developed in a Taylor series expansion. More recently empirical studies have introduced the third order co-moment, i.e. the co-skewness of the securities' rates of return with the market. We will apply this "moment" approach to the firms' criteria by expanding the functions $\Phi^{*}(Y_s)$ and $\Phi^{*\prime}(Y_s)$ which appear in the criteria (7) and (8) in Taylor series expansions. We will find that there are two advantages to this approach: first a "moment" representation of the criteria (7) and (8) expresses in a more intuitive way how a firm should choose its production plan so that it fits optimally with what other firms are doing, using " prices" for co-moments which can be deduced from the security prices by simple regression; the second is that the two objectives (7) and (8), which appear to be rather different criteria in Proposition 1, become the same function in the moment representation.

Before exploring the moment approach to the objectives of the firms, we first derive the moment formula for the prices of the securities in an exchange equilibrium (x^*, z^*, q^*) .

3. Moment Formula for Security Prices

Two properties of the convolution function Φ^* in (4) are needed to obtain a moment representation of the pricing formula (6) and the criteria (7) and (8): Φ^* must be smooth and the Taylor series expansion of $\Phi^*(Y_s^*)$ must converge to $\Phi^*(Y_s^*)$ in a neighborhood of the mean aggregate output. The assumption that the agents' date 1 utility functions are smooth and strictly concave ensures that Φ^* is smooth, a necessary condition for writing the Taylor expansion series, but it does not guarantee that the series converges and coincides with the value of the function.

Let $A = \{a = (a_1, \ldots, a_K) \in \mathbb{R}^K_+ \mid \sum_{k \in K} a_k < \sum_{i \in I} \omega_0^i\}$ denote the set of feasible date 0 investments by the firms.

Assumption TS. For all $a^* \in A$, y^* feasible given a^* , and $s \in S$

$$\Phi^{*}(Y_{s}^{*}) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{*(n)}(\overline{Y}^{*})(Y_{s}^{*} - \overline{Y}^{*})^{n}$$

$$\Phi^{*'}(Y_{s}^{*}) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^{*})(Y_{s}^{*} - \overline{Y}^{*})^{n}$$

where $Y_s^* = \sum_{k \in K} y_s^{*k}$, $\overline{Y}^* = E(Y^*)$ in the \mathcal{SN} model and $\overline{Y}^* = E_{a^*}(Y)$ in the \mathcal{P} model.

Example. Assumption TS is an assumption on both the preferences and the technology. For it requires the functions Φ^* and $\Phi^{*'}$ to be analytic at \overline{Y}^* , which is an assumption on the preferences, and it requires that Y_s^* is in the radius of convergence around \overline{Y}^* , which limits the variability of aggregate output. To see how stringent this restriction is, consider the familiar case where the consumer sector of the economy can be summarized by a representative agent with a CRRA utility function

$$U(Y) = \frac{1}{1-\alpha} \left(Y_0^{1-\alpha} + \delta E(Y^{1-\alpha}) \right)$$

for $\alpha > 0$: when $\alpha = 1$, the Bernouilli index is $\ln(Y)$ and the reasoning is the same as with $\alpha \neq 1$. If (a^*, y^*) is feasible and $Y_0^* = \sum_{i \in I} \omega_0^i - \sum_{k \in K} a_k^*$, then

$$\Phi^*(Y_s^*) = \frac{\delta}{1 - \alpha} \frac{(Y_s^*)^{1 - \alpha}}{(Y_0^*)^{-\alpha}}, \qquad s \in S$$

and the Taylor series expansion of $\Phi^*(Y_s^*)$ around \overline{Y}^* is of the form $\sum_{n=0}^{\infty} v_n$ with

$$v_n = \delta(Y_0^*)^{\alpha} (-1)^{n+1} \frac{\alpha(\alpha+1)\dots(\alpha+n-2)}{n!} \left(\overline{Y}^*\right)^{-\alpha-n+1} \left(Y_s^* - \overline{Y}^*\right)^n, \ n \ge 2$$

 v_0 and v_1 are easily calculated but they do not influence the convergence of the series.

$$\left|\frac{v_{n+1}}{v_n}\right| = \frac{\alpha + n - 1}{n+1} \frac{\left|Y_s^* - \overline{Y}^*\right|}{\overline{Y}^*}$$

so that the series converges if $\frac{\left|Y_{s}^{*}-\overline{Y}^{*}\right|}{\overline{Y}^{*}} < 1$, namely if $Y_{s}^{*} \in (0, 2\overline{Y}^{*})$.⁴ The terms in the Taylor expansion are of the order of $\left(\frac{Y_{s}^{*}-\overline{Y}^{*}}{\overline{Y}^{*}}\right)^{n}$, so that the smaller the fluctuations in aggregate output around its mean, the faster the terms in the Taylor expansion become negligible.

Since in practice variations in aggregate output of the order of 20% are considered very large, the limit placed by Assumption TS of a 100% variation in aggregate output around its mean does

⁴A more abstract proof consists in noting that the power function in the complex plane is analytic except at zero, with a radius of convergence equal to the distance to the closest singularity, which is at zero.

not seem too restrictive. Of course realism would require that the model is extended to an infinite horizon, but we restrict the analysis of this paper to the simpler two-period model.

Under Assumption TS, the pricing formula (6) for an exchange equilibrium implies that for all $j \in J$

$$q_{j}^{*} = E^{*} \left(\Phi^{*\prime}(Y^{*}) \left((V^{j*} - \overline{V}^{j*}) + \overline{V}^{j*} \right) \right) \\ = \frac{\overline{V}^{j*}}{1 + r^{*}} + E^{*} \left(\sum_{n=1}^{\infty} \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^{*}) \left(Y^{*} - \overline{Y}^{*} \right)^{n} \left(V^{j*} - \overline{V}^{j*} \right) \right)$$
(9)

where the interest rate r^* is defined by the price of the riskless income stream, $\frac{1}{1+r^*} = E^*(\Phi^{*'}(Y^*))$, and where $\overline{V}^{j*} = E^*(V^{j*})$.

For a pair of random variables (x, y) with means \bar{x} and \bar{y} and a pair of non-negative integers (m, n) the central co-moment (m, n) is defined by

$$como^{(m,n)}(x,y) = E\left((x-\bar{x})^m(y-\bar{y})^n\right)$$
 (10)

For (m, n) = (1, 1) the co-moment is the covariance; for (m, n) = (2, 1) the co-moment is the coskewness of y with respect to x (we adopt the convention of the finance literature that the squared deviation is in the first variable); for (m, n) = (3, 1) the co-moment is the co-kurtosis of y with respect to x. For (m, n) = (2, 0) the co-moment is the variance of x, for (m, n) = (3, 0) it is the (un-normalized) skewness of x, and for (m, n) = (4, 0) it is the (un-normalized) kurtosis of x. The pricing formula introduces the co-moments (m, 1) for $m \ge 1$, and we will see that the moment representation of the criteria (7) and (8) introduces all co-moments (m, n) with $m \ge 0, n > 0$.

With this notation, the pricing formula (9) can be written as

$$q_j^* = \frac{\overline{V}^{j*}}{1+r^*} + \sum_{n=1}^{\infty} c_{n+1}^* \operatorname{como}^{(n,1)}(Y^*, V^{j*}), \quad \text{with} \quad c_{n+1}^* = \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^*)$$
(11)

(11) expresses how the random income stream V^{j*} is priced in equilibrium; the risk premium $\frac{\overline{V}^{j*}}{1+r^*} - q_j^*$ depends on the way the purely risky part $V^{j*} - \overline{V}^{j*}$ of the income stream co-moves with the purely risky part $Y^* - \overline{Y}^*$ of aggregate output, as expressed by the c_{n+1}^* -weighted sum of the co-moments como^(n,1) (Y^*, V^{j*}) , c_{n+1}^* reflecting the weight attached by the social welfare function to the co-moments of order n + 1 at \overline{Y}^* .

Since the infinite series in (11) converges, co-moments of high order can be neglected. As shown in the example above, the smaller the fluctuations in aggregate risk, the faster the terms become negligible. In the finance literature the valuation expression (in rate-of-return form) is typically restricted to the first 3 or 4 terms. If we neglect the co-moments of order 5 or more, the price q_j^* can be expressed as

$$q_j^* = \frac{\overline{V}^{j*}}{1+r^*} + c_2^* \operatorname{cov}(Y^*, V^{j*}) + c_3^* \operatorname{co-skew}(Y^*, V^{j*}) + c_4^* \operatorname{co-kurt}(Y^*, V^{j*})$$

with $c_2^* = \Phi^{*''}(\overline{Y}^*)$, $c_3^* = \frac{1}{2}\Phi^{*'''}(\overline{Y}^*)$, $c_4^* = \frac{1}{6}\Phi^{*(4)}(\overline{Y}^*)$. The coefficients c^* can be expressed as functions of the preferences of the agents, their weights in the social welfare function (their wealth) and the derivatives of their shares $x^i(\overline{Y}^*)$ in the solution of the allocation problem (4) at (\overline{Y}^*) . Taking derivatives of the function Φ^* defined in (4) and using the first-order conditions gives

$$c_{2}^{*} = \Phi^{*''}(\overline{Y}^{*}) = \sum_{i \in I} \alpha_{i}^{*} u_{1}^{i\,''}(x^{i}(\overline{Y}^{*})) \left(x^{i'}(\overline{Y}^{*})\right)^{2}$$

$$c_{3}^{*} = \frac{1}{2} \Phi^{*'''}(\overline{Y}^{*}) = \frac{1}{2} \sum_{i \in I} \alpha_{i}^{*} u_{1}^{i\,(3)}(x^{i}(\overline{Y}^{*})) \left(x^{i'}(\overline{Y}^{*})\right)^{3}$$

In an efficient allocation of the random aggregate output, each agent's consumption stream $x^i(Y)$ is co-monotone with aggregate output, so that $x^{i'}(\overline{Y}^*) > 0, i \in I$. Thus if agents are risk averse, $(u^{i''} < 0)$, and have preference for positive skewness —or dislike negative skewness i.e. are "prudent"— $(u^{i'''} > 0)$, then $c_2^* < 0$ and $c_3^* > 0$. The term $c_4^* = \frac{1}{6}\Phi^{*(4)}(\overline{Y}^*)$ does not necessarily inherit the sign of the fourth derivative of the agents' utility functions since

$$\Phi^{*(4)}(\overline{Y}^{*}) = \sum_{i \in I} \alpha_{i}^{*} \left(u_{1}^{i\,(4)} \left(x^{i}(\overline{Y}^{*}) \right) \left(x^{i\prime}(\overline{Y}^{*}) \right)^{4} + 3u_{1}^{i\,(3)} \left(x^{i}(\overline{Y}^{*}) \right) \left(x^{i\prime}(\overline{Y}^{*}) \right)^{2} x^{i\,\prime\prime}(\overline{Y}^{*}) \right)$$

Thus even if $u_1^{i(3)} > 0$ and $u_1^{i(4)} < 0$, since $\sum_{i \in I} x^{i''}(\overline{Y}^*) = 0$, the sign of $\Phi^{*(4)}$ is ambiguous if the sharing rule is not linear $(x^{i''}(\overline{Y}^*) \neq 0$ for some i).

Formula (11) with agent-specific coefficients was first derived in the finance literature by Rubinstein (1973) and by Kraus-Litzenberger (1976, 1981) as an equilibrium formula stopped at the co-moment of order 3, generalizing the CAPM formula. Pricing formulae in finance are typically expressed in return form, which is more convenient for empirical analysis. If R^{j*} denotes the return on security j, defined by $R^{j*} = \frac{V^{j*}}{q_j^*}$, if $R^{M*} = \frac{Y^*}{q_M^*}$ with $q_M^* = \sum_{J \in J} q_j^*$ denotes the return on the market portfolio of all equity, and $R^* = 1 + r^*$ denotes the return on a riskless bond, then (11) can be written as

$$E^*(R^{j*}) - R^* = -\sum_{n=1}^{\infty} \gamma_{n+1}^* \operatorname{como}^{(n,1)}(R^{M*}, R^{j*}), \quad \text{with} \quad \gamma_{n+1}^* = (q_M^*)^n c_{n+1}^*(1+r^*)$$
(12)

If for some \bar{n} the terms of order $n > \bar{n}$ are negligible and the number of securities exceeds \bar{n} , then in principle the coefficients γ_{n+1}^* and thus c_{n+1}^* can be obtained by linear regression of the excess returns of the securities on the co-moments of these returns with the market portfolio. Recently there has been a revival of interest in the three co-moment version of the pricing formula since adding a preference for skewness over and above mean-variance preferences can help to explain "puzzles" in observed investors' portfolios and in the pricing of assets which cannot be explained by the standard mean-variance model. An analysis of portfolio holdings of 60,000 investors at a large discount brokerage firm in the period 1991-1996 revealed that the majority of investors hold very undiversified portfolios, typically with only a few securities (Mitton and Vorknik (2007)). This study served to confirm results of earlier studies (e.g. Blume and Friend (1975)) which found that the diversification in investors' portfolios is far less than that prescribed by CAPM. Although a number of explanations for this phenomenon have been proposed, the simplest and most natural from a theoretical point of view is based on the observation that many investors have a preference for positive skewness, and with such preferences optimal portfolios are less diversified than those which are optimal with mean-variance preferences: see Briec-Kertens-Jokung (2007) for an analysis of the efficient frontier of mean-variance-skewness portfolios.

The standard mean-variance model which stops the excess return formula (12) at the covariance term does a poor job of explaining observed excess returns in the postwar period (see e.g. Fama and French (1992)). Harvey and Siddique (2000) have shown that taking into account the coskewness term in (12) greatly improves the fit of the model: typically securities with higher than average expected returns (e.g. small company stocks or those with high book to market value) have negative coskewness, while those with lower than average expected returns (e.g. large companies) have positive coskewness.

4. Moment Representation of Criterion for the State-of-Nature Model

We are now in a position to derive the moment representation of a firm's criterion which leads to Pareto optimality in the state-of-nature model. To express the criterion in a natural way we decompose the aggregate output into the contribution y^k of firm k and the contribution $Y^{-k} = \sum_{k' \neq k} y^{k'}$ of the other firms: $Y = y^k + Y^{-k}$.

Proposition 2. Let (a^*, y^*) be a choice of production plans by the firms and (x^*, z^*, q^*) an associated exchange equilibrium. The consumption-production plan (x^*, a^*, y^*) is a Pareto optimal allocation of the production economy in which Assumptions EE and TS hold, if and only if for each firm $k \in K$ the production plan (a_k^*, y^{k*}) is an extremum of the co-moment criterion

$$\widetilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} \frac{c_{n+1}^*}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \operatorname{como}^{(n+1-j,j)}(Y^{-k*}, y^k) - a_k$$
(13)

Proof. By Proposition 1 (x^*, a^*, y^*) is Pareto optimal if and only if, for each firm $k \in K$, the plan (a_k^*, y^{k*}) maximizes

$$M(a_k, y^k) = E(\mu^* y^k) - a_k = E(\mu^*)E(y^k) + E\left(\mu^*(y^k - E(y^k)) - a_k = \frac{E(y^k)}{1 + r^*} + E\left(\mu^*(y^k - E(y^k)) - a_k\right) - a_k = E(\mu^* y^k) - a_k = E(\mu^* y^k$$

subject to the constraint $T_k(-a_k, y^k) \leq 0$. Thus the first-order conditions, which are necessary and sufficient for optimality,

$$D_{y^{k}}M(a_{k}^{*}, y^{k*}) = \lambda_{k}D_{y^{k}}T_{k}(-a_{k}^{*}, y^{k*}), \qquad 1 = \lambda_{k}\frac{\partial}{\partial y_{0}^{k}}T_{k}(-a_{k}^{*}, y^{k*})$$
(14)

where $D_{y^k}M = \left(\frac{\partial M}{\partial y^k_s}\right)_{s \in S}$, must be satisfied for some $\lambda_k > 0$. Since by (5) $\mu_s^* = \Phi^{*\prime}(Y_s^*), s \in S$, developing $\Phi^{*\prime}(Y_s^*)$ in Taylor series about the mean \overline{Y}^* gives

$$M(a_{k}, y^{k}) = \frac{E(y^{k})}{1 + r^{*}} + E\left(\sum_{n=1}^{\infty} c_{n+1}^{*} \left(Y^{*} - \overline{Y}^{*}\right)^{n} \left(y^{k} - E(y^{k})\right)\right) - a_{k}$$

$$= \frac{E(y^{k})}{1 + r^{*}} + E\left(\sum_{n=1}^{\infty} c_{n+1}^{*} \left(Y^{-k*} - \overline{Y}^{-k*} + y^{k*} - \overline{y}^{k*}\right)^{n} \left(y^{k} - E(y^{k})\right)\right) - a_{k}$$

$$= \frac{E(y^{k})}{1 + r^{*}} + \sum_{n=1}^{\infty} c_{n+1}^{*} \sum_{j'=0}^{n} \binom{n}{j'} E\left[\left(Y^{-k*} - \overline{Y}^{-k*}\right)^{n-j'} \left(y^{k*} - \overline{y}^{k*}\right)^{j'} \left(y^{k} - E(y^{k})\right)\right] - a_{k}$$
(15)

with $c_{n+1}^* = \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^*)$, for $n \ge 1$. Note that when differentiating $M(a_k, y^k)$ with respect to y^k (to check that the FOCs (14) are satisfied) the firm must take the term $(y^{k*} - \overline{y}^{k*})^{j'}$, which comes from the development of $(Y^* - \overline{Y}^*)^n$, as given, while differentiating only the term $y^k - E(y^k)$: this is the competitive assumption for this model. It is however an awkward expression for a firm to "maximize" since it must only take into account a part of the terms involving its actions. A more natural expression can be obtained by noting that the product terms in (15) satisfy

$$D_{y^{k}}E\left[\left(Y^{-k*} - \overline{Y}^{-k*}\right)^{n-j'}\left(y^{k*} - \overline{y}^{k*}\right)^{j'}\left(y^{k} - E(y^{k})\right)\right]_{y^{k} = y^{k*}}$$
$$= \frac{1}{j'+1}D_{y^{k}}E\left[\left(Y^{-k*} - \overline{Y}^{-k*}\right)^{n-j'}\left(y^{k} - E(y^{k})\right)^{j'+1}\right]_{y^{k} = y^{k*}}$$

so that the FOCs (14) are satisfied if and only if

$$D_{y^k}\widetilde{M}(a_k^*, y^{k*}) = \lambda_k D_{y^k} T_k(-a_k^*, y^{k*}), \qquad 1 = \lambda_k \frac{\partial}{\partial y_0^k} T_k(-a_k^*, y^{k*})$$

where

$$\widetilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c_{n+1}^* \sum_{j=1}^{n+1} \frac{1}{j} \binom{n}{j-1} E\left[\left(Y^{-k*} - \overline{Y}^{-k*} \right)^{n-j+1} \left(y^k - E(y^k) \right)^j \right]$$

with
$$j = j' + 1$$
 in (15). Since $\frac{1}{j} \binom{n}{j-1} = \frac{1}{n+1} \binom{n+1}{j}$, $\widetilde{M}(a_k, y^k)$ is the function in (13). \Box

In the state-of-nature model, the market value that a firm should maximize can also be written as

$$M(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c_{n+1}^* \operatorname{como}^{(n,1)}(Y^{-k*} + y^{k*}, y^k) - a_k$$
(16)

As noted in the proof of Proposition 2 this does not provide a natural criterion for firm k, since on the one hand it needs to choose y^k , but on the other hand it needs to take its contribution y^{k*} as given. The awkward nature of the criterion can be seen in extreme form in the case where firm k's output is independent of the output of the other firms and the preferences are mean-variance, for in this case (16) reduces to

$$M(a_k, y^k) = \frac{E(y^k)}{1 + r^*} + c_2^* \operatorname{cov}(y^{k*}, y^k) - a_k$$
(17)

This expression gives a way of checking whether the known choice y^{k*} (feasible with a_k^*) is optimal, but it does not give a way of finding the optimal choice (a_k^*, y^{k*}) , in essence because it does not provide a criterion for evaluating the trade-off between return and risk (mean and variance) in production.⁵ The Arrow-Debreu criterion in Proposition 1(i), usually interpreted as the "perfectly competitive" criterion, requires that the firm takes the present-value prices π_s^* as given. When expressed in co-moment form, this price taking requirement is equivalent to taking the coefficients $(c_{n+1}^*)_{n\geq 1}$ and the aggregate output Y^* as given. Given the pricing formula (11), the coefficients $(c_{n+1}^*)_{n\geq 1}$ can be considered as the "prices" of the co-moments, and it seems natural for firms to take these price coefficients as given. However taking Y^* as given when evaluating the effect of their plan (a_k, y^k) on the co-moments leads to the problem mentioned above. To avoid this awkward aspect of the competitive criterion (17), Proposition 2 transforms it into the criterion

$$\widetilde{M}(a_k, y^k) = \frac{E(y^k)}{1 + r^*} + \frac{c_2^*}{2} \operatorname{var}(y^k) - a_k$$
(18)

which leads to the same FOCs as $M(a_k, y^k)$. The advantage of expressing the firm's criterion in this form is that the coefficients which appear in \widetilde{M} only depend on the prices (r^*, c^*) and the

⁵The problem is not visible in the traditional finance literature which assumes that the economy is mean-variance and that firms choose among projects with exogenously given expected returns and covariance with aggregate output.

production Y^{-k*} of the other firms, variables which are clearly outside the domain of choice of firm k.

If the analysis were to begin with the firm's value expressed via the co-moment pricing formula (11) rather than starting with Proposition 1(i) which gives the abstract form of the market-value criterion for a firm, then it might seem natural that the competitive criterion for a firm which takes the prices (r^*, c^*) and the production Y^{-k*} of the other firms as given, be expressed as

$$\frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c_{n+1}^* \operatorname{como}^{(n,1)}(Y^{-k*} + y^k, y^k) - a_k$$
(19)

In a well known paper, Stiglitz (1972) took it for granted— in the mean-variance framework $(c_{n+1}^* = 0 \text{ for } n \ge 2)$ —that (19) is indeed the natural criterion for the firm: he arrived at the controversial conclusion that, when the CAPM is generalized to a setting with production, a competitive system of markets leads to a misallocation of investment since the resulting equilibrium is not Pareto optimal. In the mean-variance case where firms are independent (one of the cases considered by Stiglitz) (19) reduces to

$$\frac{E(y^k)}{1+r^*} + c_2^* \operatorname{var}(y^k) - a_k \tag{20}$$

with a weight on the variance which is twice the weight in (18), the correct criterion from the normative point of view. Because in a CAPM equilibrium firms' shareholders hold diversified portfolios, a firm's own risk should not be weighted as heavily as in the naive criterion (20).

One consequence of transforming a firm's criterion into the co-moment form is that convexity of the maximum problem of a firm may be lost, since \widetilde{M} may not be a concave function of y^k . For example in the mean-variance-skewness model ($c_{n+1}^* = 0$ for $n \ge 3$), the transformed criterion is

$$\widetilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + c_2^* \left(\operatorname{cov}(Y^{-k*}, y^k) + \frac{1}{2} \operatorname{var}(y^k) \right) \\ + c_3^* \left(\operatorname{coskew}(Y^{-k*}, y^k) + \operatorname{coskew}(y^k, Y^{-k*}) + \frac{1}{3} \operatorname{skew}(y^k) \right) - a_k$$

While in the mean variance model $(c^2 < 0, c_3 = 0) \widetilde{M}(a_k, y^k)$ is concave in y^k , if agents have a preference for skewness $(u^{i''} > 0)$ and weight the mean-skewness trade-off sufficiently relative to the mean-variance trade-off, then the terms $\operatorname{coskew}(y^k, Y^{-k*})$ and $\operatorname{skew}(y^k)$ may make \widetilde{M} a non concave function of y^k .

Since the criterion $M(a_k, y^k)$ is linear and the technology T_k is convex, the FOCs (14) are necessary and sufficient for maximizing M subject to T_k : since any extremum of \widetilde{M} subject to the technology constraint satisfies (14), it is an optimal choice. The cost of the non-concavity of (13) is that not all the Pareto optimal allocations can be found by firms maximizing the objective functions $\widetilde{M}(a_k, y^k), \ k \in K.^6$

5. Moment Representation of Criterion for the Probability Model

In the \mathcal{P} model *s* denotes a realization of the firms' outputs $y_s = (y_{s_1}^1, \ldots, y_{s_K}^K)$, with aggregate output $Y_s = \sum_{k \in K} y_{s_k}^k$. The firms' choices of investment $a = (a_1, \ldots, a_K)$ at date 0 determine the probability distribution $(\rho_s(a))_{s \in S}$ of the firms' outcomes at date 1. If a^* is a vector of investment for the *K* firms and if (x^*, z^*, q^*) is an associated exchange equilibrium satisfying Assumption EE in which agents correctly anticipate the probability distribution $\rho(a^*)$, then agents will have the same stochastic discount factor $\mu^* = \Phi^{*'}(Y)$, so that x_s^* is the optimal distribution of the aggregate output Y_s among consumers for the social welfare function Φ^* . Proposition 1 then asserts that a necessary condition for the combined production-consumption plan (a^*, x^*) to be Pareto optimal is that the investment a_k^* of each firm *k* maximizes the expected contribution of firm *k* to social welfare

$$V(a_k) = E_{(a_k, a_{k_k}^*)} \Phi^*(Y) - a_k$$

where $E_{(a_k,a^*_{-k})}$ denotes the expectation operator for the probability distribution $\rho(a_k, a^*_{-k})$. Our objective is to find an alternative way of expressing this criterion, which is more readily implementable by a firm, i.e. which can be expressed in terms of variables which are observable. As in the previous section we find this expression by applying a Taylor series expansion to Φ^* : this leads to an expression involving co-moments between the production of firm k and the production of the other firms, which firm k's choice of investment a_k influences by changing the probability of the outcomes rather than by changing quantities as in the previous section. The resulting criterion will have the same form as the criterion \widetilde{M} in Proposition 2 if we assume that the probabilities of the outcomes of the firms other than k are not influenced by firm k's choice of investment, so that a_k has no external effect on the other firms.

Assumption NE. For all $a \in A$ and each $k \in K$

$$\sum_{s_k \in S_k} \rho_{(s_k, s_{-k})}(a_k, a_{-k}) \text{ does not depend on } a_k$$

⁶The situation is analogous to using a weighted sum of agents' utility functions to find a Pareto optimal allocation in the standard GE model, when agents' utility functions are quasi-concave but not concave. Any extremum of the weighted sum of utility functions (the social welfare criterion) subject to the feasibility constraints satisfies the FOCs for Pareto optimality, and in a convex economy is Pareto optimal, but it is not necessarily a maximum of the social welfare function.

Note that Assumption NE does not imply independence among the firms' outcomes: Assumption NE is satisfied when firms' outcomes are correlated but conditionally independent (see Magill-Quinzii (2007)).

To indicate that the co-moments between y^k and the output Y^{-k} of other firms depend on the choice a_k of firm k given the investment a_{-k} of the other firms, we write

$$\operatorname{como}^{(m.n)}(Y^{-k}, y^k; a_k, a_{-k}) = E_{(a_k, a_{-k})} \left(Y^{-k} - E_{a_{-k}}(Y^{-k}) \right)^m \left(y^k - E_{a_k}(y^k) \right)^n$$

Proposition 3. Let a^* be choice of investment by the firms, (x^*, z^*, q^*) an associated exchange equilibrium, and let Assumptions EE, TS, and NE be satisfied. The consumption-investment plan (x^*, a^*) is Pareto optimal if and only if, for each firm $k \in K$, the investment a_k^* is an extremum of the co-moment criterion

$$\widetilde{V}(a_k) = \frac{E_{a_k}(y^k)}{1+r^*} + \sum_{n=1}^{\infty} \frac{c_{n+1}^*}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \operatorname{como}^{(n+1-j,j)} \left(Y^{-k}, y^k; a_k, a_{-k}^*\right) - a_k$$
(21)

where the coefficient c_{n+1}^* are those of the pricing formula (11), $c_{n+1}^* = \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^*), n \ge 1$.

Proof: Under Assumption TS, for all $s \in S$, $\Phi^*(Y_s)$ coincides with its Taylor expansion series around \overline{Y}^* . Since a_k^* maximizes (8), the first-order condition

$$\frac{\partial}{\partial a_k} E_{(a_k,a_{-k}^*)} \left(\Phi^*(\overline{Y}^*) + \sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^*)}{n!} (Y - \overline{Y}^*)^n \right)_{a_k = a_k^*} - 1 \le 0, \quad \text{with equality if } a_k^* > 0$$

must be satisfied, where $\Phi^*(Y_s)$ has been developed in Taylor series. The first term can be omitted since for all a, $E_a(\Phi^*(\overline{Y}^*)) = \Phi^*(\overline{Y}^*)$ and is independent of a_k . Expanding $(Y - \overline{Y}^*)^n$ into the terms which depend on the output of firm k and those which depend on the output of all other firms, and using the convention that for any random variable x, \overline{x}^* denotes the expectation under the probability distribution $\rho(a^*)$, gives

$$\frac{\partial}{\partial a_k} E_{(a_k, a_{-k}^*)} \left(\sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^*)}{n!} \sum_{j=1}^n \binom{n}{j} \left(Y^{-k} - \overline{Y}^{-k*} \right)^{n-j} \left(y^k - E_{a_k^*}(y^k) \right)^j \right)_{a_k = a_k^*} - 1 \le 0$$
(22)

where by Assumption NE the term with index j = 0 has been omitted since the distribution of Y^{-k} is not influenced by a_k . The function in (22) whose derivative with respect to a_k must be non-positive at a_k^* (and zero if $a_k^* > 0$) can be used to check whether the choice a_k^* is optimal, but it does not provide an objective function which the firm can maximize to find its optimal investment: for the decision a_k^* must already be known to evaluate $E_{a_k^*}(y^k)$ and only some of the

effect of the action a_k must be taken into account. To transform (22) into the first-order condition for an objective function for firm k, we need to find a function $\tilde{V}(a_k)$ in which no prior choice a_k^* appears, with the property that if $\frac{d}{da_k}\tilde{V}(a_k) = 0$ for $a_k = a_k^*$, then (22) is satisfied. To obtain the appropriate transformation note that

$$\frac{\partial}{\partial a_{k}} E_{(a_{k},a_{-k}^{*})} \left(Y^{-k} - \overline{Y}^{-k*}\right)^{n-j} \left(y^{k} - E_{a_{k}^{*}}(y^{k})\right)^{j} \Big|_{a_{k}=a_{k}^{*}} = \frac{\partial}{\partial a_{k}} E_{(a_{k},a_{-k}^{*})} \left(Y^{-k} - \overline{Y}^{-k*}\right)^{n-j} \left(y^{k} - E_{a_{k}}(y^{k})\right)^{j} + E_{(a_{k},a_{-k}^{*})} j \left(Y^{-k} - \overline{Y}^{-k*}\right)^{n-j} \left(y^{k} - E_{a_{k}}(y^{k})\right)^{j-1} \frac{d}{da_{k}} E_{a_{k}}(y^{k}) \Big|_{a_{k}=a_{k}^{*}} \tag{23}$$

Substituting (23) into (22), we find that

$$\frac{\partial}{\partial a_k} \sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^*)}{n!} \sum_{j=1}^n \binom{n}{j} \operatorname{como}^{(n-j,j)} \left(Y^{-k}, y^k; a_k, a_{-k}^*\right) + \left[\sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^*)}{n!} \sum_{n=1}^{\infty} j\binom{n}{j} E_{(a_k, a_{-k}^*)} \left(Y^{-k} - \overline{Y}^{-k*}\right)^{n-j} \left(y^k - E_{a_k}(y^k)\right)^{j-1}\right] \frac{d}{da_k} E_{a_k}(y^k) - 1 \le 0$$
(24)

holds at a_k^* if and only if (22) is satisfied. Consider the term in square brackets which multiplies $\frac{d}{da_k}E_{a_k}(y^k)$. Since $j\binom{n}{j} = n\binom{n-1}{j-1}$, when it is evaluated at a_k^* it can be written as

$$\Phi^{*'}(\overline{Y}^{*}) + \sum_{n=2}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^{*})}{n!} n \sum_{j=1}^{n} \binom{n-1}{j-1} E_{a^{*}} \left(Y^{-k} - \overline{Y}^{-k*}\right)^{n-1-(j-1)} \left(y^{k} - \overline{y}^{k*}\right)^{j-1}$$
(25)

Setting j' = j - 1 in the second sum in (25) gives

$$\sum_{j'=0}^{n-1} \binom{n-1}{j'} E_{a^*} \left(Y^{-k} - \overline{Y}^{-k^*} \right)^{n-1-j'} \left(y^k - \overline{y}^{k^*} \right)^{j'} = E_{a^*} (Y - \overline{Y})^{n-1}$$

so that (25) reduces to

$$\Phi^{*'}(\overline{Y}^*) + \sum_{n=2}^{\infty} \frac{\Phi^{*(n)}(\overline{Y}^*)}{(n-1)!} E_{a^*}(Y - \overline{Y})^{n-1} = E_{a^*}(\Phi^{*'}(Y)) = \frac{1}{1 + r^*}$$

Now consider the first term in (24) consisting of the sum of the co-moments. For n = 1 the term is

$$\operatorname{como}^{(0,1)}\left(Y^{-k}, y^k; a_k, a_{-k}^*\right) = E_{(a_k, a_{-k}^*)}\left(y^k - E_{a_k}(y^k)\right) = 0$$

so that this term can be omitted, the summation beginning at n = 2. Thus the index can be shifted from n to n + 1 to begin with n = 1 and, since $c_{n+1}^* = \frac{\Phi^{*(n+1)}(\overline{Y}^*)}{(n)!}$, (24) can be written as $\frac{d}{da_k} \widetilde{V}(a_k) \leq 0$ where $\widetilde{V}(a_k)$ is the criterion in Proposition 3.

Proposition 1(ii) tells us that in order to obtain a Pareto optimum in the probability model each firm must choose its investment a_k to maximize the expected value of its contribution to social welfare $V(a_k) = E_{(a_k, a_{k,k}^*)} \Phi^*(Y) - a_k$, which in particular implies that the firm must "know" the social welfare function Φ^* . This looks like a very demanding requirement since in essence it requires that firms know the utility functions $(u^i)_{i \in I}$ and the distribution of income summarized by the coefficients $(\alpha_i^*)_{i \in I}$. Even though the model assumes that there are rich financial markets, in the representation V^k of a firm's criterion, the informational role of prices—by which they convey the requisite information to each firm to make its optimal investment decision— seems to be lost. The co-moment representation \widetilde{V}^k of the firm's criterion gives a simpler way of expressing the criterion when assumption TS holds, exhibiting the market information that can be used by a firm to infer the function Φ^* and make its (socially) optimal choice of investment: in addition to the probability distribution $\rho(a_k, a^{-k})$ and the investment decisions a_{-k}^* of the other firms, firm k needs to know the derivatives of the social welfare function Φ^* at the mean aggregate output, namely the pricing coefficients $c_{n+1}^* = \frac{1}{n!} \Phi^{*(n+1)}(\overline{Y}^*)$. If the risks are not too high and a good approximation for both the security prices and the firms' criterion can be obtained by keeping only a finite number of terms, then the needed coefficients c_{n+1}^* can be deduced (by regression) from the security prices using (11). In this way 'markets' once again provide firms with the requisite information. Both the criteria V^k and \widetilde{V}^k require that the firm know enough about Φ^* in an appropriate neighborhood of the mean aggregate output \overline{Y}^* . The real advantage of the co-moment representation \widetilde{V}^k is that it provides a simple way of obtaining an approximation of Φ^* in a neighborhood of \overline{Y}^* by recovering the value of the derivatives at \overline{Y}^* from the prices: a truncated Taylor series expansion then provides an approximate value of $\Phi^*(Y)$.

The striking feature of the two criteria \widetilde{M} for the state-of-nature model and \widetilde{V} for the probability model is that they are the same function of the co-moments, the difference being only in the way a firm's investment influences the function—through quantities in the SN model (in which probabilities are fixed) and through probabilities in the \mathcal{P} model (in which outcomes are fixed). To understand why the same criterion emerges from these two distinct ways of modeling uncertainty in production, let us simplify the technology of the SN model by writing it in parametric form $y^k(a) = (y_s^k(a_k))_{s \in S}, k \in K$. Another way of expressing the optimality of the investment decision is to note that (x^*, a^*, y^*) is Pareto optimal if and only if for each $k \in K$, a_k^* maximizes

$$E\left(\Phi^*(Y^{-k*} + y^k(a_k)) - a_k\right)$$
(26)

This can be readily deduced from the first-order conditions for Pareto optimality, which are sufficient in a convex economy: as in the probability model, to be optimal the investment of firm k must

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maximize its contribution to the expected discounted social welfare, net of its investment. Note that a_k^* maximizes (26) if and only if

$$E\left(\Phi^{*'}(Y^{-k*} + y^k(a_k^*))y^{k'}(a_k^*)\right) - 1 = 0$$

which is equivalent to the property that a_k^* maximizes $E\left(\Phi^{*'}(Y^*)y^k(a_k)\right) - a_k$, which is just the market value criterion (7). In the probability model, the criterion (8) for the firm is directly taken to be the maximization of the firm's contribution to social welfare, since this is no longer equivalent to a form of profit maximization. Thus since in the two models firm k's optimal investment maximizes the same social welfare function, only by different channels, the co-moment criteria \widetilde{M} and \widetilde{V} end up being the same expression. The proof of Proposition 3 could probably be adapted to cover the two models at the same time. However since most readers are likely to be more familiar with the $S\mathcal{N}$ model than the \mathcal{P} model and the market-value criterion is well accepted in the economics and finance literature, a separate treatment of the two models seems clearer and more helpful. Going through the steps which lead from the market value M to the co-moment criterion \widetilde{M} helps explain why, when market-value maximization is expressed in a "Nash/competitive" form where firms take the security prices and the actions of the other firms as given, the coefficients are not those that are obtained by a naive use of the pricing formula (11).

Alternative Representations of Technology. For both the SN and the P model we have chosen a representation for the firms' technologies which leads to a simple derivation of the co-moment criterion—a transformation function T_k for each firm in the SN model and a joint probability distribution $\rho(a)$ for the firms' outcomes in the P model. The results can be extended to alternative representations of the firms' technologies which are more realistic. For the SN model, even if "states of nature" or primitive causes which influence a firm's profit outcomes could be known, it is unlikely that a firm could vary its production in each state independently, as implied by the increasing differentiable transformation function T_k . The proof of Proposition 2 (and the objective \widetilde{M}) does not however depend on the number of constraints that limit the production possibilities of firm k, and would go through if for example the firm could invest in several projects with fixed risk characteristics, or at the extreme if its technology was described by a production function $y^k(a_k) = (y_s^k(a_k))_{s \in S}$.

For the probability model we have assumed that each firm chooses a single parameter a_k which can be thought of as a scale parameter, but the possibility of a choice of technique has not been taken into account. A more developed model in which NE is satisfied can be obtained by assuming that the probability distribution of each firm's output depends on a vector of parameters ν^k chosen

6. Conclusion

by the firm (choice of technique) and a vector of exogenous shocks $\eta = (\eta_1, \ldots, \eta_m)$, which can be either economy wide or sectoral and affect firms or sub-groups of firms: conditional on the value of the exogenous shocks, the firms' probability distributions are independent. The choice of parameters ν^k which affect the expected value, variance and more generally the moments of the firms' outputs given η , has associated with it a cost $a_k = C_k(\nu^k)$ which is incurred at date 0. Given the choice of parameters $\nu = (\nu^1, \ldots, \nu^K)$ by each of the firms, the joint probability distribution of firms' outcomes is given by

$$\rho_s(\nu) = \int \rho_{s_1}^1(\nu^1|\eta) \dots \rho_{s_K}^K(\nu^K|\eta) dG(\eta)$$

where $G(\cdot)$ is the probability distribution for the economy wide and sectoral shocks η . For a vector of choices $(\nu^{k*})_{k\in K}$ to be optimal, each firm $k\in K$ must choose ν^{k*} which maximizes

$$E_{(\nu^{k}, \nu^{-k*})}\Phi(Y) - C_{k}(\nu^{k})$$

and the same reasoning as in Proposition 3 leads to the criterion

$$\widetilde{V}(\nu^{k}) = \frac{E_{\nu^{k}}(y^{k})}{1+r^{*}} + \sum_{n=1}^{\infty} \frac{c_{n+1}^{*}}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \operatorname{como}^{(n+1-j,j)} \left(Y^{-k}, y^{k}; \nu^{k}, \nu^{-k*}\right) - C_{k}(\nu^{k})$$

6. Conclusion

There are two basic models of production under uncertainty, the state-of-nature model and the probability model which differ in the way they express how decisions by firms affect the random outcomes. The criteria that firms should maximize in order to be led to Pareto optimal outcomes are different in the two models, but the information needed to implement these criteria is essentially the same: firms need to know the stochastic discount factor of the economy.

Although in principle with full spanning of the securities the stochastic discount factor can be recovered from security prices by inverting the payoff matrix, this is infeasible in practice given the number of securities and states involved. We thus go back to an idea originally explored by Rubinstein (1973) and Kraus-Litzenberger (1976, 1981), developing the social welfare function in Taylor series expansion around the mean aggregate output. They used the development in Taylor series expansion to obtain an alternative way of expressing the price of an asset in terms of comments of its payoff with aggregate output: we take it a step further by showing how a co-moment criterion for a firm can be derived, whose extrema give the optimal investment/production decision

References

for the firm. The criterion has the interesting property that it is the same function for the two models, thus providing a unifying framework for the two different way of modeling production risks.

The analysis has been developed in a setting where the production risks of the firms can be fully diversified by all agents in the economy. The analysis needs to be extended to the more realistic setting where some stakeholders in a firm can not diversify their positions—managers and workers for incentive reasons, or large shareholders for signaling reasons. We conjecture that a criterion of similar nature can be derived in this case but with larger coefficients on the terms involving high powers for the firm's idiosyncratic risks, reflecting the increased importance of these risks for social welfare—a topic that we leave for future research.

References

Blume, M.E. and I. Friend (1975), "The Asset Structure of Individual Portfolios and Some Implications for Utility Functions," *The Journal of Finance*, 30, 585-603.

Briec, W., Kerstens, K. and O. Jokung (2007), "Mean-Variance-Skewness Portfolio Performance Gauging: A General Shortage Function and Dual Approach," *Management Science*, 53, 135-149.

Fama, E.F. and K.R. French (1992) "The Cross-Section of Expected Stock Returns," *Journal of Finance*, 47, 427-465.

Harvey, C.R. and A. Siddique (2000), "Conditional Skewness in Asset Pricing Tests," *Journal of Finance*, 55, 1263-95.

Judd, K.L. and S.M. Guu (2001), "Asymptotic Methods for Asset Market Equilibrium Analysis," *Economic Theory*, 18, 127-157.

Kraus A. and R. Litzenberger (1976), "Skewness Preference and the Valuation of Risk Assets," *Journal of Finance*, 31, 1085-1100.

Kraus A. and R. Litzenberger (1983), "On the Distributional Conditions for a Consumption-Oriented Three Moment CAPM," *Journal of Finance*, 38, 1381-1991.

Magill, M. and M. Quinzii (2006), "Normative Properties of Stock Market Equilibrium with Moral Hazard," forthcoming *Journal of Mathematical Economics*.

Magill, M. and M. Quinzii (2007), "The Probability Approach to General Equilibrium with

Production," forthcoming *Economic Theory*.

Mitton, T. and K. Vorkink (2007), "Equilibrium Underdiversification and the Preference for Skewness," *Review of Financial Studies*, 20, 1255-1288.

Rubinstein, M.E. (1973), "The Fundamental Theorem of Parameter-Preference Security Valuation," *Journal of Financial and Quantitative Analysis*, 8, 61-69.

Simkowitz, M. A. and W.L. Beedles (1978), "Diversification in a Three Moment World," *Journal of Financial and Quantitative Analysis*, 13, 927-941.

Stiglitz, J. E. (1972), "On the Optimality of the Stock Market Allocation of Investment," *Quarterly Journal of Economics*, 86, 25-60.