Informational Robustness of Competitive Equilibria

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Abstract

Consider an exchange economy with complete information. We perturb this economy by assuming that each agent's observation about the true state of the world is noisy. The paper investigates the robustness of equilibria of the complete information economy with respect to incomplete information. We provide conditions under which complete information equilibria are limits of equilibria of the economies with incomplete information, as the noise in the signal converges to zero.

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1 Introduction

The model of an exchange economy with complete information is best viewed as an idealized description of a world with "sufficiently small" informational asymmetries. Abstracting from "small" informational asymmetries is only legitimate if the equilibria of the complete information model are close to those of a model in which all informational asymmetries are accounted for. In other words, the competitive equilibrium should not change too much when a small amount of asymmetric information is introduced, and the economy with small informational asymmetries should be almost indistinguishable from a complete information economy. We refer to this desirable property as robustness. The objective of this paper is to characterize when complete information competitive equilibria are robust.

Consider a model of an exchange economy where agents have complete information about an underlying state. In other words, think of agents receiving a signal about the state that is completely accurate. Incomplete information then means that agents' signals are no longer perfect, but rather are subject to some noise. We will investigate under what conditions the complete information competitive equilibria are limits of equilibria of the economies with incomplete information, as the noise in the signal converges to zero.

For economies with incomplete information, equilibrium concepts can be defined in many different ways. Because the objective of this paper is to show that economies with small informational asymmetries are almost indistinguishable from complete information economies, the equilibrium concept used for the incomplete information economies should resemble as closely as possible that of a standard competitive equilibrium. That is,

1. trade should occur at linear prices;

2. the outcome should be Pareto efficient;

3. only minimal intervention of a central authority should be necessary.

What type of interventions are considered to be minimal? All interventions will be in the form of lump-sum taxes or subsidies. However, these transfers should only be of a limited size. For example, the central authority cannot implement a particular outcome by threatening large penalties, even if the penalties only occur with a small probability in equilibrium.

In Section 2 we show by means of examples that competitive equilibria need not be robust for some specifications of noise. One of the difficulties is that even if the noise in each agent's signal is small, an agent can still have more accurate information than the remaining agents, and can therefore manipulate the outcome. In the terminology of McLean and Postlewaite (1999), such agents are not "informationally small."

The main results of our paper, Theorem 2 and Theorem 3, show that the McLean and Postlewaite (1999) informational smallness condition is the critical property for robustness of competitive equilibria. Theorem 2 shows that complete information equilibria are robust, if all agents become arbitrarily small informationally as the noise in the signals vanishes. In contrast, Theorem 3 shows that one can always find complete information economies with non-robust equilibria if the informational smallness conditions is violated. In other words, informational smallness is in essence a necessary and sufficient condition for robustness of competitive equilibria with respect to incomplete information.

The proof of Theorem 2 uses Theorem 1, a result which provides sufficient conditions for the supportability of Pareto efficient allocations as incentive compatible competitive equilibria. Proposition 1 shows that if a particular allocation itself is not supportable, we can make it supportable by modifying it on a small set provided a condition combining agents' informational smallness and the size of gains from trade is fulfilled. [1]

Our main results do not only hold in the case where each agent's information becomes complete as the noise in the signal vanishes. Rather, each agent's information in the limit can be incomplete, as long as the pooled information of all but any two agents in the economy is complete. If this condition on agents' information holds then the equilibria of the incomplete information economy will resemble the competitive equilibria of a complete information economy, despite the fact that each agent's information is incomplete even if there is no noise. This demonstrates that there is a large class of economies with asymmetric information for which the abstraction of a complete information Arrow-Debreu economy is applicable.

By addressing the question how models of exchange economies with complete information perform as approximation of economies with incomplete information, our work is most closely related to Gul and Postlewaite (1992), and McLean and Postlewaite (1999). They show that the informational smallness condition mentioned above is sufficient to implement allocations that are approximately ef-

[1] It is interesting to note that Heifetz and Minelli (1997) address the question, whether rational expectations equilibria themselves become incentive compatible as economies are replicated. Clearly, such a result requires stronger conditions than when allocations are modified (i.e., when transfers are allowed).

ficient. The motivation for this line of research is to identify situations in which asymmetries of information are not significant, and a mechanism designer could therefore ignore issues of misrepresentation and incentive compatibility. As mentioned earlier the main motivation for our work is to identify situations in which informational asymmetries do not matter for a particular market institution, the competitive mechanism. In order to remain in a competitive framework, the allocations in incomplete information economies must be supportable by prices, and hence must be ex-post efficient. In contrast, given the motivation for the results in Gul and Postlewaite (1992), and McLean and Postlewaite (1999), approximate ex-post efficiency is all that is required and therefore analyzed. In addition, given our question of robustness of competitive equilibria, supportability by prices is not sufficient. Rather, as indicated in item 3 above, only small lump-sum transfers can be used. Nevertheless, despite the different motivation for our work, informational smallness turns out to be again the key condition for the results.

Starting with Hurwicz (1972), there is a substantial literature which investigates to what extend the complete information competitive equilibrium can be viewed as the limit of economies in which agents can act strategically and influence market prices (see for example Roberts and Postlewaite (1976), Otani and Sicilian (1990), Jackson (1992), Rustichini, Satterthwaite and Williams (1994), Jackson and Manelli (1997)). In the perturbed economies that we consider, agents could also influence the market because of asymmetric information. However, our results show that close to complete information, competitive behavior can already be obtained through an appropriate system of (small) lump-sum transfers.

Our analysis was motivated by Kajii and Morris (1997), who address the question of robustness of another economic solution concept with respect to incomplete information, the Nash equilibrium. Specifically, Kajii and Morris provide conditions under which the Nash equilibrium of a complete information game with two players is close to that of an informationally slightly perturbed game. Similar to our paper, they demonstrate that small noise alone is not sufficient for robustness. Rather, stronger conditions are needed. Kajii and Morris (1998) investigate the lower-hemicontinuity of Bayesian Nash equilibria, when ε-equilibria are used in the perturbed games. Their paper, like ours, stresses the importance of a condition which ensures that agents' updated priors are close in the perturbed games.

Our paper is organized as follows. Section 2 provides some intuition for the main results by means of examples. The description of the model and definitions are in Sections 3 and 4. The main results are in Section 5.

2 Some Examples

2.1 The Lemons Problem

Consider the classic lemons problem. There is an agent who wants to sell a car. The quality of the car is private information of the seller. For example, assume that the buyer's and the seller's valuations are as follows:

	Buyer	Seller
good quality	5	4
bad quality	2	1

If the quality of the car is known to the buyer and the seller, then trade would occur at a state contingent price in both states. In particular, if the car is of good quality the price must be between 4 and 5. If the car is a lemon, the price must be between 1 and 2. However, if only the seller is informed, then he would always have the incentive of announcing that the car is not a lemon in order to receive the higher price. Thus, given the informational assumptions in the lemons problem, it is not possible to credibly reveal the quality of the car.

It seems that the informational assumptions in the lemons example are rather extreme. In many cases a seller's information will not be perfect. Moreover, in many markets (e.g., arts, antiques) some buyers have at least as much information as the seller, although both agents' information may be incomplete. As mentioned above, if information is complete, both high and low quality goods will be traded and Pareto efficient allocations will be obtained. Given this equilibrium, robustness with respect to incomplete information means that the lemons problem disappears provided the informational asymmetry is not too extreme. However, robustness does not hold for the above example as we now demonstrate.

2.2 Informed Buyers and Sellers

Assume that both the buyer and the seller have both noisy information about the quality of a good. Let $\omega \in \Omega = \{g, b\}$ be the true quality, and assume that each of the two states g or b occurs with the same probability. Rather than observing ω directly, the buyer and the seller receive noisy signals $s_B, s_S \in \{g, b\}$ about the true quality ω. Let $\pi\left(s_B \mid \omega\right)$ and $\pi\left(s_S \mid \omega\right)$ be the probabilities that signals s_B, s_S are received if ω is the true quality. Assume that for given ω the realizations of s_S

and s_B are independent and identically distributed. Moreover, let $\pi(s \mid \omega) = \delta$ for $\omega \neq s$. That is, δ is the probability that the signal differs from the true state.

Assume that for sufficiently small δ, there exists an equilibrium with no lemons problem. That is, agents reveal their information truthfully, trade always occurs, and only small lump sum transfers are needed to support the equilibrium. Let $s = (s_B, s_S)$. Assume that the price $p(s) = (v_B(s) + v_S(s))/2$, where $v_B(s)$ and $v_S(s)$ are the buyer's and the seller's valuation respectively (for other prices the argument is similar). The valuations depend on the updated prior, $\pi(\omega \mid s_B, s_S)$ given by

	$s_B = s_S = g$	$s_B \neq s_S$	$s_B = s_S = b$
$\omega = g$	$\frac{(1-\delta)^2}{(1-\delta)^2+\delta^2}$	0.5	$\frac{\delta^2}{(1-\delta)^2+\delta^2}$
$\omega = b$	$\frac{\delta^2}{(1-\delta)^2+\delta^2}$	0.5	$\frac{(1-\delta)^2}{(1-\delta)^2+\delta^2}$

Assume for example that the buyer and the seller report different signals. If the reports are truthful, both agents assign probability 0.5 to each state $\omega = g, b$. As a consequence, the buyer's valuation is 3.5, the seller's valuation is 2.5, and the price is $p(s) = 3$. Now assume that both agents report the good signal. Then as δ converges to 0, the price converges to 4.5. As a consequence, the buyer can lower the price by reporting that the car is of bad quality, even if he observed the good signal. In fact, as $\delta \to 0$ this false report lowers the price by 1.5. As a consequence, a lump sum transfer of at least 1.5 would be required to induce the buyer to report truthfully. Therefore even if the noise is small, only equilibria will large transfers could possibly support allocations that are close to the complete information equilibrium. A similar argument can be made for the seller.

What generates the difference between complete information and cases of "almost complete information" (i.e., where δ is small)?

The complete information case can be interpreted as a case where both agents receive a signal with noise $\delta = 0$. Thus, agents know the true state and in order to trade it should not be necessary for them to announce their private information. However, what would happen if the seller made a false report? In order to support the complete information equilibrium we must assume that the buyer does not change his updated beliefs. The buyer must therefore assume that the seller is lying. Now consider the case where $\delta > 0$. Then the buyer's updated prior will be affected as we demonstrated above. This is the case because the buyer knows that there is some noise in his own signal. As a consequence, a different quality report by the seller can be an indication that the buyer's own signal is incorrect. The buyer's updated prior will therefore change and misreports can have an effect.

2.3 A Robustness Example

The main reason why the above economy is not robust is that each agent can have a large influence on the price even if the noise in the economy is small. With a larger number of agents, one would expect this influence to become smaller as the noise is reduced.

Consider again the lemons example, but assume that there are many pairs of buyers and sellers. In addition, for each buyer and seller, there is another agent, C, who has information about the quality of the good. The price of each good will then be a function, $p(s_B, s_S, s_C)$, of the announcement of three agents who have information about the particular good. Each agent's noise is δ. Let $v_S(s)$ and $v_B(s)$ be the valuation of the seller and the buyer, respectively, given the vector of signals s.

Assume that if all reported signals are the same, trade occurs at the average of the buyer's and seller's valuation. On the other hand, if two of the reported signals are the same and the third agent reports a different signal, then the agent whose report differs must make a lump sum transfer to the other agents.

Now assume for example that the seller observes b. If δ is sufficiently small than the seller will expect with a probability close to 1 that the other agents will also have observed the bad signal. If all reported signals are b, trade will occur at a price of approximately $p = 1.5$. This results in a utility of approximately 0.5 for the seller.

Now assume that the seller falsely claims that the good is of high quality. Then $\pi\big(\omega \bigm| s_B = s_C = b, s_S = g\big)$ is δ for $\omega = g$ and $1-\delta$ for $\omega = b$. Therefore $p = (v_S(g,b,b) + v_B(g,b,b))/2 = 1.5 + 3\delta$. Thus, a lump sum tax of 3δ is sufficient to deter misreports by the seller. Similarly, one can show that all other misreports can also be deterred by small transfers. The size of the transfer converges to zero as the noise in the signals vanishes. Thus, an incomplete information economy with small noise δ will closely resemble a complete information economy.

The lump sum taxes and subsidies are imposed to penalize or reward agents for certain types of reports. Agents will be penalized if their reports substantially differ from the reports of other agents. Theorem 1 and Proposition 1 provide a second Welfare Theorem for economies with asymmetric information. In particular, conditions are provided under which Pareto efficient allocations can be supported by an incentive compatible equilibrium with lump sum transfers. This result is then used to show under what conditions complete information equilibria are robust. The example also seems to suggest that robustness holds in the lemons problem if there are at least three informed agents for each good. However, it turns out that this is not the case in general. In particular, it is essential that each of the agents is informationally small. Informational smallness is not automatically guaranteed by having small noise and sufficiently many informed agents.

2.4 A Non Robustness Example

We now show, that for more general specification of agents' noise, the complete information case is not robust. In other words, the lemons problem persists even if the noise is arbitrarily small.

Consider again the example with three informed agents per good. However, assume that noise in the seller's signal is δ^2. The noise in the signal of the other agents is δ. Thus, the seller has slightly superior information.

Now note that if δ is sufficiently small then $\pi\big(\omega \bigm| s_B = s_C = b, s_S = g\big)$ is approximately 0.5. Thus, as $\delta \to 0$, $v_S(g,b,b)$ converges to 2.5, and $v_B(g,b,b)$ converges to 3.5. The price given these reports is therefore approximately 3. On the other hand if all agents report b, then the price will be approximately 1.5, provided δ is small. Therefore, in order to deter a misreport, the seller would have to pay a transfer of at least 1.5. This transfer does not become small as δ is small. Hence the complete information competitive equilibrium is not robust with respect to this type of noise in the agents' signals.

As in Section 2.2 complete information competitive equilibria are not robust, because agents are able to influence the updated prior even if the noise is small. In Example 3 we show that agents might not be informationally small even if the noise is i.i.d. and symmetric (in the example of this section the noise is not identically distributed). In Theorem 3 we show that lack of informational smallness leads in general to robustness problems.

3 The Model

Consider an exchange economy with $i = 1, \dots, n$ agents. There is uncertainty over the state of nature $\omega \in \Omega$, where Ω is assumed to be finite. Each agent i receives a noisy signal $s_i \in S_i$ about ω. Let $S = \prod_{i=1}^n S_i$. Any $s = (s_1, \dots, s_n) \in S$ will also be denoted by (s_{-i}, s_i). Let π be a probability on $\Omega \times S$ which is the common prior of all agents over states and signals. Assume there are ℓ goods. Let $X_i = \mathbb{R}^\ell_+$

be the consumption space of agent i. Each agent i's preferences are given by a state dependent von Neumann Morgenstern utility function $u_i \colon \Omega \times X_i \to \mathbb{R}$. Note that an agent's utility depends directly only on consumption and the true state ω. However, utility will indirectly depend on the signals, as agents use their signal to update their prior on Ω.

A consumption bundle for agent i is therefore given by $x_i \colon S \to \mathbb{R}_+^{\ell}$. Note that consumption depends only on s but not on ω since only s is observable.

Agent i's endowment is e_i. For simplicity we assume that e_i is state independent. By assuming state independence of the endowments, agents cannot learn the true state by observing their endowments.[2]

We assume that each agent i's signal, s_i, is private information to agent i.

4 The Equilibrium Concept

When considering economies with incomplete information, we must avoid the problem of lack of incentive compatibility in the standard rational expectations equilibrium. Thus, we consider mechanisms which have basic features of decentralized markets. That is, the planner's authority is restricted to lump sum taxes and subsidies which are used to discourage false reports. As explained in the introduction, these mechanism should look like standard competitive markets for incomplete information economies which are "close" to a complete information economy, and use only small lump-sum transfers.

We can imagine that the economy proceeds as follows. The "planner" announces a system of lump sum transfers $M_i(s)$, $i \in I$ and prices $p(s)$. Then nature selects a state ω that is not directly observable. Rather, each agent observes her signal s_i about ω, and makes a report s_i' which becomes known to everyone. Each agent i then maximizes utility subject to her budget constraint given $p(s')$ and $M_i(s')$, requesting a consumption bundle $x_i(s')$. If $\sum_{i\in I} x_i(s') \neq \sum_{i=1}^{n} e_i$, some rationing must occur.[3] As a consequence, each agent will receive a level of consumption $x_i'(s')$ and therefore a payoff $u_i(\omega, x_i'(s'))$.

Thus, we have game with differential information. An equilibrium of this

[2] State dependent endowments can be accommodated. However, agents can then reveal information about the state by showing their endowment (see Hurwicz, Maskin and Postlewaite (1995)).

[3] More formally, a rationing rule maps each collection of consumption bundles $x_i(s)$, $i \in I$, where $x_i(s) \in \mathbb{R}_+^{\ell}$, into consumption bundles $x_i'(s)$, $i \in I$, where $x_i'(s) \in \mathbb{R}_+^{\ell}$ and $\sum_{i\in I} x_i'(s) = \sum_{i\in I} e_i$ (feasibility). If $x_i(s)$, $i \in I$ is feasible then no rationing occurs, i.e., $x_i'(s) = x_i(s)$, for all $s \in S$, $i \in I$.

game consists of reporting strategies for all agents that constitute a Bayesian Nash equilibrium.

In our model, the planner does not have enough authority to implement an explicit rationing scheme that he desires. Rather, agents have ex-ante expectations about rationing. These expectations are not known by the planner. As a consequence, the choice of mechanism and its equilibrium should be immune to different choices of rationing expectations. Formally, consider a mechanism $p(\cdot)$, $M_i(\cdot)$, $i \in I$ and reporting strategies for each agent. Then

1. the reporting strategies should be a Bayesian Nash equilibrium independent of the rationing rules;
2. the equilibrium allocation should be the same for all rationing rules.

We can apply the revelation principle. Thus, without loss of generality arbitrary mechanisms can be replaced by direct truthtelling mechanisms. Such mechanisms are described in Definition 1 below. Before stating Definition 1, we describe the optimization problem faced by a consumer i given a signal profile s.

In particular, agent i's optimal consumption $x_i(s)$ solves

$$\max_{x \in X_i} E_\Omega\big(u_i(\omega, x) \bigm| s\big) \text{ s.t. } p(s)x \leq p(s)e_i + M_i(s); \tag{1}$$

Definition 1 $\left\{x_i, p, M_i \bigm| i = 1, \ldots, n\right\}$ *is an* ***incentive compatible competitive equilibrium*** *if and only if*

❶ *Each $x_i(s)$ solves the consumer optimization problem (1) for all $s \in S$, given prices $p(s)$ and lump sum transfer $M_i(s)$.*

❷ $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$.

❸ *For all agents i, signals s_i, $s_i' \in S_i$, and for all $0 \leq \hat{x}_i(s_{-i}, s_i') \leq \sum_{\in I} e_i$ with $p(s_{-i}, s_i')\hat{x}_i(s_{-i}, s_i') \leq p(s_{-i}, s_i')e_i + M_i(s_{-i}, s_i')$ it follows that*

$$E_{\Omega \times S_{-i}}\Big[u_i\big(\omega, x_i(\omega, s_i)\big) \bigm| s_i\Big] \geq E_{\Omega \times S_{-i}}\Big[u_i\big(\omega, \hat{x}_i(\omega, s_i')\big) \bigm| s_i\Big].$$

The first two conditions of Definition 1 follow because there cannot be rationing in equilibrium. Otherwise, the equilibrium would depend on the rationing rule. The last condition follows because the Bayesian Nash equilibrium must hold for all rationing rules. As a consequence, misreporting by an agent i should not be optimal, even if agent i uses the most "optimistic" rationing rule, where he believes that all other agents are rationed first, and he is only rationed if his demand of a good exceeds aggregate supply.

Although our formal definition of a mechanism includes the price vector, $p(s)$, our informal interpretation is that the government simply announces the lump sum transfers and then agents go to the market where equilibrium prices are determined. Thus, our model is as decentralized as the standard competitive model, requires minimal interference of the planner (provided the $M_i(s)$ are small), and has minimum possible informational requirements for the planner and agents.

We conclude this section with some comments.

1. Our mechanism assume that the reports s_i' become public information. Alternatively one could assume that agents only know their own signal, s_i', the price function, $p(\cdot)$, and the lump sum transfer function, $M_i(\cdot)$. An agent can then learn about s by inverting these two functions. However, because $p(\cdot)$ is generically fully revealing under standard assumptions, all results in this paper would hold at least generically under this alternative formulation.

2. The incentive compatibility constraint in Definition 1 is stronger than the standard notion of Bayesian incentive compatibility in a model for implementing Pareto efficient allocations. Bayesian incentive compatibility only requires that for all agents i, and signals s_i, $s_i' \in S_i$

$$E_{\Omega\times S_{-i}}\Big[u_i\big(\omega, x_i(\omega, s_i)\big) \;\big|\; s_i\Big] \geq E_{\Omega\times S_{-i}}\Big[u_i\big(\omega, x_i(\omega, s_i')\big) \;\big|\; s_i\Big]. \tag{2}$$

   Thus, Bayesian incentive compatibility implicitly assumes that the "planner" can determine the net-trades of agents. That is, after the signals s_i are reported, the planner induces agents to make the trades $x_i(s) - e_i$. Implicitly, this assumes that the planner has a more powerful role than one would like to have in a model where markets should be decentralized. In contrast, our incentive compatibility concept allows agents the possibility of trading from $x_i(\cdot, s_i')$ to some other consumption bundle $\hat{x}_i(\cdot, s_i')$. Our incentive compatibility notion is used in all positive results. In the negative (i.e., non robustness) result of Theorem 3 we use the standard Bayesian incentive compatibility. We do this in order to strengthen the result, because Theorem 3 obviously holds also under a stronger incentive notion.

3. All lump sum transfers must add up to zero, i.e., $\sum_{i=1}^{n} M_i = 0$. This follows immediately from 1 and 2 of Definition 1.

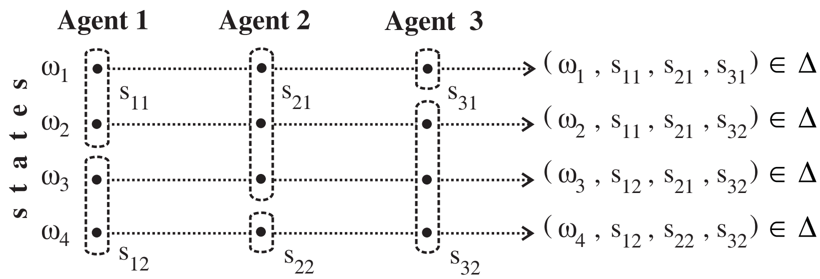

Figure 1: An Example for the Definition of Δ

The set Δ is the collection of signal profiles and states that are consistent if signals were deterministic.

5 Robustness of Equilibria

5.1 Definition of Robustness

The examples in Section 2 consider economies where all agents are fully informed in the limit. Our model allows also a more general interpretation. In particular, we can start with a noiseless limit economy in which not all agents are fully informed. Consider the Arrow-Debreu equilibrium of this economy, ignoring the informational asymmetries. We provide conditions under which this Arrow-Debreu equilibrium is the limit of incentive compatible competitive equilibria of economies where the signals are not only incompletely informative but also noisy.

In order to describe a noiseless economy with incompletely informative signals, we can identify each S_i with a partition of Ω. In such an economy we wish to identify the collection, Δ, of states and signal profiles, $(\omega, s_1, \dots, s_n)$, which are consistent. Formally,

$$\Delta = \left\{ (\omega, s_1, \dots, s_n) \;\middle|\; \omega \in s_i \text{ for all } i \in I \right\}$$

For example, assume the signals are completely informative, i.e., each S_i is the fine partition of Ω. Then Δ consists of all $(\omega, s_1, \dots, s_n)$ such that each $s_i = \{\omega\}$ (with a slight abuse of notation we will write $s_i = \omega$ in the following). If there is no noise in signal (i.e., $\pi(\Delta) = 1$) then prices, p, and consumption plans, x_i, can be written as functions of $\omega \in \Omega$ alone, and the economy therefore corresponds to a standard complete information economy.

As another example, illustrated in Figure 1, assume that Ω consists of four states. The partitions representing the signal spaces are indicated by the dotted lines. For example, agent 1 receives signal s_{11} both in state ω_1 and ω_2, whereas his signal is s_{12} in states ω_3 and ω_4.

We next describe what it means for equilibria of incomplete information economies to converge to an equilibrium of the complete information economy. As mentioned above, a complete information economy is an economy where each agent can observe ω without noise, and hence prices and the consumption bundles can be written as functions of ω only.

Definition 2 *Let $x_i\colon \Omega \to X_i$ and $p\colon \Omega \to \mathbb{R}^\ell_+$. Then $\{x_i, p \mid i = 1, \dots, n\}$ is a* ***complete information competitive equilibrium*** *if and only if*

- ❶ *Each $x_i(\omega)$ solves $\max_{x \in X_i} u_i(\omega, x)$ s.t. $p(\omega)x \leq p(\omega)e_i$;*
- ❷ *$\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$ for all $\omega \in \Omega$.*

We perturb the complete information economy by allowing signals that are both noisy and incomplete. Thus for fixed S and Ω, we consider sequences of priors π^k, $k \in \mathbb{N}$ such that there is no noise in the limit. Formally, $\lim_{k\to\infty} \pi^k(\Delta) = 1$. We say that an equilibrium of the complete information economy is robust, if there exists a sequence of incentive compatible competitive equilibria of the perturbed economies that converges to the to the competitive equilibrium of the original economy and for which all lump sum transfers converge to zero. We now provide the formal definition of robustness.

Definition 3 *Assume $\bigvee_{i\in I} S_i$ is the fine partition of Ω, and let π^k be a sequence of probabilities on $\Omega \times S$ with $\lim_{k\to\infty} \pi^k(\Delta) = 1$.*

A competitive equilibrium $\{x_i, p_i \mid i \in I\}$ of the complete information economy $\mathcal{E} = \{u_i, e_i, \Omega \mid i = 1, \dots, n\}$ is ***robust*** *with respect to π_k, $k \in \mathbb{N}$ if and only if there exists a sequence of incentive compatible competitive equilibria $\{x_i^k, p^k, M_i^k \mid i = 1, \dots, n\}$, $k \in \mathbb{N}$ of the economies $\mathcal{E}^k = \{u_i, e_i, \Omega, S_i, \pi^k \mid i = 1, \dots, n\}$ that converges to $\{x_i, p_i \mid i \in I\}$, i.e.,*

- ❶ *$x_i^k(s)$, $p^k(s)$ converge to $x_i(\omega)$, $p(\omega)$ for all s, ω with $\lim_{k\to\infty} \pi^k(\omega \mid s) = 1$.*
- ❷ *$\lim_{k\to\infty} M_i^k(s) = 0$ for all $s \in S$.*

5.2 Existence of Incentive Compatible Equilibria

Theorem 1 (below) is the main technical result which we use to prove robustness of equilibria with respect to incomplete information. The Theorem provides conditions under which rational expectations equilibria with lump sum transfers are incentive compatible. Proposition 1 (below) will be used as a first step to construct the lump sum transfers in our main robustness result (Theorem 2).

5.2.1 Notation and Assumptions

We assume that agent i's signal space S_i is a partition of Ω. If S_i is the fine partition $\{\{\omega_1\},\ldots,\{\omega_n\}\}$, then with slight abuse of notation we write $s_i = \omega$ instead of $s_i = \{\omega\}$. Let $S = \prod_{i\in I} S_i$, and $S_{-i} = \prod_{j\neq i} S_j$.

For every $s_i \in S_i$, let $\sigma_{-i}(s_i)$ be the set of all reports of agents $j \neq i$ that are consistent with agent i's signal, provide all signals are deterministic. Formally,

$$\sigma_{-i}(s_i) = \Big\{ s_{-i} \;\Big|\; \exists \omega \in \Omega \colon (\omega, s_{-i}, s_i) \in \Delta \Big\}.$$

To illustrate this definition, assume that $S_j = \{\{\omega_1\},\ldots,\{\omega_n\}\}$ for all agents, then $\sigma_{-i}(s_i) = (s_i, s_i, \ldots, s_i)$. Thus, if there is no noise, then all signals must be the same. Now consider the example of Figure 1. Then $\sigma_{-3}(s_{31}) = \{(s_{11}, s_{21})\}$, and $\sigma_{-3}(s_{32}) = \{(s_{11}, s_{21}), (s_{12}, s_{21}), (s_{12}, s_{23})\}$.

Finally, we define

$$\varepsilon_i = \max_{\substack{s_i, s_i' \in S_i \\ s_{-i} \in \sigma_{-i}(s_i)}} \sum_{\omega\in\Omega} \Big| \pi(\omega \mid s_{-i}, s_i) - \pi(\omega \mid s_{-i}, s_i') \Big|.$$

Then ε_i determines agent i's ability to influence the updated prior. ε_i is related to the notion of information smallness introduced in McLean and Postlewaite (1999).

Throughout this section we will use the following assumptions.

Assumption A.

❶ $u_i(\omega, \cdot)$ is continuous, strictly concave, and strictly monotone for all agents $i \in I$, and states $\omega \in \Omega$.

❷ The aggregate endowment $e \in \mathbb{R}^\ell_{++}$

Finally, note that we can normalize agents' utility functions such that $0 \leq u_i(\omega, x) \leq 1$, for all agents $i \in I$, and for all $0 \leq x \leq e$.

5.2.2 The Existence Theorem

We now state conditions under which an incentive compatible competitive equilibrium with lump sum transfers exists.

Theorem 1. *Consider an economy that fulfills assumption A. Let $x(s)$, $s \in S$ be a Pareto efficient allocation. Assume that*

$$\pi\big(\sigma_{-i}(s_i) \bigm| s_i\big) \geq \frac{1}{1+\gamma_i - 2\varepsilon_i}, \tag{3}$$

where

$$\gamma_i = \min_{\substack{s_i \neq s_i' \in S_i \\ s_{-i} \in \sigma_{-i}(s_i)}} E_\Omega\Big(u_i\big(\cdot, x_i(s_{-i}, s_i)\big) \Bigm| s_{-i}, s_i\Big) - E_\Omega\Big(u_i\big(\cdot, x_i(s_{-i}, s_i')\big) \Bigm| s_{-i}, s_i\Big).$$

Then $x_i(s)$, $i \in I$ can be supported as an incentive compatible competitive equilibrium.

Theorem 1 applies to arbitrary signal spaces S_i, but assume for the moment that $S_i = \Omega$. In this case, all agents will observe the same signal if there is no noise. Thus, $\sigma_{-i}(s_i) = (s_i, \ldots, s_i)$. Hence γ_i measures the difference in each agent i's utility between the case where all agents observe and report the same signal, and the case where all agents except agent i report the same signal. The left-hand side of (3) is the conditional probability that all other agents observe the same signal as agent i. If the noise in the economy is small, this probability will be close to 1. Thus, the inequality is more likely to hold the larger γ_i and the smaller ε.

Also note that (3) never holds if $\gamma_i < 0$. Otherwise, the right-hand side of (3) is strictly greater than 1, whereas the left-hand side is always less or equal to 1. Thus, the agent must be worse off in the case where all signals except for his signal are the same compared to the case where all agents' signals are the same.

Theorem 1 has therefore the following intuitive interpretation. $\gamma_i \geq 0$ simply means that we have "incentive compatibility" when everyone observes the same signal. Then if condition (3) also holds, we have incentive compatibility for all signal profiles.

Now assume that we have a Pareto efficient allocation $x(s)$, $s \in S$ which does not fulfill the conditions of Theorem 1. In Proposition 1 we show conditions under which we can change $x(s)$ on a "small" set D such that the resulting allocation is supportable as an incentive compatible equilibrium. The changed allocation is still individually rational, but it might require relatively large transfers. Theorem 2

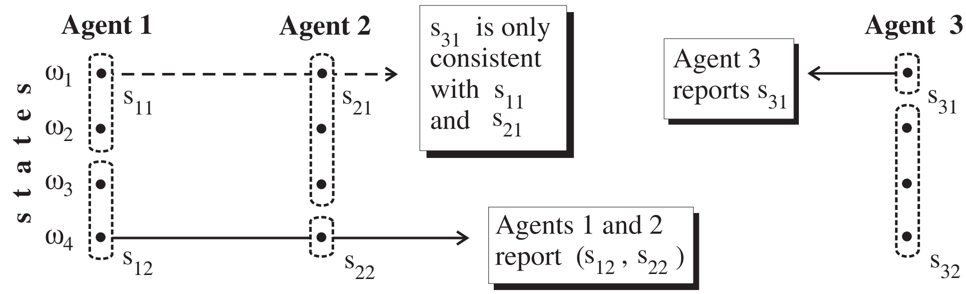

Figure 2: An Example for the Definition of D_i

D_3 is the set of all signal profiles (s_1, s_2, s_3) for which agent 1 and 2's signal are mutually consistent, but are not consistent with agent 3's signal (if signal were deterministic). In this example, (s_{12}, s_{22}, s_{31}) is in D_3.

uses the argument of Proposition 1 and then constructs allocations for which the transfers becomes small.

The set D is the union of sets D_i for each agent i. Each D_i is set of all signal profiles, where all signals, except agent i's signal, are consistent, if signals are noiseless. Formally,

$$D_i = \left\{ (s_{-i}, s_i) \;\middle|\; s_{-i} \notin \sigma_{-i}(s_i) \text{ and } \exists s_i' \neq s_i \colon\, s_{-i} \in \sigma_{-i}(s_i') \right\}, \text{ and } D = \bigcup_{i \in I} D_i.$$

For example, if $S_j = \{\{\omega_1\}, \ldots, \{\omega_n\}\}$ for all agents, then D_1 consists of all signal profiles, where all $s_2 = \ldots = s_n$, and $s_1 \neq s_2$. In the example of Figure 2, $s_i = s_{31}$, $s_i' = s_{32}$, and $s_{-i} = (s_{12}, s_{22})$. It follows that $(s_{12}, s_{22}, s_{31}) \in D_3$, because $(s_{12}, s_{22}) \notin \sigma_{-3}(s_{31})$ but $(s_{12}, s_{22}) \in \sigma_{-3}(s_{32})$.

Both in Proposition 1 and in Theorem 2 below, agent i will be penalized by an appropriate lump sum transfer if and only if $s \in D_i$, that is if all but agent i's report are consistent. In all other cases, no lump sum transfers are imposed. Thus, the informational requirements for determining when to impose lump sum transfers are rather minimal.

We also define

$$G_i = \min_{\substack{s_i \in S_i \\ s_{-i} \in \sigma_{-i}(s_i)}} E_\Omega\Big(u_i\big(\cdot, x_i(s_{-i}, s_i)\big) \;\Big|\; s_{-i}, s_i\Big) - E_\Omega\Big(u_i(\cdot, e_i) \;\Big|\; s_{-i}, s_i\Big),$$

for any consumption bundle x. Note that G_i is the minimum gains from trade for agent i, taken over all states where the agents receive the same signal.

The additional assumption in Proposition 1 is that any two agents are informationally redundant.

Proposition 1. *Consider an economy that fulfills assumption A. Moreover, for all agents $i, j \in I$ assume that $\bigvee_{k \neq i,j} S_k$ corresponds to complete information.*

Let $x(s)$, $s \in S$ be a Pareto efficient allocation. Assume that

$$\pi\big(\sigma_{-i}(s_i) \;\big|\; s_i\big) \geq 1/(1 + G_i - 2\varepsilon_i), \tag{4}$$

for all $s_i \in S_i$, $i \in I$. Then there exists an incentive compatible competitive equilibrium with lump sum transfers $\big\{\tilde{x}_i, \tilde{p}, \tilde{M}_i \;\big|\; i \in I\big\}$, where $\tilde{x}_i(s) = x_i(s)$, for all $s \in D^c$.

Proof. First, let $s \in D$ and hence in D_i for some i. In this case we choose $\tilde{x}(s)$ to be a Pareto efficient allocation with $E_\Omega(u_i(\cdot, \tilde{x}_i(s))|s) = E_\Omega(u_i(\cdot, e_i)|s)$. Now let $s \notin D$. Then we choose $\tilde{x}(s) = x(s)$.

In order for this definition to be consistent, we must show that $D_i \cap D_j = \emptyset$ for all $i \neq j$.

Let ω, ω' be arbitrary. Then an element of D_i is of the form $(s_{-i}, \hat{s}_i)$, where s_{-i} is the unique profile of signals containing $(\omega, \ldots, \omega)$. Similarly, an element of D_j is of the form $(s'_{-j}, \hat{s}'_j)$, where s'_{-j} is the unique profile of signals containing $(\omega', \ldots, \omega')$. Now assume by way of contradiction that two such elements are the same. Then we must have $s_{-i,-j} = s'_{-i,-j}$.

If signals are deterministic, then after removal of agents i and j, the pooled information of the remaining agents $\bigvee_{k \neq i,j} S_k$ is by assumption complete. Therefore, $\omega = \omega'$. Moreover, for the two elements to be same we must also have $\hat{s}_i = s'_i$ and $\hat{s}'_j = s_j$. This and $\omega = \omega'$ implies that $\hat{s}_i$ contains ω which contradicts $(s_{-i}, \hat{s}_i) \in D_i$.

Thus, $\tilde{x}(s)$ is well defined. The allocation fulfills the conditions of Theorem 1 and can therefore be supported as an incentive compatible equilibrium. ■

Note that there is a tradeoff between the size of the set D on which transfers are made and condition (4). More precisely if we increase the size of D we can make condition (4) more slack. For example, we could invoke the transfers as long as less than half of the agents report inconsistent signals. An agent who reports an inconsistent signal gets penalized by a consumption close in utility to the endowment. However, at the same time we would have to assume that the pooled information of more than half of the agents is always complete.

5.3 The Robustness Results

We now use Theorem 1 and Proposition 1 to provide sufficient conditions for robustness. Theorem 2 shows that any complete information competitive equilibrium is robust with respect to a sequence of priors which fulfills condition (5) below. This condition says that each agent's ability to influence the updated prior converges to zero. In the language of McLean and Postlewaite (1999), the condition ensures that agents become informationally arbitrarily small as the noise in signals disappears. Theorem 2 therefore implies that if we perturb an economy in such a way that agents are informationally sufficiently small then any equilibrium of the complete information economy will be close to an equilibrium of the economy with incomplete information. The Theorem does not require that each agent's information becomes complete as the noise in the signal disappears. Rather, it is sufficient to assume that in the limit no two agents together have information that none of the remaining agents has. Formally, this means that $\bigvee_{k\neq i,j} S_k$ corresponds to complete information, i.e., to the fine partition $\Omega = \big\{\{\omega_1, \ldots, \omega_n\}\big\}$.

Section 2 indicates the importance of the assumption that in the noiseless economy no two agents have information which is exclusive to them. In particular, consider again the lemons market of Section 2.2. If there were no noise, then exactly two agents, the buyer and the seller, would be informed about the quality of a particular good. We have shown that this results in non robustness. In contrast, in Section 2.3 three agents are informed about each good. As a consequence, in the noiseless economy, $\bigvee_{k\neq i,j} S_k$ corresponds to complete information.

Theorem 2. *Let $\{x_i, p \mid i \in I\}$ be a complete information competitive equilibrium. Assume that*

❶ *The economy fulfills Assumption A;*

❷ *The gains from trade are strictly positive, i.e., $u_i(\omega, x_i(\omega)) - u_i(\omega, e_i) > 0$;*

❸ *$\bigvee_{k\neq i,j} S_k$ is the fine partition of Ω, for any $i, j \in I$.*

Let π^k, $k \in \mathbb{N}$ be a sequence of priors with $\lim_{k\to\infty} \pi^k(\Delta) = 1$ and

$$\lim_{k\to\infty} \left| \pi^k(\omega \mid \sigma_{-i}(s_i), s_i) - \pi^k(\omega \mid \sigma_{-i}(s_i), s_i') \right| = 0, \qquad (5)$$

for all $s_i, s_i' \in S_i$, $\omega \in \Omega$, and for all agents i.

Then the competitive equilibrium is robust with respect to π^k, $k \in \mathbb{N}$.

We now provide three examples that illustrate Theorem 2.

Examples 1 and 2 indicate cases in which informational smallness holds, and complete information competitive equilibria are therefore robust. In Example 1, each agent receives a noisy signal about the entire state. Example 2 extends this to the case also covered by Theorem 2 where agents' signals are noisy and incomplete. It covers the case discussed in Section 2.3, where three agents always have information about the quality of a particular good.

Example 1. Assume that there are n agents and m states. Each agent i's probability of receiving the correct signal $s_i = \omega$ is $\pi^{\delta}(s_i|\omega) = 1-\delta$. Moreover, each signal $s_i' \neq \omega$ has the same probability $\pi^{\delta}(s_i'|\omega) = \delta/(m-1)$. Let $\mu(\omega)$ be the probability of state ω.

It is now easy to check that condition (5) holds as $\delta \to 0$. In particular, fix $\omega, \omega' \in \Omega$ with $\omega \neq \omega'$. Choose $s \in S$ with $s_i = \omega'$, and $s_j = \omega$ for all $j \neq i$. Then

$$\pi^{\delta}(\omega|s) = \frac{\delta(1-\delta)^{n-1}\mu(\omega)}{\delta(1-\delta)^{n-1}\mu(\omega) + (1-\delta)\delta^{n-1}\mu(\omega') + \sum_{\omega'' \neq \omega, \omega'} \delta^n \mu(\omega'')}$$

Clearly, $\pi^{\delta}(\omega|s)$ converges to 1 as δ becomes small. This implies that $\pi^{\delta}(\tilde{\omega}|s)$ converges to 0 as δ becomes small, for any $\tilde{\omega} \neq \omega$. Thus, agent i cannot influence the updated prior in the limit. This means that condition (5) holds. Therefore for small δ, there exist incentive compatible equilibria with small transfers that are close to the equilibria of the complete information economy.

Example 2. Now assume that S_i is a partition of Ω. That is, even if there is no noise in the signal, agents do not have complete information. Assume that the pooled information of all except two agents is complete if signals were deterministic, i.e., condition 2 of Theorem 2 holds.

Let ω be the true state. Then agent i's signal s_i is correct if $\omega \in s_i$ (recall that $s_i \in S_i$ can be identified with a subset of Ω, because S_i is a partition of Ω). As in example 1, we assume that the probability of receiving the correct signal is $1-\delta$, and that the probability of each incorrect signal is the same. Thus, agent i's probability of receiving a particular incorrect signal is $\delta/(|S_i|-1)$, where $|S_i|$ denotes the number of elements of S_i. Again, let $\mu(\omega)$ be the probability of state ω.

We now check that condition (5) holds as $\delta \to 0$. Again fix $\omega, \omega' \in \Omega$ with

$\omega \neq \omega'$. Choose $s \in S$ with $s_i \ni \omega'$, and $s_j \ni \omega$ for all $j \neq i$. Then

$$\pi^\delta(\omega|s) = \frac{\frac{\delta}{|S_i|-1}(1-\delta)^{n-1}\mu(\omega)}{\frac{\delta}{|S_i|-1}(1-\delta)^{n-1}\mu(\omega) + \sum_{\omega''\neq\omega}\prod_{k\in I}\mu(s_k|\omega'')\mu(\omega'')}.$$

Next we show that $\omega'' \notin s_k$ for at least two agents k with $k \neq i$.

Assume by way of contradiction that $\omega'' \notin s_k$ for at most one agent k. By assumption $\bigvee_{j\neq i,k} S_j$ corresponds the fine partition of Ω. Thus, all agents except i and k should together be able to distinguish between ω and ω'' if signals were deterministic. That is, ω and ω'' are in different elements of the partition S_j for all $j \neq i,k$. However, for all these agents we have $\omega'' \in s_k$. Moreover, by assumption $\omega \in s_k$ for all $k \neq i$. Thus, ω cannot be distinguished from ω'' with respect $\bigvee_{j\neq i,k} S_j$, a contradiction.

Because $\omega'' \notin s_k$ for at least two agents k, we can conclude that

$$\prod_{k\in I}\mu(s_k|\omega'')\mu(\omega'') \leq \frac{\delta^2}{|\Omega|-1}(1-\delta)^{n-2}.$$

Thus, we get

$$\pi^\delta(\omega|s) \geq \frac{(1-\delta)^{n-1}\mu(\omega)}{(1-\delta)^{n-1}\mu(\omega) + \delta\frac{|S_i|-1}{|\Omega|-1}\sum_{\omega''\neq\omega}(1-\delta)^{n-2}\mu(\omega'')}.$$

Therefore $\lim_{\delta\to 0}\pi^\delta(\omega|s) = 1$. Hence $\lim_{\delta\to 0}\pi^\delta(\tilde{\omega}|s) = 0$ for all $\tilde{\omega} \neq \omega$. Thus, condition (5) of Theorem 2 holds.

Example 2 illustrates that individual agents do not need to be fully informed in order for the resulting equilibrium to be similar to a complete information competitive equilibrium. The main property which must be fulfilled in Example 2 is that the pooled information of all agents is complete, and that there do not exist two agents who have information that none of the remaining agents has. This is sufficient to ensure that the conditions of Theorem 2 hold. As mentioned above, an example in which the sets S_i fulfill the required condition is discussed in Section 2.3. In the example there are always at least three agents who are informed about the quality of a particular good. Thus, if there is no noise, then we can remove any two agents in the economy and the pooled information of the remaining agents will still be complete.

In Section 2 we saw that independence of the noise in agents' signals is not sufficient for condition (5) to hold. Example 3 shows that condition (5) can be violated even if the assumptions on the distributions are strengthened to i.i.d. and symmetric. This, together with Theorem 3 demonstrates that complete information competitive equilibria may not be robust with respect to an arbitrarily small noise, even if the noise is i.i.d. and symmetric.

Example 3. Assume that there are three agents and three states. $S_i = \Omega$ and the noise is i.i.d. and symmetric. In order to abbreviate the notation, let $b_{ij} = \pi(s = \omega_i|\omega_j)$ for $1 \leq i, j \leq 3$. Assume that the b_{ij} are given by

	ω_1	ω_2	ω_3
$s = \omega_1$	$1-\delta-\delta^{\frac{1}{4}}$	$\delta^{\frac{1}{4}}$	δ
$s = \omega_2$	$\delta^{\frac{1}{4}}$	$1-2\delta^{\frac{1}{4}}$	$\delta^{\frac{1}{4}}$
$s = \omega_3$	δ	$\delta^{\frac{1}{4}}$	$1-\delta-\delta^{\frac{1}{4}}$

Now assume for example that the agents receive signals $s_1 = \omega_1$ and $s_2 = s_3 = \omega_3$.

Then

$$\begin{aligned}\pi(\omega_3|s_1 = \omega_1, s_2 = s_3 = \omega_3) &= \frac{b_{13}b_{33}^2\mu(\omega_3)}{b_{11}b_{31}^2\mu(\omega_1) + b_{12}b_{32}^2\mu(\omega_2) + b_{13}b_{33}^2\mu(\omega_3)} \\ &= \frac{\delta b_{33}^2\mu(\omega_3)}{b_{11}\delta^2\mu(\omega_1) + \delta^{\frac{3}{4}}\mu(\omega_2) + \delta b_{33}^2\mu(\omega_3)}.\end{aligned}$$

Clearly, $\lim_{\delta\to 0}\pi^{\delta}(\omega_3|s_1 = \omega_1, s_2 = s_3 = \omega_3) = 0$. On the other hand, one can check immediately that $\lim_{\delta\to 0}\pi^{\delta}(\omega_3|s_1 = s_2 = s_3 = \omega_3) = 1$. Thus, agent 1 has a non-trivial influence on the updated prior and his information does not become negligible.

It is interesting to note that Example 3 can immediately be extended to an example where informational smallness is violated for any number of agents n. In particular, replace every occurrence of $\delta^{1/4}$ by a function $f(\delta)$ which has the property that $f(\delta)^n/\delta$ converges to 0 as $\delta \to 0$. Such a function can be shown to exist by standard arguments.

The final result of the paper, Theorem 3, shows that the concept of informational smallness of McLean and Postlewaite (1999) is the key determinant for robustness. In particular, we show that if condition (5) is violated, then one can always find economies with non-robust equilibria. It should be noted that non-robustness already holds for rather simple economies in which all agents have Cobb-Douglas preferences.

In Theorem 3 we can replace 4 of Definition 1 by the weaker Bayesian incentive compatibility notion (2). This yields a stronger result.

Theorem 3. *For given Ω and $S = \prod_{i \in I} S_i$, let $\pi^k, k \in \mathbb{N}$ be a sequence of priors on $\Omega \times S$ which violates the informal smallness condition (5). Then there exists a complete information economy $\mathcal{E} = \left\{u_i, e_i, \Omega \;\middle|\; i = 1, \ldots, n\right\}$ such that*

❶ *$\mathcal{E}$ fulfills Assumption A;*

❷ *$\mathcal{E}$ has a unique competitive equilibrium with strictly positive gains from trade (i.e., $u_i(\omega, x_i(\omega)) - u_i(\omega, e_i) > 0$ for all agents $i \in I$);*

❸ *The competitive equilibrium of $\mathcal{E}$ is not robust with respect to π^k, $k \in \mathbb{N}$.*

6 Concluding Remarks

In this paper we perturb complete information economies by assuming that agents' information about the true state of the world is noisy. The main results of the paper characterize for what type of noise complete information equilibria are robust. Roughly speaking, a complete information equilibrium is robust, if it is almost indistinguishable from an equilibrium of the incomplete information economy when the noise is small.

Theorems 2 and 3 shows that an informational smallness condition is the key determinant for robustness. These Theorems do not only apply to the case where each agent's signal is a noisy description of the complete state. Rather, the informational smallness condition holds as long as there do not exist two agents, who have information which is exclusive to them. Clearly, this condition is not fulfilled in a model with independent private values, or in the classic lemons problem where only one agent is informed, but it will hold if there is sufficient correlation between agents' signals. Because the equilibrium concept for economies with incomplete information used in this paper closely resembles that of a decentralized competitive market, the theorems therefore demonstrate that the complete information Arrow-Debreu model remains a useful abstraction in those cases where information is incomplete but informational smallness prevails.

7 Appendix

In the following let $V_i(s;y) = E_\Omega\big[u_i(\omega,y) \mid s\big]$ be agent i's expected utility from consuming y, given $s \in S$. We also use the notation introduced in Section 5.2.1.

Proof of Theorem 1. By the second Welfare Theorem there exist lump-sum transfers $M_i(s)$ and competitive equilibrium prices $p(s)$ which support $x(s)$.

We must now check incentive compatibility.

First, note that

$$\Big|V_i(s_{-i},s_i;y) - V_i(s_{-i},s_i';y)\Big| \leq \varepsilon_i, \tag{6}$$

for all $s_i, s_i' \in S_i$, $s_{-i} \in \sigma_{-i}(s_i)$ and for any consumption bundle y, with $0 \leq y \leq e = \sum_{i \in I} e_i$. This follows immediately from the definition of ε_i and from the utility normalization which ensures that $0 \leq u(\omega,y) \leq 1$.

Now suppose that agent i receives signal s_i. Then a truthful report is optimal if

$$\sum_{s_{-i}\in S_{-i}} V_i\big(s_{-i},s_i;x_i(s_{-i},s_i)\big)\pi(s_{-i}|s_i) \geq \sum_{s_{-i}\in S_{-i}} V_i\big(s_{-i},s_i;\hat{x}_i(s_{-i},s_i')\big)\pi(s_{-i}|s_i);$$

where $\hat{x}_i(s_{-i},s_i')$ is the agent's optimal consumption from reporting s_i' although s_i is the true signal. Recall that when an agent reports s_i' instead of s_i we allow arbitrary trades at prices $p(s_{-i},s_i')$ as long as the resulting consumption does not exceed the aggregate endowment (see Definition 1). As a consequence the optimal consumption, $\hat{x}_i$, from reporting s_i' instead of s_i is a solution to

$$\max_{x\in\mathbb{R}^\ell_+} V_i(s_{-i},s_i;x) \text{ s.t. } (i)\ p(s_{-i},s_i')(x-e_i) \leq M_i(s_{-i},s');$$
$$(ii)\ x \leq \sum_{i\in I} e_i.$$

To abbreviate the notation, let $x' = x_i(s_{-i},s_i')$ and $\hat{x}' = \hat{x}_i(s_{-i},s_i')$. Note that both x' and $\hat{x}'$ are affordable for agent i at prices $p(s_{-i},s_i')$. Thus, $V_i(s_{-i},s_i;\hat{x}') \geq V_i(s_{-i},s_i;x')$, and $V_i(s_{-i},s_i';x') \geq V_i(s_{-i},s_i';\hat{x}')$. Hence $V_i(s_{-i},s_i;\hat{x}') - V_i(s_{-i},s_i';x') \leq V_i(s_{-i},s_i;\hat{x}') - V_i(s_{-i},s_i';\hat{x}') \leq \varepsilon_i$. Similarly, we get $V_i(s_{-i},s_i';x') - V_i(s_{-i},s_i;\hat{x}') \geq V_i(s_{-i},s_i';x') - V_i(s_{-i},s_i;x') \geq -\varepsilon_i$. Thus,

$$\Big|V_i(s_{-i},s_i;\hat{x}') - V_i(s_{-i},s_i';x')\Big| \leq \varepsilon_i. \tag{7}$$

Now note that (6) implies $\left|V_i(s_{-i},s_i;x') - V_i(s_{-i},s_i';x')\right| \le \varepsilon_i$. Using (7) we can therefore conclude that $\left|V_i(s_{-i},s_i;\hat{x}') - V_i(s_{-i},s_i;x')\right| \le 2\varepsilon_i$. Now let $x = x_i(s_{-i},s_i)$ be agent i's consumption from reporting truthfully. Then

$$\begin{aligned} V_i(s_{-i},s_i;x) - V_i(s_{-i},s_i;\hat{x}') &= [V_i(s_{-i},s_i;x) - V_i(s_{-i},s_i;x')] \\ &\quad + [V_i(s_{-i},s_i;x') - V_i(s_{-i},s_i;\hat{x}')] \\ &\ge \gamma_i - 2\varepsilon_i, \end{aligned}$$

for all $s_{-i} \in \sigma_{-i}(s_i)$. Hence,

$$\begin{aligned} &\sum_{s_{-i}\in S_{-i}} \Big(V_i\big(s_{-i},s_i,\tilde{x}(s_{-i},s_i)\big) - V_i\big(s_{-i},s_i,\hat{x}(s_{-i},s_i')\big)\Big)\pi(s_{-i} \mid s_i) \\ &\qquad \ge \pi(\sigma_{-i}(s_i) \mid s_i)[\gamma_i - 2\varepsilon_i] - (1 - \pi(\sigma_{-i}(s_i) \mid s_i)) \\ &\qquad = \pi(\sigma_{-i}(s_i) \mid s_i)[1 + \gamma_i - 2\varepsilon_i] - 1. \end{aligned}$$

Consequently, if $\pi(\sigma_{-i}(s_i) \mid s_i)[1+\gamma_i - 2\varepsilon_i] - 1 \ge 0$, truthful reports are optimal. ■

Proof of Theorem 2. Note that $\lim_{k\to\infty}\pi^k(\Delta) = 1$ implies $\lim_{k\to\infty}\pi^k(\sigma_{-i}(s_i)|s_i) = 1$. Hence, the left-hand side of (3) converges to 1.

Let $\{x_i(\omega), p(\omega) | i \in I, \omega \in \Omega\}$ be the competitive equilibrium of the complete information economy. Let D and D_i be defined as in Proposition 1. For all $\omega \in \Omega$ and $s \in S \setminus D$ with $\lim_{k\to\infty}\pi^k(\omega|s) = 1$ let $\hat{x}_{ik}(s) = x_i(\omega)$. Let $x_i(s)$, $i \in I$ be a Pareto efficient allocation with $V_i(s;x_i(s)) \ge V_i(s;\hat{x}_i(s))$, for all agents i.

Before defining $x_i(s)$ for the remaining states, note that $G_{ik} > 0$ for all agents $i \in I$ and for sufficiently large k, where G_{ik} is agent i's gains from trade given π^k.[4] This follows because $u_i(\omega, x_i(\omega)) - u_i(\omega, e_i) > 0$, and $V_{ik}(s;x)$ converges to $u_i(\omega,x)$ as $k \to \infty$ for any $x \in \mathbb{R}^\ell_+$ and $(\omega,s) \in \Delta$. Let ε_{ik} be defined as in Section 5.2.1, given prior π^k. Then (5) implies $\lim_{k\to\infty}\varepsilon_{ik} = 0$. Thus, $\limsup_k 1/(1+G_{ik} - 2\varepsilon_{ik}) < 1$. Since the left-hand side of (4) converges to 1, (4) holds for all sufficiently large k. We can therefore find a sequence $\tilde{G}_{ik}$ with $\lim_{k\to\infty}\tilde{G}_{ik} = 0$, such that (4) holds for all sufficiently large k (replacing G_{ik} by $\tilde{G}_{ik}$).

We now define $x_{ik}(s)$ for all remaining s. First, assume that $s \in D_j$. Let $\bar{s}$ be the corresponding diagonal element.[5] Let ω denote the unique element in

[4] G_i is defined immediately before the statement of Proposition 1 as the gains from trade given prior π.

[5] Recall that $s \in D_j$ if $s = (s_{-j}, s_j)$ where $s_{-j} \in \sigma_{-j}(s_j')$, for some $s_j' \ne s_j$. Then the corresponding diagonal element $\bar{s} = (s_{-j}, s_j')$. Moreover, note that $\bar{s}$ is the only element of S with the property, because the pooled information of all agents excluding agent j is complete.

$\bigcap_{i=1}^{n} \bar{s_i}$. Now choose $\hat{x}_j(s)$ as a convex combination of $e_j(\omega)$ and $x_j(\omega)$ such that $V_{jk}(\bar{s}; x_j(\omega)) - V_{jk}(\bar{s}; \hat{x}_{jk}(s)) = \tilde{G}_{jk}$.

For $i \neq j$ we can find α_{ik} with $0 \leq \alpha_{ik} \leq 1$ and $\sum_{i \neq j} \alpha_{ik} = 1$ such that $\hat{x}_{ik}(s) = x_i(\omega) + \alpha_{ik}\big(x_j(\omega) - \hat{x}_{jk}(s)\big) \in \mathbb{R}_+^{\ell}$.

Let $x_{ik}(s)$ be a Pareto efficient allocation with $V_{ik}(s; x_{ik}(s)) \geq V_{ik}(s; \hat{x}_{ik}(s))$, such that the equality holds for agent j.

Finally, for all remaining states s, choose $x_i(s)$, $i \in I$ to be a competitive equilibrium allocation in the economy with utilities $V_{ik}(s; \cdot) = E_\Omega[u_i(\omega, \cdot)|s]$.

Because the assumptions of Theorem 1 are fulfilled, there exist M_i, $i \in I$, such that $\{x_{ik}(s), p_k(s), M_{ik}(s) | i \in I, s \in S\}$, is an incentive compatible equilibrium with lump sum transfers. It remains to prove that $\lim_{k \to \infty} M_{ik}(s) = 0$ for all s, and that prices and allocations converge to those of the complete information competitive equilibrium.

We first prove convergence of the allocations. Let $\omega \in \Omega$, $s \in S \setminus D$ be such that $\lim_{k \to \infty} \pi^k(\omega|s) = 1$. By construction $V_{ik}(s; x_{ik}(s)) \geq V_{ik}(s; x_i(\omega))$. Now take any subsequence $x_{ik_t}(s)$, $t \in \mathbb{N}$ that converges for all agents i. Such a sequence must exist, because agents' consumption fulfills $0 \leq x_{ik}(s) \leq e$, where e is the aggregate endowment. Denote the limit by $\bar{x_i}(s)$. Then $\lim_{t \to \infty} V_{ik_t}(s; x_{ik_t}(s)) = u_i(\omega, \bar{x_i}(s))$. Thus, $u_i(\omega, \bar{x_i}(s)) \geq u_i(\omega, x_i(\omega))$, for all agents i. Now note that because $x_i(\omega)$ is a competitive equilibrium, it is also Pareto efficient. Therefore $u_i(\omega, \bar{x_i}(s)) = u_i(\omega, x_i(\omega))$. Since u_i is strictly concave it therefore follows that $\bar{x_i}(s) = x_i(\omega)$, for all agents i. Because the subsequence k_t, $t \in \mathbb{N}$ was chosen arbitrarily, we can conclude that $x_{ik}(s)$ converges and that $\lim_{k \to \infty} x_{ik}(s) = x_i(\omega)$.

Now assume that $s \in D_j$. Then since $\lim_{k \to \infty} \tilde{G}_{ik} = 0$, it follows that $\hat{x}_{jk}(s)$ converges to $x_j(\omega)$. Thus, $\hat{x}_{ik}(s)$ converges to $x_i(\omega)$ for all agents i. We can therefore use the above argument to show that $x_{ik}(s)$ converges to $x_i(\omega)$ for all agents $i \in I$.

Next, we show that prices converge. Consider ω, s with $\lim_{k \to \infty} \pi^k(\omega|s) = 1$. Because $x_i(\omega)$ is the competitive equilibrium at price $p(\omega)$ and $\lim_{k \to \infty} x_{ik}(s) = x_i(\omega)$ it follows by the smoothness of preferences that $\lim_{k \to \infty} p_{ik}(s) = p(\omega)$.

Finally, we show that the lump sum transfers converge to 0. Let s, ω be such that $\lim_{k \to \infty} \pi^k(\omega|s) = 1$. For other s, ω there is nothing to prove since the allocation is already a competitive equilibrium with $M_i(s) = 0$. Note that $p_{ik}(s)(x_{ik}(s) - e_i(s)) = M_{ik}(s)$. If we take the limit for $k \to \infty$ on both sides of this equality we get $p(\omega)(x_i(\omega) - e_i(\omega)) = \lim_{k \to \infty} M_{ik}(s)$. However, since $x_i(\omega)$ is a competitive equilibrium allocation, $p(\omega)(x_i(\omega) - e_i(\omega)) = 0$. Thus, $\lim_{k \to \infty} M_{ik}(s) = 0$. ■

Proof of Theorem 3. We provide the argument for the case where $S_i = \Omega$. The general case where S_i is a partition of Ω is similar.

If π^k, $k \in \mathbb{N}$ violates condition 2 then there exists an agent j and states ω, ω', and $\bar{c} > 0$ such that $\limsup_{k\to\infty}\left|\pi(\omega \mid \sigma_{-j}(s_j), s_j\big) - \pi(\omega \mid \sigma_{-j}(s_j), s'_j\big)\right| \geq \bar{c} > 0$, for $s_j = \omega$ and $s'_j = \omega'$. Without loss of generality we can assume that $j = 1$. Now consider the following economy.

The agents' endowments are $e_i = (1,1)$, $i \in \mathbb{N}$. For $y \in \mathbb{R}^2_+$, let $u_1(\omega, y) = v(y) = \log y_1 + \log y_2$. For all $i > 1$ let

$$u_i(\omega, y) = \begin{cases} 2\log y_1 + \log y_2 & \text{if } \omega = \omega_1; \\ 3\log y_1 + \log y_2 & \text{otherwise.} \end{cases}$$

Now consider the economy with prior π^k. Let s be the signal of all agents. Then the unique equilibrium allocation will only depend on $\alpha = \pi^k(\omega_1 \mid s)$ and can therefore be denoted by $x(\alpha)$. Moreover, one can easily verify that $E\big(u_1(\cdot, x_1(\alpha)) \mid s\big) = v(x_1(\alpha))$ is a strictly decreasing function of α.

Now assume that agent 1 observes $s_1 = \omega_1$. Let $s = (s_1, \ldots, s_1)$. Define $\alpha_k = \pi\big(\omega_1 \mid s\big)$ and $\tilde{\alpha}_k = \pi\big(\omega_1 \mid \sigma_{-1}(s_1), s'_1\big)$. Then there exist $k > 0$ such that $\limsup_{k\to\infty}\big|v(x_1(\alpha_k)) - v(x_1(\tilde{\alpha}_k))\big| \geq k$. Thus, in order to induce agent 1 to report truthfully, one must choose $M^k_i(\sigma_{-1}(s_1), s'_1)$ such that $\limsup_{k\to\infty} M^k_i(\sigma_{-1}(s_1), s'_1) > 0$. However this implies that the economy does not converge to the complete information competitive equilibrium. ■

REFERENCES

G. AKERLOF (1970), "The Market for Lemons," *Quarterly Journal of Economics* **89**, 488-500.

M. JACKSON (1991), "Bayesian Implementation," *Econometrica* **59**, 461–478.

F. GUL AND A. POSTLEWAITE (1992), "Asymptotic Efficiency in Large Economies with Asymmetric Information," *Econometrica* **60**, 1273–1292.

A. HEIFETZ AND E. MINELLI (1997), "Informational Smallness in Rational Expectations Equilibria," mimeo.

L. HURWICZ (1972), "On Informationally Decentralized Systems," in McGuire and Radner eds., Decision and Organization, A Volume in Honor of Jacob Marschak, North Holland: Amsterdam.

L. HURWICZ, E. MASKIN AND A. POSTLEWAITE (1995), "Feasible Nash Implementation of Social Choice Rules When the Designer Does Not Know Endowments or Production Sets," in John Ledyard ed., Essays in Honor of Stanley Reiter, Kluwer: Boston.

M. JACKSON AND A. MANELLI (1997), "Approximately Competitive Equilibria in Large Finite Economies," *Journal of Economic Theory* **77**, 354–376.

M. JACKSON (1992), "Incentive Compatibility and Competitive Allocations," *Economic Letters* **40**, 299–302.

A. KAJII AND S. MORRIS (1997), "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283–1309.

A. KAJII AND S. MORRIS (1998), "Payoff Continuity in Incomplete Information Games," *Journal of Economic Theory* **82**, 267–276.

R MCLEAN AND A. POSTLEWAITE (1999), "Informational Size and Incentive Compatibility," CARESS Working Paper #99-14, University of Pennsylvania.

Y. OTANI AND J. SICILIAN (1990), "Limit Properties of Equilibrium Allocations of Walrasian Strategic Games," *Journal of Economic Theory* **51**, 295–312.

D. ROBERTS AND A. POSTLEWAITE (1976), "The Incentives for Price-Taking in Large Economies," *Econometrica* **44**, 115–128.

A. RUSTICHINI, M. SATTERTHWAITE AND S. WILLIAMS (1994), "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica* **62**, 1041–1063.