Incompleteness as a Constraint in Contract Design*

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Abstract

We develop a model that encompasses both the incomplete contracts that are used in practice and the idealized complete contracts that address all contingencies. The objectives of the paper are: (i) to identify properties of agents’ preferences that determine whether or not incompleteness causes inefficiency; (ii) to examine the extent of the inefficiency caused by the constraint of contractual incompleteness; (iii) to analyze the implications of the incompleteness constraint on optimal contracts in principal-agent and bilateral bargaining models.

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1 Introduction

Contracts are necessarily limited in length. As a consequence, they may be incomplete in the sense that they fail to address all of the different contingencies that may arise. This incompleteness can lead to inefficiency in the contractual outcome, as evidenced by legal disputes or costly renegotiation. In this paper we develop a model that encompasses both the incomplete contracts that are used in practice and the idealized complete contracts that address all contingencies. The goals of this paper are to characterize properties of agents’ preferences that determine whether or not incompleteness causes inefficiency and to examine the extent of the inefficiency caused by the constraint of contractual incompleteness and to

Two instances of a principal-agent model illustrate the differences among contracting problems with respect to contractual incompleteness. First, consider a homeowner who hires a contractor to build a home. Building a home requires a multitude of decisions in design, material, and construction. Suppose the homeowner can specify a brief list of features that matter greatly to him. In this case, he is likely to be able to reach a mutually satisfactory agreement with the contractor by writing a contract specifying those features and leaving all other features at the discretion of the contractor. If the homeowner instead cares greatly about a multitude of details in the construction of the home, then he is unlikely to be satisfied with the outcome of any incomplete contract. This illustrates how the effectiveness of an incomplete contract can depend on the preferences of the contracting parties, specifically with regard to whether or not unspecified contingencies can be regarded as mere “details” that have only a minor effect on utility. Asymptotic decreasing importance is a feature of our model that captures this intuition.

Second, consider a central administration (or center) and a division of a firm. The issue to be contracted is the production plan of the division for an upcoming year. Relevant contingencies include the type realizations (e.g., high or low quality) of the employees. The division manager has private information about the workers’ types. The center’s problem of finding the optimal work assignments for the division is a principal-agent problem with hidden information, in which the center is the principal and the division manager is the agent. Optimal contracts in models of this problem are complete and highly type specific. In reality, however, a contract between a center and a division tends to be much simpler. For example, it may specify only payoffs to the division as a function of output along with performance targets for a few key employees. All other decisions are delegated to the division manager.

It is quite possible that beyond a certain number of employees the private information about types of all other employees is insignificant to the performance of the division. If this is the case, then asymptotic decreasing importance holds. This
suggests that an incomplete contract between the center and the division may adequately approximate the optimal production plan. An approximating incomplete contract, however, might still be quite lengthy. Asymptotic decreasing importance alone thus cannot explain the prevalence of contracts that are nearly state independent, such as delegation of decision making to the division manager. A second condition is needed to explain this common but extreme form of contractual incompleteness.

Our second condition explains the fact that the performance of only few key employees might be contracted upon by the center and the division. A particular contingency is reversible if other contingencies may be realized in such a way as to undo the utility effects of the given contingency. Reversibility of a particular employee’s type in this example means that, regardless of whether that employee is of high or low quality, the types of other employees can be realized in such a way as to undo the effect upon production of the selected employee’s type. Conversely, a particular employee’s type is irreversible if its effect cannot be masked by the type realizations of other employees. An irreversible contingency is thus easy to contract upon and a reversible contingency permits misrepresentation that may be difficult to address. Theorems 5 and 6 show that the center can effectively contract upon the performance of only those employees who are key in the sense that their types are irreversible. Authority over those employees whose types are reversible is therefore delegated to the division manager. If no employees are key in this sense, then the contract between the center and the division will not explicitly depend on the performance of any individual.

Our model is as follows. There are two agents and a countably infinite number of contingencies that may affect their utilities of the agents and over which they may contract. For any \( i \in \mathbb{N} \), the realization \( a_i \) of contingency \( i \) is an element of \{0, 1\}. This can be interpreted as a “no” or “yes” answer to whether contingency \( i \) has occurred. A state of the world \( \alpha \) is a sequence \( \alpha = (a_i)_{i \in \mathbb{N}} \) that specifies a realization of each contingency \( i \). A contract is a function \( f : A \rightarrow C \) from the set of states of the world \( A \) into a set \( C \) of possible collective choices for the agents. The utility \( u_i(\alpha, c) \) of each agent is a function of the state \( \alpha \) and the choice \( c \). A contract is incomplete if it only depends on a finite number of contingencies. A contract that does not have this property is called complete (even though it may not depend upon the realization of each contingency).

Our distinction between an incomplete and a complete contract is therefore determined by whether the contract is finite or infinite in length. A feasible contract in our paper is an incomplete contract and a complete contract is at best a useful abstraction. This is analogous to using an infinite time horizon to model decision making by agents who can not foresee an ending date, or an infinite number of traders to model perfect competition. People in reality do not live forever and mar-
kets do not truly have an infinite number of traders; similarly, we are not proposing that the contingencies in a contracting problem are in reality infinite. Rather, an infinity of contingencies models the unattainable complexity of addressing every contingency without making ad hoc assumptions about the feasible length of a contract or the costs of lengthening it.

In the same sense that a perfectly competitive market is useful as an abstraction to the extent that it can be approximated by finite markets, a complete contract is an idealization whose meaningfulness depends upon it being the limit of a sequence of incomplete contracts. A contract that can be approximated in this sense is recordable. Recordability is a rather weak requirement to impose on a complete contract as a way to model the fact of contractual incompleteness; it is far less severe than the alternative of simply restricting attention to incomplete contracts. Recordability simply requires that a contract can be approximated arbitrarily closely with incomplete contracts. The main results of this paper address the recordability of complete contracts, first in the case of complete information in which each agent fully observes the state of the world, and then in the case of asymmetric information in which one or both agents privately observes the realizations of some contingencies. We first show in the case of complete information that asymptotic decreasing importance is a sufficient condition on preferences under which all complete contracts are recordable. This, however, is not true when information is asymmetric and the constraint of incentive compatibility becomes relevant: although recordability is a minimal requirement, it has major implications when information is incomplete.

We demonstrate this point in two common contracting problems with asymmetric information. The first problem is an extension of the Chatterjee-Samuelson (1983) bilateral bargaining problem and the second is a principal-agent problem with hidden information. Reflecting the presence of asymmetric information, we consider the recordability of those complete contracts that are optimal subject to the constraints of incentive compatibility and interim individual rationality. At issue is whether or not such an optimal contract is recordable with the additional requirement that the incomplete contracts in the sequence converging to the optimal contract must satisfy incentive compatibility and interim individual rationality. Recordability is shown to depend crucially upon the agents’ preferences, and in particular upon the reversibility of contingencies. Similar results hold in each of the two problems:

(i) The optimal contract is complete.

(ii) If all contingencies are reversible, then any recordable contract is necessarily state independent. The optimal contract in this case is thus not recordable.
(iii) Conversely, if all contingencies are irreversible, then the optimal contract is recordable.

The most provocative of these results is (ii), for it describes a case in which recordability is a severe constraint on ex ante contracting. We clarify this point in the context of the bilateral bargaining problem. A choice $c \in C$ in this problem consists of a decision of whether or not the trade of a good or service takes place between a seller and a buyer along with a price at which the transaction may occur. The set $A$ of states of the world consists of all attributes of the good that are utility relevant and about which the agents may wish to contract (e.g., time and date of delivery, aspects of warranty, and physical characteristics of the good). The optimal contract in this problem is characterized in Myerson and Satterthwaite (1983) and is highly state dependent. As noted in (ii) above, if all contingencies are reversible, then every recordable contract is state independent. A recordable contract thus cannot in any way address the actual realization of contingencies. Ex ante contracting is therefore completely ineffective. A practical interpretation of this result is that ex ante contracting will be dispensed with completely in such a case and replaced with interim negotiation of the terms of trade.

As explained above, we see recordability as a minimal requirement on a contract in models with asymmetric information. Depending upon the agents’ preferences, however, the two models analyzed in this paper show that this requirement can cause an extreme form of contractual incompleteness. The state dependent contracts that have commonly been deemed optimal in models of incomplete information may not be recordable and may thus overstate the performance potential of ex ante contracting. As suggested in the bilateral bargaining problem above, the viability of ex ante contracting must itself be questioned in such cases.

1.1 Related Work

There are two components of the contracting literature. The first is the incomplete contracting literature, which starts with the assumption that certain contingencies can be contracted upon and others can not.\(^1\) This is commonly motivated by the observation that some contingencies may be difficult to characterize unambiguously, even though within the model they may not differ in any mathematical sense from those contingencies that are assumed to be contractible. The standard criticism of this approach is that specifying certain contingencies as uncontractible in this way seems arbitrary. We address this criticism in our paper by identifying properties of

\(^1\)This is typified by Grossman and Hart (1986), Hart and Moore (1988), Aghion and Bolton (1992), and Aghion, Dewatripont and Rey (1994).
preferences that distinguish those contingencies that can be contracted upon from those that cannot.

The second component is the complete contracting literature, which derives contractual incompleteness from primitives of the model such as asymmetric information and limited commitment. Our work contributes to this strand of the literature by showing that asymmetric information together with reversibility of contingencies are primitive frictions that may cause an extreme form of contractual incompleteness.

Our paper originates most directly in the work of Anderlini and Felli (1994), who grounded the theory of contractual incompleteness in the theory of computational complexity. The most fundamental idea that we draw from their paper is the identification of incomplete contracts with finiteness and complete contracts with infiniteness. This idea stems from regarding the process of writing a contract as a computational problem that is addressed by a Turing machine. The state space in their paper is an interval on the real line with a point in the interval interpreted through its binary expansion as a sequence of “yes” or “no” answers to an infinite number of questions. Incomplete contracts can only depend on a finite number of these questions and are therefore identified with step functions on the state space. More recent work based on this approach includes Anderlini and Felli (1998) and Al-Najjar and Casadesus-Masanell (1999). These papers, like ours, avoid assumptions concerning the costs of adding contingencies to a contract by assuming instead that all incomplete contracts are equally feasible and all complete contracts are infeasible.

The main distinction between our approach and these papers is that we start with sequences of “yes” and “no” answers as our primitive and explore the relationship between the efficiency of incomplete contracts and the functions that determine the agents’ utilities from the contingencies. These other papers assume the specific relationship of the binomial expansion between contingencies and the agents’ utilities. Our approach allows us to identify properties of the utility functions (asymptotic decreasing importance and reversibility) that distinguish contracting problems in which incomplete contracts can be almost efficient from those in which they cannot. Asymptotic decreasing importance and irreversibility of contingencies are implicitly assumed by Anderlini and Felli (1994) by their choice of

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3In contrast, Dye (1985) models the impact of positive costs of writing contingencies on optimal contracts. Batigalli and Maggi (2000) provide a detailed theory of the relationship between the type of a particular contingency and the cost of specifying it in a contract. Both of these papers show that a contract will be incomplete if the costs of specifying an additional contingency outweigh the benefits.
the binomial expansion; consistent with our results, they show that a complete contract can be approximated by incomplete contracts and is therefore recordable in the sense of our paper.

2 The Model

We start with a probability space \((A, \mathcal{A}, \pi)\), where \(A = \{0, 1\}^\mathbb{N}\) is the set of states of the world, \(\mathcal{A}\) is the \(\sigma\)-algebra of measurable sets, and \(\pi\) is the common prior of the two agents. A state of the world is denoted as \(\alpha = (a_i)_{i \in \mathbb{N}}\). All cylinder sets of the form \(\{a_1\} \times \cdots \times \{a_n\} \times \{0, 1\}^\mathbb{N}\) are in \(\mathcal{A}\), which permits the evaluation of probabilities conditional on the realization of a finite set of contingencies. Let \(C\) be the choice set. Agent \(j\)'s utility function is given by \(u_j : A \times C \rightarrow \mathbb{R}\).

**Definition 1** A contract \(f\) is **incomplete** if and only if there exists \(n \in \mathbb{N}\) such that \(f(\alpha) = f(\alpha')\) for all \(\alpha = (a_i)_{i \in \mathbb{N}}\) and \(\alpha' = (a'_i)_{i \in \mathbb{N}}\) with \(a_i = a'_i\) for \(0 \leq i \leq n\). A contract is **complete** if it is not incomplete.

Recordability indicates whether or not a complete contract overstates the potential of contracting in the sense that its performance cannot be approximated by a sequence of incomplete contracts. A precise definition is given in the context of each problem considered in the remaining sections. Intuitively, a contract \(f\) is **recordable** if there exists a sequence of incomplete contracts \((f_m)_{m \in \mathbb{N}}\) such that

\[
\lim_{m \to \infty} \|f_m\| \geq \|f\|,
\]

where “\(\|\cdot\|\)” denotes a measure of performance that is appropriate in the particular contracting problem. Each contract \(f_m\) in the sequence is also required to have properties appropriate to the problem. In the principal-agent problem of section 6, for instance, “\(\|f\|\)” is the ex ante expected payoff to the principal determined by a given wage contract \(f\) that he offers to the agent, while in the bilateral trade problem of section 7 it is the ex ante gains from trade achieved by the traders in the contract \(f\). The agent and the traders must willingly participate and fulfill the contract. Each \(f_m\) must therefore be both interim individually rational and incentive compatible in these problems. If \(f\) is not recordable, then all incomplete contracts are bounded below \(f\) according to the performance measure for the problem, i.e.,

\[
\|f^\star\| < \|f\| + k,
\]

for some constant \(k > 0\) and any incomplete contract \(f^\star\). An contract \(f\) that is not recordable therefore overstates the performance potential of contracting.
Our notion of recordability is based on approximating a contract \( f \) with a sequence of incomplete contracts that in the limit performs at least as well as \( f \). One might insist that the sequence \((f_m)_{m \in \mathbb{N}}\) converges in some stronger sense, e.g., that for large \( m \) the incomplete contract \( f_m \) approximately implements the same choices as \( f \) in each state of the world. Ours is an appropriate notion of approximation given our interest in the efficiency loss associated with contractual incompleteness. Requiring a stronger notion of convergence would weaken our most interesting results, namely those in which recordability is shown to be a binding constraint in contract design. All such results of course also hold for more demanding notions of convergence. Theorems 6 and 9 below, however, which prove recordability of particular contracts, do so by constructing a sequence of incomplete contracts that converges pointwise to the given contract. Incomplete contracts in these results thus not only match the performance of the recordable contract but also approximately implement the same social choices.

3 Complete Information

This section concerns the case in which both agents fully observe the state \( \alpha \) at the interim stage. This provides a simple setting for identifying the significance of asymptotic decreasing importance.

Optimal contracting problems between two agents in this case are distinguished by both the timing of the participation constraint and by the time at which contracting occurs. In the case of ex ante optimization together with an ex ante participation constraint for agent 2, the optimization problem is given by

\[
\max_{f : A \to C} \int u_1(\alpha, f(\alpha)) d\pi(\alpha) \text{ s.t. } \int u_2(\alpha, f(\alpha)) d\pi(\alpha) \geq r. \quad (\text{Ex Ante})
\]

Alternatively, one can require interim or ex post participation constraints together with maximization at the ex ante, interim or ex post stage. The solution to any such problem must also be a solution to the ex post optimization problem with an ex post participation constraint,

\[
\max_{f : A \to C} u_1(\alpha, f(\alpha)) \text{ s.t. } u_2(\alpha, f(\alpha)) \geq r(\alpha), \text{ for all } \alpha \in A, \quad (\text{Ex Post})
\]

where \( r(\alpha) \) is agent 2’s reservation utility in state \( \alpha \). A solution \( f \) of (Ex Post) is recordable if there exists a sequence of incomplete contracts \((f_n)_{n \in \mathbb{N}}\) that satisfy the ex post participation constraint and such that \( \lim_{n \to \infty} u_1(\alpha, f_n(\alpha)) \geq u_1(\alpha, f(\alpha)) \) for all \( \alpha \in A \).

As mentioned in the introduction, asymptotic decreasing importance holds for in a contracting problem if all but a finite number of contingencies can be regarded
as “details” that have only a minor effect on utility. Theorem 1 below identifies asymptotic decreasing importance as a sufficient condition for insuring that a solution of (Ex Post) is recordable. This result immediately implies recordability for any solution of (Ex Ante), and as a consequence also for solutions of the other optimization problems.

**Definition 2** The contingencies are asymptotic decreasing in importance if and only if for every \( \varepsilon > 0 \), there exists an \( n \in \mathbb{N} \) such that

\[
|u_j(\alpha, c) - u_j(\alpha', c)| < \varepsilon
\]

for each agent \( j \), all \( c \in C \), and all \( \alpha = (a_i)_{i \in \mathbb{N}}, \alpha' = (a'_i)_{i \in \mathbb{N}} \in A \) with \( a_i = a'_i \) for \( 0 \leq i \leq n \). If this condition holds, then the contracting problem satisfies asymptotic decreasing importance.\(^4\)

A regularity condition is also needed for Theorem 1. Define the maximum function

\[
M(\alpha, r) = \max_{c \in C} u_1(\alpha, c) \text{ s.t. } u_2(\alpha, c) \geq r.
\]

We assume in Theorem 1 that this function is well-defined\(^5\) and that

\[
M(\alpha, r) - M(\alpha, r + \Delta) \leq K \cdot \Delta \text{ for all } \alpha \in A \text{ and } \Delta \in [0, \gamma],
\]

for some constants \( K, \gamma > 0 \). This is a Lipschitz condition on \( M(\alpha, r) \); it is satisfied (for instance) if \( \frac{\partial M(\alpha, r)}{\partial r} \) exists and is uniformly bounded in a neighborhood of \( r \) for all \( \alpha \).\(^6\) It holds in many models in the contracting literature. The proof of Theorem 1 is in the Appendix.

**Theorem 1** Let \( \hat{f} \) denote a solution to Problem (Ex Post) in a contracting problem that has the following properties.

1. Asymptotic decreasing importance holds;

2. Agent 2’s reservation utility \( r(\cdot) \) depends upon only a finite number of contingencies and the optimal contract \( \hat{f} \) satisfies the ex post individual rationality constraint with equality;

\(^4\)Asymptotic decreasing importance implies continuity of the valuation function in the product topology on \( A \). This issue is discussed in more detail after the statement of Lemma 1 in the Appendix.

\(^5\)Sufficient conditions are that \( C \) is compact and that \( u_j(\alpha, c), j = 1, 2 \) is continuous in \( c \) for each \( \alpha \).

\(^6\)Conditions on an optimization problem to insure that \( M(\alpha, r) \) satisfies (2) are routine in the comparative statics literature and so we do not dwell on the topic here.
3. \( M(\alpha, r) \) is well defined and satisfies (2).

Then \( \hat{f} \) is recordable.

Theorem 1 requires that the reservation utility \( r(\alpha) \) of agent 2 depends only on a finite number of contingencies. Because an incomplete contract can only address a finite number of contingencies, it cannot hold agent 2 to his reservation utility \( r(\alpha) \) in every state \( \alpha \) if this function depends upon an infinite number of contingencies. A solution to (Ex Post) in the case of such an individual rationality constraint will thus not be recordable.

It is instructive to consider an example in which asymptotic decreasing importance does not hold and an optimal contract is therefore not recordable.\(^7\)

**Example 1** Consider a principal-agent problem in which the principal (agent 1) is a patient and the agent (agent 2) is a physician. A state \( \alpha \in A \) is a sequence of answers to an infinite number of questions concerning the condition of the patient. To model the complexity of diagnosis and selection of the appropriate treatment, we assume that each sequence of answers \( \alpha = (a_i)_{i \in \mathbb{N}} \) requires its own course of action. We thus identify \( C \) with \( A \). The utility functions of both the patient and the physician are of the form

\[
u_j(\alpha, c) = \begin{cases} 
1 & \text{if } \alpha = c; \\
0 & \text{if } \alpha \neq c.
\end{cases}
\]

Utility equal to 1 corresponds to the good health of the patient and utility equal to 0 corresponds to illness. The interests of the patient and the physician are thus perfectly aligned and there are consequently no agency problems.

A tail of contingencies \( (a_i)_{i \geq n} \), no matter how large the value of \( n \), can never be regarded as mere details in this example; for a given choice of \( c \in C \), the realization of the tail \( (a_i)_{i \geq n} \) in any state \( \alpha = (a_i)_{i \in \mathbb{N}} \) always matters in determining whether the utility of the patient and the physician equal 1 or 0. Consequently, asymptotic decreasing importance does not hold.

Suppose \( r(\alpha) = 0 \) so that the participation of the physician is assured. The identity contract \( \hat{f}(\alpha) = \alpha \) is the unique solution to the (Ex Post) problem. An incomplete contract (no matter how long) can not even approximately define the correct action as a function the state. This is shown by the following argument. Let

\(^7\)Example 3 fulfills all conditions of Theorem 1 except for the Lipschitz condition (2). To ensure that it holds as well, we can introduce “money” that enters each agent’s utility linearly. This implies \( \frac{\partial M(\alpha, r)}{\partial r} = -1 \), and (2) holds. We choose not to do this in order to keep the example as simple as possible.
let \((f_n)_{n \in \mathbb{N}}\) be a sequence of incomplete contracts. For every \(n \in \mathbb{N}\), there exists at most a finite number of states \(\alpha\) at which \(f_n(\alpha) = \hat{f}(\alpha)\). Thus,

\[
\limsup_{n \to \infty} u_i(\alpha, f_n(\alpha)) = 0 < 1 = u_i(\alpha, \hat{f}(\alpha)),
\]

for all but at most a countable subset of states \(\alpha \in A\), while \(A\) has the same cardinality as \(\mathbb{R}\). As a consequence, \(\hat{f}\) is not recordable.

Contracts between patients and physicians describing courses of action are uncommon, in part because of differences in expertise. Such differences in expertise alone, however, cannot explain the almost complete absence of contracts between physicians and patients; instead, they would give rise to the second-best contracts of a principal-agent model. This example points to the impossibility of adequately specifying the action as the reason why ex ante contracting is uncommon in this setting. It is instead common to delegate the choice of the action to the physician upon observation of the state. This leads to the first best outcome within the example because the interests of the doctor and the patient are perfectly aligned.

A shortcoming of this last example is that it does not model the asymmetry of information that typifies a patient-physician relationship. We begin in the next section to address contracting subject to this constraint.

### 4 Asymmetric Information

For \(j = 1, 2\), let \(A_j\) denote agent \(j\)'s type space, which is the set of contingencies that agent \(j\) observes. The set of states of the world is \(A = A_1 \times A_2\).

**Definition 3** A contract \(f\) is **incentive compatible** if and only if

\[
E_{A_{-j}} [u_j(\alpha, f(\alpha)) | \alpha_j] \geq E_{A_{-j}} [u_j(\alpha, f(\alpha', \alpha_{-j})) | \alpha_j]
\]

for \(j = 1, 2\) and for all \(\alpha_j, \alpha'_j \in A_j\).

The revelation principle justifies restricting attention to contracts that are incentive compatible. Imposing the constraint of incompleteness on incentive compatible contracts is motivated by a variation of the revelation principle that we now discuss. Our purpose with this variation is to ground incomplete contracts in a theory of limited communication between the agents.\(^8\)

\(^8\)Other examples of this approach include Dow (1991), Meyer (1991), and Rubinstein ((1993), (1998, Chapter 5)). In their work, an agent can observe a state \(\omega\) (e.g., a price), but actions can only be contingent on a set \(P(\omega)\) that contains \(\omega\) (whether or not the price is high or low), because communication among agents is limited.
A strategy of agent $j$ in a game is a mapping $\gamma_j : A_j \to M_j$ from his type space $A_j$ into his strategy space $M_j$. Strategy $\gamma_j$ is incomplete if it depends upon only a finite number of contingencies. The version of the revelation principle that we now state motivates incentive compatible incomplete contracts by showing that they arise as equilibria in incomplete strategies.

**Revelation Principle for Incomplete Contracts.** An incomplete contract is incentive compatible if and only if it can be implemented as a Bayesian Nash equilibrium of a game using incomplete strategies.

This result is proven by a straightforward modification of the proof of the standard revelation principle. The proof is therefore omitted.

A fundamental assumption underlying this paper is that communication between agents is finite. The rationale of a Bayesian Nash equilibrium requires that each agent knows the strategy of his opponent. If communication is finite, then agents can only communicate incomplete strategies to each other, which motivates Bayesian Nash equilibria in incomplete strategies. An equilibrium of this form is resistant to arbitrary deviations by either agent, however, reflecting the fact that an agent having observed his type is capable of reporting any of his available messages. This again emphasizes a theme of this paper, namely the distinction between the complexity of communicating state-contingent messages in the ex ante stage as opposed to the simplicity of selecting a single message in the interim stage.

Our discussion of asymmetric information concerns an independent private value model. It is assumed for the remainder of the paper that:

1. The types of the agents are independent.
2. Each agent $j$’s utility is quasilinear in the sense that
   \[ u_j(\alpha, c) = h_j(c)v_j(\alpha_j) + t_j(c). \] (3)

The function $v_j(\alpha_j)$ is agent $j$’s valuation function, which represents the portion of his utility that is determined directly by his type. The function $t_j(\cdot)$ is a monetary transfer to agent $j$ and $h_j(\cdot)$ is a level or fraction of $v_j(\alpha)$ that agent $j$ receives as a result of the social choice. The form of utility in (3) is common in the mechanism design and the contracting literatures.

Let $\pi_j$ denote the probability distribution on $A_j$. We write

\[ H_j(\alpha_j^*) = E \left[ h_j(f(\alpha)) \middle| \alpha_j = \alpha_j^* \right], \quad \text{and} \]
\[ T_j(\alpha_j^*) = E \left[ t_j(f(\alpha)) \middle| \alpha_j = \alpha_j^* \right]. \] (4) (5)
Independence of types insures that each function $H_j(\alpha^*_j)$ and $T_j(\alpha^*_j)$ depends only upon the reported type $\alpha^*_j$ of agent $j$ and not upon his observed type $\alpha_j$. A contract $f$ is thus incentive compatible if

$$H_j(\alpha^*_j) v_j(\alpha^*_j) + T_j(\alpha^*_j) \geq H_j(\alpha_j) v_j(\alpha_j) + T_j(\alpha_j)$$

for $j = 1, 2$ and all $\alpha^*_j, \alpha_j \in A_j$. Let $r_j$ denote agent $j$’s reservation utility. The contract $f$ is interim individually rational for agent $j$ if

$$H_j(\alpha_j) v_j(\alpha_j) + T_j(\alpha_j) \geq r_j$$

for all $\alpha_j \in A_j$. We assume for the remainder of the paper that

$$v_i(A_i) \subseteq [\underline{v}_i, \overline{v}_i] \text{ for } i = 1, 2.$$

Finally, let $\mu_j$ denote the induced probability distribution on $[\underline{v}_j, \overline{v}_j]$ defined by $\pi_j$ and $v_j(\cdot)$ and let $v : A \rightarrow [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2]$ denote the valuation mapping

$$v(\alpha) = (v_1(\alpha_1), v_2(\alpha_2)).$$

### 4.1 Contracts and Mechanism Design

The main goal of the remainder of this paper is to investigate how the property of reversibility of contingencies determines whether or not an optimal contract is recordable. We wish to apply standard results from the literature on optimal mechanisms to characterize optimal contracts. The results in this section connect our approach to this literature.

We now distinguish a “contract” from a “mechanism.” Throughout this paper, a contract is a function $f : A \rightarrow C$ that determines a social choice $f(\alpha)$ for each state $\alpha$. Consistent with the terminology of mechanism design, a mechanism is defined for our purposes here as a function $\hat{f} : v_1(A_1) \times v_2(A_2) \rightarrow C$ that determines a social choice $f(v_1, v_2)$ for each pair of valuations $(v_1, v_2)$. As depicted in Figure 1, a mechanism $\hat{f}$ defines a contract $f$ through composition with the valuation mapping $v$; a contract $f$ defines a mechanism $\hat{f}$, however, only if it selects the same social choice for all states that determine the same pair of valuations. The set of contracts is larger than the set of mechanisms, and an optimal contract may thus in principle surpass the performance of an optimal mechanism. This is precisely the point that must be addressed.

We begin with a simple result whose proof is immediate.

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9In the broader literature, “contracts” and “mechanisms” are not distinguished as they are here by their domains; the words are used almost interchangeably, depending upon the subject of the model. The distinction we make here is purely for expositional purposes.
Theorem 2 If the mechanism \( \hat{f} \) and the contract \( f \) satisfy \( f = \hat{f} \circ v \), then:

1. \( \hat{f} \) is incentive compatible with respect to the revelation of valuations if and only if \( f \) is incentive compatible with respect to revelation of types;

2. \( \hat{f} \) is interim individually rational for each valuation of each agent if and only if \( f \) is interim individually rational for each type of each agent.

We now describe a case in which optimal mechanisms define optimal contracts and vice-versa. Let

\[
\Phi : C \times [\bar{u}_1, \bar{v}_1] \times [\bar{u}_2, \bar{v}_2] \rightarrow \mathbb{R}
\]

be the objective of the contracting problem. In a principal-agent model, for instance, \( \Phi \) is the principal’s ex post payoff, and in a bilateral trading problem \( \Phi \) is the ex post gains from trade. The optimal contract problem is

\[
\max_f E_a [\Phi (f(\alpha), v(\alpha))] \text{ s.t. } IC \text{ and } IIR,
\]

and the optimal mechanism problem is

\[
\max_f E_{[\bar{u}_1, \bar{v}_1] \times [\bar{u}_2, \bar{v}_2]} [\Phi (\hat{f}(v), v)] \text{ s.t. } IC \text{ and } IIR,
\]

where \( IC \) and \( IIR \) should be interpreted appropriately in the case of the optimal mechanism problem. Theorem 3 allows us in the following sections to characterize optimal contracts using the theory of optimal mechanisms.

Theorem 3 Suppose that:

1. \( C \) is convex and the objective \( \Phi(c, v_1, v_2) \) is concave in \( c \) for each \( v_1 \) and \( v_2 \).
2. The functions $h_j$ and $t_j$ are affine\(^{10}\) in $c$ for each agent $j$ that has private information (i.e., $A_j \neq \emptyset$).

Then the following statements are true:

1. For every incentive compatible and interim individually rational contract $f$, there exists an incentive compatible and interim individually rational mechanism $\hat{f}$ that ex ante weakly dominates $f$:

$$E_{\mathbb{E}_j, \overline{\pi}_1} \left[ \Phi \left( \hat{f} (v), u \right) \right] \geq E_A \left[ \Phi (f (\alpha), v(\alpha)) \right].$$  \hspace{1cm} (6)

2. As a consequence of statement 1., if a given mechanism $\hat{f}$ solves the optimal mechanism problem, then the induced contract $f = \hat{f} \circ v$ solves the optimal contract problem. Conversely, if a contract $f$ solves the optimal contract problem, then the mechanism $\hat{f}$ defined in the proof of statement 1 solves the optimal mechanism problem.

Theorem 3 is proven in the Appendix. The mechanism $\hat{f}$ that establishes statement 1. is defined by the equation $\hat{f} \circ v = g$, where the contract $g(\alpha)$ is defined by averaging $f(\alpha^*)$ over all states $\alpha^*$ for which $v(\alpha^*) = v(\alpha)$. Theorem 3 thus reflects the familiar result of mechanism design that there may be no gains in ex ante performance from introducing lotteries into the operation of a mechanism. The lottery in this case is the dependence of the choice $f(\alpha^*)$ upon $\alpha^*$ given that $v(\alpha^*) = v(\alpha)$. Given the assumptions of the theorem, ex ante expected performance can only improve by replacing each lottery over choices $\{f(\alpha^*) | v(\alpha^*) = v(\alpha)\}$ with its certainty equivalent, which corresponds to a mechanism.

5 Reversibility

Reversibility is the key property of preferences in models with asymmetric information. It is significant in both (i) determining whether or not a complete contract is recordable, and (ii) identifying those contingencies that can be addressed by incentive compatible incomplete contracts. Property (i) means that reversibility identifies those contracting problems in which ex ante contracting is highly inefficient and hence likely to be replaced by contracting at the interim stage. Property (ii) means that reversibility can distinguish the finite number of contingencies that should be addressed by an incentive compatible incomplete contract from the infinite number that cannot.

\(^{10}\)That is, $h_j(\beta c + (1 - \beta)c) = \beta h_j(c) + (1 - \beta)h_j(c)$ for all $c \in C$ and $\beta \in [0, 1]$, and similarly for $t_j$.\]
The following notation is needed in the definition of reversibility: for any type \( \alpha_j = (a_{j,i})_{i \in \mathbb{N}} \) observed by agent \( j \), let \( \alpha_{j,n^-} = (a_{j,i})_{i < n} \) denote the initial string of contingencies that precede contingency \( n \) and let \( \alpha_{j,n^+} = (a_{j,i})_{i > n} \) denote the tail of contingencies that follow contingency \( n \).

**Definition 4** Contingency \( n \) is reversible for agent \( j \) if for any initial string \( \alpha_{j,n^-} \) one can select at least two pairs of tails

\[
\left\{ (\alpha_{j,n^+}^k, \overline{\alpha}_{j,n^+}^k) \mid k = 1, 2 \right\}
\]

so that the following two properties hold.

1. Each pair of tails \( (\alpha_{j,n^+}^k, \overline{\alpha}_{j,n^+}^k) \) perfectly reverses the effect upon agent \( j \)'s valuation of contingency \( n \): for \( k = 1, 2 \),

\[
v_j(\alpha_{j,n^-}, 0, \alpha_{j,n^+}^k) = v_j(\alpha_{j,n^-}, 1, \overline{\alpha}_{j,n^+}^k). \quad (7)
\]

2. The pairs of tails are distinguishable in their effects upon agent \( j \)'s valuation:

\[
v_j(\alpha_{j,n^-}, 0, \alpha_{j,n^+}^1) \neq v_j(\alpha_{j,n^-}, 0, \alpha_{j,n^+}^2). \quad (8)
\]

Reversibility of contingency \( n \) is depicted in Figure 2. We now define irreversibility.

**Definition 5** Contingency \( n \) is irreversible for agent \( j \) if for every initial string \( \alpha_{j,n^-} \) there does not exist a pair of tails \( \left\{ (\alpha_{j,n^+}^k, \overline{\alpha}_{j,n^+}^k) \mid k = 1, 2 \right\} \) that satisfies (7) and (8).
Definition 5 is not the exact opposite of Definition 4. In order to negate Definition 4 the “for every” quantifier in Definition 4 must be replaced by a “there exists” quantifier. Our purpose in defining irreversibility in this way is to simplify the statements of Theorems 6 and 9 below.

The following example uses a particular family of formulas for an agent’s valuation function to illustrate reversibility.

Example 2 Let \( v^\gamma (\alpha) = \sum_{i=1}^{\infty} \gamma^i a_i \) for \( 0 < \gamma < 1 \). We explore in this example how reversibility of contingency \( n \) depends upon the choice of the parameter \( \gamma \) that specifies the valuation function \( v^\gamma \). Let \( \alpha_{n-} \) be any initial string. The inequality

\[
v^\gamma (\alpha_{n-}, 0, \alpha_{n+}) - v^\gamma (\alpha_{n-}, 1, \alpha_{n+}) \leq v^\gamma (\alpha_{n-}, 0, 1, \ldots) - v^\gamma (\alpha_{n-}, 1, 0, \ldots) = \gamma^n \left( \frac{2\gamma - 1}{1 - \gamma} \right) \tag{10}
\]

holds for any two tails \( \alpha_{n+}, \alpha_{n+} \). The reversibility of contingency \( n \) depends upon whether \( \gamma \) is less than or exceeds 0.5:

1. For \( \gamma = 0.5 \), the right side of (10) equals 0 and (9) is strict except when \( \alpha_{n+} = (1, 1, \ldots) \) and \( \alpha_{n+} = (0, 0, \ldots) \). Condition (7) in the definition of reversibility thus holds only for this one choice of \( \alpha_{n+} \) and \( \alpha_{n+} \). Because \( \alpha_{n-} \) is arbitrary, each contingency \( n \) is therefore irreversible.

2. For \( 0 < \gamma < 0.5 \), the right side of (9) is negative. Each contingency \( n \) is therefore again irreversible.

3. For \( 0.5 < \gamma < 1 \), the right side of (9) is positive. It is also true that

\[
v^\gamma (\alpha_{n-}, 0, \alpha_{n+}) - v^\gamma (\alpha_{n-}, 1, \alpha_{n+}) = -\gamma^n.
\]

Holding the initial string \( \alpha_{n-} \) constant, \( v^\gamma (\alpha_{n-}, 0, \alpha_{n+}) - v^\gamma (\alpha_{n-}, 1, \alpha_{n+}) \) covers an interval on the real line that contains 0 as \( \alpha_{n+} \) is varied. Choose \( \alpha_{n+}^1 \) and \( \alpha_{n+}^2 \) so that (8) holds. For each \( \alpha_{n+}^k \) there exists an \( \alpha_{n+}^k \) so that (7) is satisfied. Contingency \( n \) is therefore reversible.

As a final note to our definition of reversibility, we point out that this concept is bypassed in Anderlini and Felli (1994). As stated earlier, their work led us to the idea of using a countable sequence of contingencies to model the complexity of states upon which contracts may be written. Anderlini and Felli start with real numbers representing agents’ valuations as their primitive and use the binary
expansion as a way to model the information contained in each number. A real number is unraveled in this way to define a sequence of “yes” or “no” answers to a countable number of questions. To avoid ambiguity in this interpretation, they assume that a single binary representation is selected for every real number. We instead start with the set of possible sequences of answers (i.e., the set of states) and consider the properties of different valuation functions that assign real numbers to such sequences. As demonstrated in the two applications that follow, we find that reversibility—the possibility of multiple states defining the same real number as the agent’s valuation—is an interpretable property of preferences on which the effectiveness of ex ante contracting may depend.

5.1 The Significance of Reversibility

Reversibility of a contingency \(a_{j,n}\) for agent \(j\) presents an opportunity for misreporting to this agent that must be policed against. Given the initial string \(a_{j,n-}\), each pair of tails \((\alpha_{j,n-}, \alpha_{j,n+})\) in the definition of reversibility imposes an additional constraint on the contracting problem. The following theorem describes a situation in which reversibility of contingency \(a_{j,n}\) is sufficient to insure that incentive compatible revelation of \(a_{j,n}\) can be achieved by an incomplete contract only at the cost of making agent \(j\) indifferent to his report of this contingency. This forces contractual incompleteness as a means of insuring incentive compatibility. It is common in mechanism design to show that an incentive compatible mechanism can be constructed in a particular problem only by appropriately building inefficiency into the social choice. Analogously, we show in the following sections that reversibility in principal-agent and bilateral trade models can force the agents to design inefficiency into their contract by making it insensitive to certain contingencies as a means of achieving incentive compatibility.

**Theorem 4** For some \(r \in \mathbb{N}\), suppose every contingency observed by agent \(j\) beyond the \(r\)th is reversible. If the contract \(f\) is incentive compatible and incomplete, then the functions \(H_j(\alpha_j)\) and \(T_j(\alpha_j)\) determined by \(f\) (defined in (4) and (5)) depend only upon contingencies \(a_{j,n}\) with \(n < r\). Consequently, agent \(j\)’s interim expected utility in the contract \(f\) depends only upon his true type \(\alpha_j\) and his report of the initial string \(\alpha_{j,r-}\).

**Proof.** Suppose that \(f\) does not depend on any contingency observed by agent \(j\) beyond the \(n\)th for some \(n \geq r\). We show here that \(H_j(\alpha_j)\) and \(T_j(\alpha_j)\) cannot depend upon \(a_{j,n}\). The theorem then follows by backwards induction. Given an initial string \(\alpha_{n-}\), reversibility of contingency \(n\) implies

\[
v(\alpha_{j,n-}, 0, \bar{\alpha}_{j,n+}) = v(\alpha_{j,n-}, 1, \bar{\alpha}_{j,n+})
\]  

(11)
for tail sequences $(\alpha_{j,n+}^k, \overline{\alpha}_{j,n+}^k), k = 1, 2$. Fixing for the moment the value of $k$, let $\underline{\alpha}_j = (\alpha_{j,n-}, 0, \underline{\alpha}_{j,n+})$ and $\overline{\alpha}_j = (\alpha_{j,n-}, 1, \overline{\alpha}_{j,n+})$. Incentive compatibility implies

$$H_j(\underline{\alpha}_j)v_j(\underline{\alpha}_j) + T_j(\underline{\alpha}_j) \geq H_j(\overline{\alpha}_j)v_j(\overline{\alpha}_j) + T_j(\overline{\alpha}_j) \iff (12)$$

and

$$H_j(\overline{\alpha}_j)v_j(\overline{\alpha}_j) + T_j(\overline{\alpha}_j) \geq H_j(\underline{\alpha}_j)v_j(\underline{\alpha}_j) + T_j(\underline{\alpha}_j) \iff (13).$$

Because $v_j(\underline{\alpha}_j) = v_j(\overline{\alpha}_j), (13)$ and $(15)$ can hold only if

$$H_j(\underline{\alpha}_j)v_j(\underline{\alpha}_j) - H_j(\overline{\alpha}_j)v_j(\overline{\alpha}_j) = T_j(\underline{\alpha}_j) - T_j(\overline{\alpha}_j).$$

Write $H_j(\alpha_j) = H_j(\alpha_{j,n-}, a_{j,n})$ and $T_j(\alpha_j) = T_j(\alpha_{j,n-}, a_{j,n})$, reflecting incompleteness. Equation $(16)$ becomes

$$H_j(\alpha_{j,n-}, 0) - H_j(\alpha_{j,n-}, 1) \iff v_j(\alpha_j) = T_j(\alpha_{j,n-}, 0) - T_j(\alpha_{j,n-}, 1).$$

Equation $(17)$ holds for distinct values of $v_j(\alpha_j)$ because contingency $n$ is reversible. This can be true only if

$$H_j(\alpha_{j,n-}, 0) - H_j(\alpha_{j,n-}, 1) = 0 = T_j(\alpha_{j,n-}, 0) - T_j(\alpha_{j,n-}, 1),$$

i.e., $H_j(\alpha_j)$ and $T_j(\alpha_j)$ do not depend upon $a_{j,n}$. 

6 Reversibility in a Principal-Agent Model

In the following two sections, we exploit the structure of two particular contracting problems to strengthen the implications drawn in Theorem 4 concerning the effects of reversibility. We begin with the simplest case in which only one of the two contracting parties has private information. Consider a principal-agent problem with hidden information.$^{11}$ The agent privately observes the state of the world $\alpha \in A$. Let $e \in [0, \overline{e}]$ denote the agent’s effort and $t \in [\underline{t}, \overline{t}]$ a monetary transfer from the principal to the agent. The principal’s utility function is $u_0(e,t) = h(e) - t$, where $h(e)$ is a continuous and strictly increasing production function. The agent’s utility function is $u_1(\alpha, e, t) = t + v(\alpha)e$, where $v(\alpha) < 0$ for all $\alpha \in A$. The state of the world $\alpha$ thus determines the disutility of the effort $e$ through the term $v(\alpha)e$.

A contract specifies an effort \( e(\alpha) \) and a transfer \( t(\alpha) \) as functions of the state \( \alpha \). The principal-agent literature typically focuses on maximizing the ex ante expected utility of the principal subject to these constraints, reflecting the idea that the principal selects the contract and offers it to the agent. The contract must provide the agent with a payoff exceeding his reservation wage \( r \) and it must be in his interest to fulfill the contract once he accepts it. The constraints \( IIR \) and \( IC \) defined above must therefore be satisfied for the agent. The optimal contract for the principal solves

\[
\max_{e(\cdot), t(\cdot)} \int h(e(\alpha)) - t(\alpha) d\pi(\alpha) \text{ s.t. } IIR \text{ and } IC.
\]

This differs from the standard optimal mechanism problem in that the variable \((e, t)\) in (19) can depend upon the state \( \alpha \) and not just the valuation \( v(\alpha) \). Theorem 3, however, implies that a solution \((\hat{e}(v), \hat{t}(v))\) to the optimal mechanism problem also determines a solution to (19) through composition with the agent’s valuation function \( v \).

Because of our interest in the losses associated with contractual incompleteness, we alter the notion of recordability for the principal-agent problem to reflect its emphasis on the ex ante utility of the principal. A contract \((e, t)\) in this problem is thus recordable if there exists a sequence of incomplete contracts \((e_n, t_n)\) satisfying \( IIR \) and \( IC \) such that

\[
\lim_{n \to \infty} \int h(e_n(\alpha)) - t_n(\alpha) d\pi(\alpha) \geq \int h(e(\alpha)) - t(\alpha) d\pi(\alpha).
\]

**Theorem 5** If there exists \( r \in \mathbb{N} \) such that every contingency \( n \geq r \) is reversible for the agent, then the following statements hold.

1. An incentive compatible and incomplete contract \((e, t)\) can only depend on contingencies \( a_n \) for \( n < r \).

2. A recordable contract \((e, t)\) is weakly ex ante dominated for the principal by an incentive compatible incomplete contract \((\hat{e}^*, \hat{t}^*)\), i.e.,

\[
\int h(e^*(\alpha)) - t^*(\alpha) d\pi(\alpha) \geq \int h(e(\alpha)) - t(\alpha) d\pi(\alpha).
\]

3. If only complete contracts solve the optimal contracting problem (19), then optimal contracts are not recordable.
Statement 1 follows immediately from Theorem 4. It is significant because it proves that a particular set of contingencies cannot be addressed by an incentive compatible and incomplete contract. Statement 3 follows from statement 2 by contradiction. This is significant for the following reason. Under standard regularity conditions on the distribution \( \mu \) of the agent’s valuation, only complete contracts are optimal.\(^{12}\) Statement 3 thus applies to show that an optimal contract is not recordable. Under the standard regularity conditions on the principal-agent problem, reversibility of all contingencies in a tail therefore causes a loss to the principal of a magnitude strictly greater than zero. This loss is attributable to the incompleteness of contracts.

**Proof.** Turning to statement 2, let \( ((e_m, t_m))_{m \in \mathbb{N}} \) be a sequence of incentive compatible, interim individually rational incomplete contracts that demonstrates the recordability of \((e, t)\). Statement 1 implies that each contract in the sequence can be written as \((e_m(\alpha_{r-}) , t_m(\alpha_{r-}))\), reflecting the fact that it is fully determined by the initial string \(\alpha_{r-}\). The sequence of points \((e_m(\alpha_{r-}) , t_m(\alpha_{r-}))\) determined by \(\alpha_{r-}\) lies within the compact set \([0, \bar{v}] \times [\underline{r}, \bar{r}]\) and thus has a convergent subsequence. Because there are a finite number of such initial strings \(\alpha_{r-}\), it is possible to select a subsequence \(((e^*_m, t^*_m))_{m \in \mathbb{N}}\) of \((e_m, t_m))_{m \in \mathbb{N}}\) that converges for each \(\alpha_{r-}\). Define \((e^*, t^*)\) as the limit of this subsequence:

\[
(e^*(\alpha), t^*(\alpha)) = \lim_{m \to \infty} (e^*_m(\alpha_{r-}), t^*_m(\alpha_{r-})).
\]

The contract \((e^*, t^*)\) is incomplete and it inherits incentive compatibility and interim individual rationality from the contracts in the sequence \(((e^*_m, t^*_m))_{m \in \mathbb{N}}\). Let \(A_{r-}\) denote the set of all possible initial strings \(\alpha_{r-}\). As to the principal’s ex ante utility, the continuity of the production function \(h\) and the inequality that demonstrates recordability of \((e, t)\) together imply the desired result:

\[
\int h(e^*(\alpha)) - t^*(\alpha) d\pi(\alpha) = \sum_{a^*_{r-} \in A_{r-}} \left[ h(e^*(\alpha_{r-})) - t^*(\alpha_{r-}) \right] \pi(\alpha^*_{r-})
\]

\[
= \lim_{m \to \infty} \sum_{a^*_{r-} \in A_{r-}} \left[ h(e^*_m(\alpha_{r-})) - t^*_m(\alpha_{r-}) \right] \pi(\alpha^*_{r-})
\]

\(^{12}\)These regularity conditions can be found in Mirrlees (1971), Fudenberg and Tirole (1991), and the references therein. The argument is as follows. If \((e(\alpha), t(\alpha))\) is an incomplete contract that is optimal, then the mechanism \((\hat{e}(v), \hat{t}(v))\) defined from \((e(\alpha), t(\alpha))\) in the proof of Theorem 3 is an optimal mechanism. The incompleteness of \((e(\alpha), t(\alpha))\) together with the definition of \((\hat{e}(v), \hat{t}(v))\) imply that the functions \(\hat{e}(v)\) and \(\hat{t}(v)\) assume only a finite number of values. The conditions on \(\mu\), however, imply that the functions \(\hat{e}(v)\) and \(\hat{t}(v)\) in an optimal mechanism are strictly increasing over \([\underline{v}, \bar{v}]\) and therefore assume an infinite number of values. This contradiction implies that only complete contracts solve (19) when these regularity conditions hold.
\[ \lim_{m \to \infty} \int h(e_m^*(\alpha)) - t_m^*(\alpha) d\pi(\alpha) \geq \int h(e(\alpha)) - t(\alpha) d\pi(\alpha). \]

Conversely, the optimal contract can be recordable if every contingency is irreversible. This is the content of the following theorem. In addition to irreversibility, the theorem also requires asymptotic decreasing importance. As Theorem 1 and Example 2 indicate, asymptotic decreasing importance insures recordability of contracts in the case of complete information. Not surprisingly, the property is also significant in insuring recordability in the case of incomplete information. The regularity condition (iii) is needed so that standard results from the literature on optimal mechanisms can be applied.

**Theorem 6** Assume that:

(i) each contingency is irreversible;

(ii) the contingencies satisfy asymptotic decreasing importance;

(iii) \( v(A) = [\underline{v}, \overline{v}] \), the induced distribution \( \mu \) is nonatomic, and \( \mu \) has \( [\underline{v}, \overline{v}] \) as its support;

(iv) every finite set of contingencies has strictly positive probability;

(v) an optimal mechanism \( (\hat{e}, \hat{t}) \) exists such that \( \hat{e} \) and \( \hat{t} \) are continuous on \( [\underline{v}, \overline{v}] \).

Let \( (e, t) \) be an optimal contract such that \( e = \hat{e}(v(\cdot)) \) and \( t = \hat{t}(v(\cdot)) \). Then \( (e, t) \) is recordable. The sequence of incomplete contracts that demonstrates recordability can be chosen to converge uniformly.

The theorem is proven in the Appendix. The intuition is as follows. Standard results from the principal agent literature characterize an optimal mechanism \( (\hat{e}(v), \hat{t}(v)) \) when (iii) holds. We show that asymptotic decreasing importance and irreversibility together imply that an incomplete contract \( (e(\alpha), t(\alpha)) \) corresponds to a mechanism \( (e^*(v), t^*(v)) \) by composition with the valuation mapping in which

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13That is, \( \mu(\{v(\alpha')|\alpha'_n = \alpha_{n-} \}) > 0 \) for all \( n \in \mathbb{N} \) and \( \alpha \in A \).

14A sufficient condition for continuity of the optimal contract is that the hazard rate \( \mu'/\mu \) is increasing (e.g., Fudenberg and Tirole (1991)).
both $e^*$ and $t^*$ are step functions on $[\underline{v}, \overline{v}]$. As the length of the incomplete contract $(e(\alpha), t(\alpha))$ increases, the steps of the functions $e^*$ and $t^*$ become shorter. Because the solution to the optimal mechanism problem are continuous functions on $[\underline{v}, \overline{v}]$, they can be approximated by these step functions. The optimal contract $(\hat{e}(v(\alpha)), \hat{t}(v(\alpha)))$ can thus be approximated by incomplete contracts, and it is therefore recordable.

Theorems 5 and 6 suggest that the irreversible contingencies are those that can be addressed by a contract. The following example illustrates this point.

**Example 3** A firm (the principal) contracts with another firm (the agent) for a service. The value of the service to the principal firm is closely tied to the employment of particular employees of the agent firm. The continued employment of these people is observable, but not their effort or dedication. A *key person* is one who cannot be replaced by changing other aspects of the production plan (i.e., the tail of contingencies relating to the employment of other people). The contingencies that describe aspects of a key person’s performance might therefore be modeled as irreversible. Our theory suggests that the contract between the principal firm and the agent firm can address exactly these contingencies. This is consistent with common practice, where contracts between firms often provide key individuals with performance incentives. Acquisitions of small software firms or financial firms, for instance, may specify bonuses (such as stock options) and penalties (such as non-compete clauses) to guarantee the performance of the key employees.

7 Reversibility in a Model of Bilateral Trade

We next consider a contracting problem with two-sided incomplete information. A seller can provide a service to a buyer. The state of the world specifies every detail that affects the value of the service to the buyer and the cost of provision to the seller. The realization of the buyer’s and the seller’s types are given by

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15The term “key person” is drawn from the insurance industry, where “key person insurance” is a form of term life insurance sold to a firm to protect itself against the loss through death or disability of an individual who is crucial to the success of the firm. The issuance of such a policy typically requires not only that the health of the individual be appraised but also that his value to the firm be documented. The insured value of such an individual may run in the tens of millions of dollars.

16For example, consider a service provider who negotiates the sale of a service contract for a computer system with a system owner. The contingencies of the owner capture all aspects of his business that determine the value he would receive from a properly maintained system, including future decisions by current and prospective customers and employees. For the service provider, the contingencies include everything that affects his cost of providing the service, including service demands from other customers, the difficulty of repairing different components of the system, and
\( \alpha_B = (a_{B,i})_{i \in \mathbb{N}} \in A_B \) and \( \alpha_S = (a_{S,i})_{i \in \mathbb{N}} \in A_S \), respectively, with \( \pi_B \) and \( \pi_S \) denoting the distributions of the traders’ types. A contract is a pair \( (p, t) \) that specifies for each \( \alpha_B \) and \( \alpha_S \) a probability \( p(\alpha_B, \alpha_S) \) that the seller provides the service to the buyer and a transfer \( t(\alpha_B, \alpha_S) \) from the buyer to the seller. Contracts are thus assumed to be ex post budget balanced throughout this discussion. The buyer’s utility is

\[
u_B(\alpha_B, p, t) = p \cdot v_B(\alpha_B) + t
\]

and the seller’s utility is

\[
u_S(\alpha_S, p, t) = t - p \cdot v_S(\alpha_S),
\]

and so utilities are of the form (3). A contract is required to satisfy IC for both the buyer and the seller because each trader has private information. A contract is required to satisfy IIR for each trader given that his default utility is 0.

Let \( \mu_B \) denote the distribution of the buyer’s valuation \( v_B \) defined by \( v_B : A_B \rightarrow \mathbb{R} \) and \( \pi_B \) and let \( \mu_S \) denote the distribution of the seller’s valuation defined by \( v_S : A_S \rightarrow \mathbb{R} \) and \( \pi_S \). A special case of our model in which densities \( \mu_B \) and \( \mu_S \) are continuous functions and have closed subintervals of the real line as their supports is the model of bilateral trade devised in Chatterjee and Samuelson (1983). Our approach extends Chatterjee and Samuelson (1983) by modeling as states of the world those aspects of the service or good that determine the payoffs from trading.

The optimal contract \((p^\star, t^\star)\) maximizes the expected gains from trade subject to IC and IIR: \((p^\star, t^\star)\) solves

\[
(20) \quad \max_{(p, t)} \int_A [v_B(\alpha_B) - v_S(\alpha_S)] p(\alpha) d\pi(\alpha) \text{ s.t. IC and IIR.}
\]

As in the principal-agent problem, Theorem 3 reduces the problem of designing optimal contracts to the problem of designing optimal mechanisms. Subject to some regularity conditions on \( \mu_B \) and \( \mu_S \), this problem is solved in Myerson and Satterthwaite (1985).

Reflecting the objective in (20) of maximizing expected gains from trade, we alter the notion of recordability here as follows. A contract \((p, t)\) is recordable if there exists a sequence of incomplete contracts \((p_m, t_m)_{m \in \mathbb{N}}\) satisfying IC and IIR such that

\[
(21) \quad \lim_{m \to \infty} \int_A [v_B(\alpha_B) - v_S(\alpha_S)] p_m(\alpha) d\pi(\alpha) \geq \int_A [v_B(\alpha_B) - v_S(\alpha_S)] p(\alpha) d\pi(\alpha).
\]

whether or not the various components require service at the interim stage.
Suppose that all but a finite number of contingencies observed by a trader are reversible. Given a recordable contract, our first theorem states that for there exists an incentive compatible, interim individually rational and incomplete contract that achieves at least as much of the potential gains from trade. This seems to suggest that contractual incompleteness does not cause inefficiency, for a recordable contract can always be replaced with an incomplete contract that is as efficient or better. The work of Myerson and Satterthwaite (1985), however, suggests that optimal contracts between buyer and seller are typically complete. Similar to Theorem 5 Theorem 7 thus implies that optimal contracts are not recordable in problems in which only complete contracts can be optimal. The theorem therefore indicates a loss in gains from trade attributable to contractual incompleteness.

**Theorem 7** For some \( b, s \in \mathbb{N} \), suppose that a contingency \( a_{B,n} \) with \( n \geq b \) and a contingency \( a_{S,n} \) with \( n \geq s \) is reversible for the trader who observes its realization. If the contract \((p, t)\) is recordable, then there exists an incentive compatible, interim individually rational and incomplete contract \((p^\star, t^\star)\) such that:

1. \((p^\star(\alpha), t^\star(\alpha))\) does not depend upon \( a_{B,n} \) for \( n \geq b \) and \( a_{S,n} \) for \( n \geq s \); 
2. the ex ante gains from trade in \((p^\star, t^\star)\) are at least as large as in \((p, t)\): for every \( \alpha_B \in A_B \) and \( \alpha_S \in A_S \),
   \[
   \int_A \left[ v_B(\alpha_B) - v_S(\alpha_S) \right] p^\star(\alpha)d\pi(\alpha) \geq \int_A \left[ v_B(\alpha_B) - v_S(\alpha_S) \right] p(\alpha)d\pi(\alpha). 
   \]

Theorem 7 is proven in the Appendix. Theorem 8 applies Theorem 7 to reveal the extreme costs of contractual incompleteness when (i) every contingency is reversible, and (ii) it is not common knowledge ex ante that trade should occur. It is shown in this case that the no gains from trade are achieved ex-ante in any recordable contract. Ex ante contracting is thus pointless in this setting. This does not mean that trade never occurs in such a contracting problem; rather, it means that trade will not be arranged through an ex ante contract but may instead be negotiated at the interim. This may cause a loss in efficiency, for it may be beneficial to contract ex ante in order to prevent opportunistic behavior of the agents at the interim or ex post stages.\(^\text{17}\)

**Theorem 8** Assume that:

\(^{17}\)For example, as in Hart and Moore (1988), suppose that the seller bears an ex ante fixed cost in order to provide the service. If seller expects that bargaining at the interim will not provide him with sufficiently high payoff, then he will not prepare to provide the service. It can thus be efficient for the agents to write a contract ex ante that guarantees a sufficiently high payoff to the seller to guard against opportunistic behavior of the buyer at the interim.
1. each contingency observed by a trader is reversible;

2. there exists types \( \alpha^*_B \) and \( \alpha^*_S \) for the buyer and the seller such that \( v_B(\alpha^*_B) < v_S(\alpha^*_S) \), so that trade should not occur for these types.

Then the ex ante expected gains from trade equal zero in any recordable contract.

**Proof.** For the recordable contract \((p, t)\), Theorem 7 implies the existence of an incentive compatible, interim individually rational contract \((p^*, t^*)\) such that: (i) \((p^*, t^*)\) is state independent and hence constant; (ii) the ex ante gains from trade in \((p^*, t^*)\) are as least as large as in \((p, t)\). Suppose \( p^* > 0 \). Interim individual rationality implies

\[
\frac{t^*}{p^*} \leq v_B(\alpha_B)
\]

for all \( \alpha_B \in A_B \) and

\[
v_S(\alpha_S) \leq \frac{t^*}{p^*}
\]

for all \( \alpha_S \in A_S \). It follows that \( v_S(\alpha_S) \leq v_B(\alpha_B) \) for all \( \alpha_B \in A_B \) and \( \alpha_S \in A_S \), which contradicts assumption 2. in the theorem. The contradiction implies that \( p^* = 0 \) and so the ex ante expected gains from trade in \((p^*, t^*)\) are zero. Property (ii) above together with interim individual rationality therefore implies that they are also zero in \((p, t)\). □

We next use the Myerson-Satterthwaite characterization of an optimal mechanism to analyze a case in which each contingency is irreversible and the optimal contract is both complete and recordable. Suppose that the densities \( \mu_B' \) and \( \mu_S' \) exist, are continuous, and have \([0, 1]\) as their common support. For \( k \in [0, 1] \), let \( H_B \) and \( H_S \) denote the functions

\[
H_B(v_B, k) = v_B + k \frac{\mu_B(v_B)}{\mu_B'(v_B)} - 1,
\]

\[
H_S(v_S, k) = v_S + k \frac{\mu_S(v_S)}{\mu_S'(v_S)}.
\]

**Theorem 9** Assume that:

(i) each contingency is irreversible;

(ii) the contingencies are of asymptotic decreasing importance;

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(iii) the distributions $\mu_B$ and $\mu_S$ have continuous, strictly positive densities on $[0, 1]$;

(iv) every finite set of contingencies has strictly positive probability;\footnote{That is, $\mu_j (\{v(\alpha_j') | \alpha_{j,n-} = \alpha_{j,n-} \}) > 0$ for all $n \in \mathbb{N}$, $\alpha_j \in A_j$ and $j = B, S$.}

(v) $H_B(\cdot, k)$ and $H_B(\cdot, k)$ (defined by (22) and (23)) are increasing on $[0, 1]$ for each choice of $k$.

Then an optimal contract $(p^*, t^*)$ is recordable. The sequence of incomplete contracts that demonstrates recordability can be chosen such that the probabilities of trade converge pointwise to $p^*$.

The hypotheses of Theorem 9 and its proof are similar to those of Theorem 6. The proof is in the Appendix.

8 Conclusion

By developing a model that encompasses both complete and incomplete contracts, we can address our main question: How effective are incomplete contracts in approximating the ideal but practically unattainable complete contract? A complete contract is useful as an abstraction only if it is recordable, which means that it can be approximated arbitrarily closely with incomplete contracts. We envision recordability as a fundamental constraint on contracts, analogous to such constraints as resource feasibility and incentive compatibility.

We identify two properties of contracting problems that determine whether a complete contract is recordable. Both properties concern the preferences of the contracting parties over the contingencies that determine the state of the world. The two properties are complementary in the sense that neither implies the other.

The first property is asymptotic decreasing importance. This property means that for any $\varepsilon > 0$, the utility consequences of all but a finite number of contingencies are together less than $\varepsilon$. If information is complete and the contracting problem satisfies certain regularity conditions, then this is a sufficient condition for ensuring that an optimal complete contract is recordable.

The second property is reversibility, which is relevant in contracting problems with incomplete information. Intuitively, contingency $n$ is reversible if contingencies following the $n$th both affect the utility of the agent but can also perfectly undo the utility consequences of contingency $n$. A reversible contingency is difficult to address in a contract because it facilitates misrepresentation. Reversibility therefore helps to identify those contingencies that may necessarily be omitted from...
contracts even though they can be described. Reversibility can this way limit the contingencies that can be addressed by an incentive compatible incomplete contract to a fixed, finite set, in which case only the most trivial of complete contracts are recordable.
9 Appendix

9.1 Complete Information

Proof of Theorem 1. For any $\varepsilon > 0$, we demonstrate the existence of an incomplete contract $f_\varepsilon$ that (i) satisfies the ex post individual rationality constraint of (Ex Post),

$$u_2 (\alpha, f_\varepsilon (\alpha)) \geq r(\alpha),$$

and (ii) provides agent 1 with an ex ante utility no less than $\varepsilon$ below that provided by $\hat{f}$,

$$u_1 (\alpha, f_\varepsilon (\alpha)) \geq u_1 (\alpha, f (\alpha)) - \varepsilon.$$

Because $u_2 (\alpha, \hat{f}(\alpha)) = r(\alpha)$, recordability of $\hat{f}$ is demonstrated with the sequence $(f_n)_{n \in \mathbb{N}}$ defined by setting $f_n = f_\varepsilon$ for $\varepsilon = 1/n$.

For the values of $K$ and $\gamma$ given in the Lipschitz condition (2), let $\delta_1, \delta_2$ satisfy $0 < \delta_2 < \delta_1 < \frac{1}{2\gamma}$ and $2 (\delta_1 K + \delta_2) < \varepsilon$. Let $f^*(\alpha)$ solve

$$\max_{f: A \to C} u_1 (\alpha, f (\alpha)) \text{ s.t. } u_2 (\alpha, f (\alpha)) \geq r(\alpha) + \delta_1,$$  \hspace{1cm} (26)

For every $\alpha = (a_i)_{i \in \mathbb{N}}$, define $\tau_n (\alpha) = (a_1, \ldots, a_n, 0, 0, \ldots)$. The projection $\tau_n$ preserves the first $n$ contingencies and sets all other contingencies equal to 0. A contract $f$ is therefore incomplete if and only if there exists $n \in \mathbb{N}$ such that $f(\tau_n(\alpha)) = f(\alpha)$ for all $\alpha \in A$.

Asymptotic decreasing importance implies that there exists an $n \in \mathbb{N}$ such that

$$\left| u_j (\alpha, c) - u_j (\tau_n (\alpha), c) \right| < \delta_2$$  \hspace{1cm} (27)

for all $\alpha \in A, c \in C$ and for both values of $j$. Because $r(\alpha)$ depends on only a finite number of contingencies, we can assume without loss of generality that $n$ is sufficiently large that $r(\tau_n (\alpha)) = r(\alpha)$. Define the incomplete contract $f_\varepsilon (\alpha) = f^*(\tau_n (\alpha))$. Inequality (27), the definition of $f_\varepsilon$, and the fact that $f^*$ solves (26) imply (24):

$$u_2 (\alpha, f_\varepsilon (\alpha)) \geq u_2 (\tau_n (\alpha), f_\varepsilon (\alpha)) - \delta_2$$

$$= u_2 (\tau_n (\alpha), f^*(\tau_n (\alpha))) - \delta_2$$

$$\geq r(\tau_n (\alpha)) + \delta_1 - \delta_2$$

$$\geq r(\alpha).$$

The remaining inequality (25) is proven by means of a contract $f^{**}(\alpha)$ that solves

$$\max_{f: A \to C} u_1 (\alpha, f (\alpha)) \text{ s.t. } u_2 (\alpha, f (\alpha)) \geq r(\alpha) + 2\delta_1, \text{ for all } \alpha \in A.$$  \hspace{1cm} (29)
Inequality (27) implies
\[
\begin{align*}
  u_2(\tau_n(\alpha), f^**(\alpha)) & \geq u_2(\alpha, f^**(\alpha)) - \delta_2 \\
  & \geq r(\alpha) + 2\delta_1 - \delta_2 \\
  & \geq r(\alpha) + \delta_1 \\
  & = r(\tau_n(\alpha)) + \delta_1,
\end{align*}
\]
for all \( \alpha \in A \). It follows from the Lipschitz condition (2) that
\[
u_1(\alpha, \hat{f}(\alpha)) - 2\delta_1 K \leq u_1(\alpha, f^**(\alpha)) \tag{31}
\]
for all \( \alpha \in A \). Inequality (25) can now be established:
\[
u_1(\alpha, f_\epsilon(\alpha)) \geq u_1(\tau_n(\alpha), f_\epsilon(\alpha)) - \delta_2 \\
= u_1(\tau_n(\alpha), f^*(\tau_n(\alpha))) - \delta_2 \\
\geq u_1(\tau_n(\alpha), f^**(\alpha)) - \delta_2 \\
\geq u_1(\alpha, f^**(\alpha)) - 2\delta_2 \\
\geq u_1(\alpha, \hat{f}(\alpha)) - 2\delta_2 - 2\delta_1 K \\
\geq u_1(\alpha, \hat{f}(\alpha)) - \epsilon.
\]
The first and third inequalities follow from (27). The second inequality is true because \( f^*(\tau_n(\alpha)) \) is the optimal choice at \( \tau_n(\alpha) \), and hence weakly better than any other choice that gives agent 2 a reservation utility of at least \( r(\tau_n(\alpha)) + \delta_1 \) in state \( \tau_n(\alpha) \). This condition is satisfied for \( f^**(\alpha) \) because of (30). The fourth inequality follows from (31). The last inequality follows from the selection of \( \delta_1 \) and \( \delta_2 \).

9.2 Results from Section 4.1

**Proof of Theorem 3.** Define the contract \( g \) by averaging \( f \) over all states that determine the same valuations: for \( \alpha^* \in A \),
\[
g(\alpha^*) = E_A[f(\alpha)|v(\alpha) = v(\alpha^*)].
\]
The mechanism \( \hat{f} \) that is sought is defined by the equation \( f \circ v = g \). Inequality (6) follows from the concavity of \( \Phi \) together with an application of Jensen’s Inequality:

\[
E_{[x_1, x_2] \times [y_1, y_2]} \left[ \Phi \left( \hat{f}(v) \cdot v \right) \right] = E_A \left[ \Phi \left( g(\alpha) \cdot v(\alpha) \right) \right]
\]

\[
= E_A \left[ \Phi \left( E_A \left[ f(\alpha^*) \cdot v(\alpha^*) = v(\alpha) \right] \right) \right]
\]

\[
\geq E_A \left[ \Phi \left( E_A \left[ f(\alpha^*) \cdot v(\alpha^*) = v(\alpha) \right] \right) \right]
\]

\[
= E_A \left[ \Phi \left( f(\alpha) \cdot v(\alpha) \right) \right].
\]

Applying Theorem 2, the mechanism \( \hat{f} \) is shown to satisfy IC and IIR by showing that the contract \( g \) has these properties. For notational simplicity, we do this for \( j = 1 \). For \( \alpha^*_1 \in A_1 \), define

\[
H_1'(\alpha^*_1) = E_{A_2} \left[ h_1(g(\alpha)) \left| \alpha = \alpha^*_1 \right. \right], \text{ and}
\]

\[
T_1'(\alpha^*_1) = E_{A_2} \left[ t_1(g(\alpha)) \left| \alpha = \alpha^*_1 \right. \right].
\]

The fact that \( h_1 \) is affine implies

\[
E_{A_1} \left[ H_1(\alpha_1) \left| v_1(\alpha_1) = v_1(\alpha^*_1) \right. \right] = E_{A_1} \left[ E_{A_2} \left[ h_1(f(\alpha)) \left| v_1(\alpha_1) = v_1(\alpha^*_1) \right. \right] \right]
\]

\[
= E_{A_1} \left[ E_{A_2} \left[ h_1(f(\alpha)) \left| v_1(\alpha_1) = v_1(\alpha^*_1) \right. \right] \right]
\]

\[
= E_{A_1} \left[ E_{A_2} \left[ h_1(f(\alpha)) \left| v(\alpha) = v(\alpha^*) \right. \right] \right]
\]

\[
= E_{A_1} \left[ h_1(E_A \left[ f(\alpha) \right| v(\alpha) = v(\alpha^*)) \right]
\]

\[
= E_{A_1} \left[ h_1(g(\alpha^*)) \right] = H_1'(\alpha^*_1).
\]

A similar argument shows that

\[
E_{A_1} \left[ T_1(\alpha_1) \left| v_1(\alpha_1) = v_1(\alpha^*_1) \right. \right] = T_1'(\alpha^*_1).
\]

These equalities are now applied to demonstrate that \( g \) satisfies IC and IIR for agent 1. Incentive compatibility of \( f \) implies

\[
H_1(\alpha^*_1) v_1(\alpha^*_1) + T_1(\alpha^*_1) \geq H_1(\alpha_1) v_1(\alpha^*_1) + T_1(\alpha_1)
\]

for all \( \alpha_1, \alpha^*_1 \in A_1 \). Because (32) holds for all \( \alpha_1 \in A_1 \), it follows that for all \( \alpha^*_1 \in A_1 \),

\[
H_1(\alpha^*_1) v_1(\alpha^*_1) + T_1(\alpha^*_1) \geq E_{A_1} \left[ H_1(\alpha_1) v_1(\alpha^*_1) + T_1(\alpha_1) \left| v_1(\alpha_1) = v_1(\alpha^*_1) \right. \right]
\]

\[
= H_1'(\alpha^*_1) v_1(\alpha^*_1) + T_1'(\alpha^*_1).
\]
Because (33) holds for all $\alpha^*_1 \in A_1$, it follows that

$$H^*_1(\alpha^*_1) = v_1(\alpha^*_1) + T^*_1(\alpha^*_1) = E_{A_1} \left[ H_1(\alpha_1) v_1(\alpha^*_1) + T_1(\alpha_1) \right]$$

and so $g$ satisfies IC. Turning to IIR, we have

$$H^*_1(\alpha^*_1) = v_1(\alpha^*_1) + T^*_1(\alpha^*_1) = E_{A_1} \left[ H_1(\alpha_1) v_1(\alpha^*_1) + T_1(\alpha_1) \right] \geq r.$$
Proof. The proof is by induction on the length \( n - 1 \) of the initial string \( \alpha_{j,n-} \). The case of \( n - 1 = 0 \) is obvious, given the assumption that \( v_j(A_j) = [\nu_j, \tau_j] \). Assume that 1. and 2. hold for strings of length \( n - 1 \). We first show below that \( D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-1)} \) are closed sets whose intersection consists of a single point. It is then shown that statement 2. holds for all pairs of distinct initial strings of length \( n \).

We begin by noting that \( D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-1)} \) are closed sets. The Tychoff Theorem implies that \( A_j \) is compact in the product topology. The sets \( [\alpha_j' \mid \alpha_j' = \alpha_j, n_-, \alpha_j' = 0] \) and \( [\alpha_j' \mid \alpha_j' = \alpha_j, n_-, \alpha_j' = 1] \) are closed subsets of \( A_j \) in this topology and are therefore also compact. The continuity of \( v_j \) then implies that \( D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-1)} \) are compact and therefore closed.

Because \( D_{\alpha_j(n-1)} \cup D_{\alpha_j(n-0)} = D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-0)} \) is a closed interval by the induction hypothesis, the connectedness of \( D_{\alpha_j(n-0)} \) implies that \( D_{\alpha_j(n-0)} \cap D_{\alpha_j(n-0)} \neq \emptyset \). Assume by way of contradiction that \( D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-1)} \) either fail to be closed intervals or else they are closed intervals whose interiors overlap. Either of these statements can be true only if there exists \( \gamma_1, \gamma_2 \in D_{\alpha_j(n-0)} \) and \( \gamma \in D_{\alpha_j(n-1)} \backslash D_{\alpha_j(n-0)} \) with \( \gamma_1 < \gamma < \gamma_2 \). Let

\[
\gamma = \sup\{y \in D_{\alpha_j(n-0)} \mid y \leq \gamma\} \quad \text{and} \quad \gamma = \inf\{y \in D_{\alpha_j(n-0)} \mid y \geq \gamma\}.
\]

The points \( \gamma, \gamma \) exist and are elements of \( D_{\alpha_j(n-0)} \) because this set is compact, and \( \gamma < \gamma \) otherwise \( \gamma = \gamma = \gamma \in D_{\alpha_j(n-0)} \). It is also the case that

\[
\gamma = \inf\{y \in D_{\alpha_j(n-1)} \mid y \geq \gamma\} \quad \text{and} \quad \gamma = \sup\{y \in D_{\alpha_j(n-1)} \mid y \leq \gamma\},
\]

and consequently \( \gamma, \gamma \in D_{\alpha_j(n-1)} \). For \( k = 1, 2 \), \( \alpha_j^k \) and \( \alpha_j \) therefore exist such that \( \alpha_j^k_{(n+1)-} = \alpha_j(n-,0) \), \( \alpha_j^k_{(n+1)-} = \alpha_j(n-,1) \), \( v_j(\alpha_j^k) = v_j(\alpha_j) = \gamma \), and \( v_j(\gamma^k) = v_j(\gamma^k) = \gamma \). This contradicts the assumption that every contingency is irreversible, and so \( D_{\alpha_j(n-0)} \) and \( D_{\alpha_j(n-1)} \) satisfy the conclusion.

Turning to statement 2. for \( \alpha_j(n+1)- \neq \alpha_j(n+1)- \), either \( \alpha_j(n-) = \alpha_j(n-) \) or \( \alpha_j(n-) \neq \alpha_j(n-) \). We have just shown in the first case that \( D_{\alpha_j(n+1)-} \cap D_{\alpha_j(n+1)-} \) contains one element. If \( \alpha_j(n-) \neq \alpha_j(n-) \), then the induction hypothesis together with the fact that \( D_{\alpha_j(n+1)-} \subset D_{\alpha_j(n-)} \) and \( D_{\alpha_j(n+1)-} \subset D_{\alpha_j(n-)} \) imply that \( D_{\alpha_j(n+1)-} \cap D_{\alpha_j(n+1)-} \) contains at most one element. 

Proof of Theorem 6. A standard argument shows that the problem of finding an optimal mechanism in the principal-agent problem is equivalent to

\[
\max_{\hat{\psi}(v), U_1(v)} \int h(\hat{\psi}(v)) + \psi(\hat{\psi}(v)) - U_1(v)d\mu(v) \quad \text{s.t.} \quad (34)
\]
1. \( \hat{e}(\cdot) \) is nondecreasing;

2. \( U_1(v) = U_1(\underline{v}) + \int_{\underline{v}}^{v} \hat{e}(s) ds \), for all \( v \in [\underline{v}, \overline{v}] \);

3. \( U_1(v) \geq r \), for all \( v \in [\underline{v}, \overline{v}] \).

Because of quasilinearity of utility, the first two constraints of (34) are equivalent to Bayesian incentive compatibility,\(^{20}\) while the third constraint (given 1 and 2) is the constraint of interim individual rationality. We show that a solution to problem (34) can be approximated by incomplete contracts.

Lemma 1 implies that the sets \( D_{m_i} \) are intervals \([\overline{x}'_i, \overline{x}'_{i+1}]\), \( i = 1, \ldots, m_n \). The intervals have strictly positive probability with respect to \( \mu \) because of assumption (iv). Thus, the fact that \( \mu \) is nonatomic (assumption (iii)) implies \( \overline{x}'_i < \overline{x}'_{i+1} \). Asymptotic decreasing importance also implies that for every \( \varepsilon > 0 \) there exists \( \overline{n} \in \mathbb{N} \) such that \( |\overline{x}'_{i+1} - \overline{x}'_i| < \varepsilon \) for all \( n \geq \overline{n} \), \( i = 1, \ldots, m_n \). Let \( \hat{e}^n = \sum_{i=1}^{m_n} \hat{e}(\overline{x}'_i) 1_{[\overline{x}'_i, \overline{x}'_{i+1}]} \). Define the transfer function \( \hat{\iota}(v) \) by the formula

\[
\hat{\iota}(v) = \overline{\tau} + \int_{\underline{v}}^{v} \hat{e}^n(s) ds - v \hat{e}^n(v).
\]

The transfer \( \hat{\iota}(v) \) is defined so that the mechanism \( (\hat{e}^n(v), \hat{\iota}(v)) \) satisfies IC and IIR. The transfer \( \hat{\iota}(v) \) is also constant on the open intervals \((\overline{x}'_i, \overline{x}'_{i+1})\), \( i = 1, \ldots, m_n \).

We now show that \( (\hat{e}^n(v), \hat{\iota}(v))_{n \in \mathbb{N}} \) converges uniformly to \( (\hat{e}(v), \hat{\iota}(v)) \). Uniform convergence of \( (\hat{e}^n)_{n \in \mathbb{N}} \) follows by construction because \( \hat{e} \) is a continuous function on a compact set and because the length of the intervals \([\overline{x}'_i, \overline{x}'_{i+1}]\) converges uniformly to 0 as \( n \to \infty \). Let \( \varepsilon > 0 \) be arbitrary. Note that \( \overline{v} - 2 \underline{v} > 0 \) because \( \underline{v} < \overline{v} < 0 \). Choose \( \overline{n} \in \mathbb{N} \) such that \( \|\hat{e}^n - \hat{e}\| < \varepsilon / (\overline{v} - 2 \underline{v}) \). Then

\[
|\hat{\iota}(v) - \hat{\iota}(v)| = \left| \int_{\underline{v}}^{v} \hat{e}^n(s) - \hat{e}(s) ds + \hat{e}(v) - v \hat{e}^n(v) \right|
\leq \left| \int_{\underline{v}}^{v} \hat{e}^n(s) - \hat{e}(s) ds \right| + |v| \left| \hat{e}^n(v) - \hat{e}(v) \right|
\leq (\overline{v} - 2 \underline{v}) \frac{\varepsilon}{\overline{v} - 2 \underline{v}} + |v| \frac{\varepsilon}{\overline{v} - 2 \underline{v}}
= (\overline{v} - 2 \underline{v}) \frac{\varepsilon}{\overline{v} - 2 \underline{v}} = \varepsilon.
\]

The transfer \( \hat{\iota}(v) \) thus converges uniformly to the optimal transfer \( \hat{\iota}(v) \).

The sequence of incomplete contracts \(((e^n, t^n))_{n \in \mathbb{N}}\) that demonstrates the recordability of the optimal contract \( (\hat{e}(v(\alpha)), \hat{\iota}(v(\alpha))) \) is defined as follows: \( (e^n(\alpha), t^n(\alpha)) \)

is the value of \((\hat{e}^n(v), \hat{t}^n(v))\) in the interior of \(D_{a_{n−}}\). The contract \((e^n(\alpha), t^n(\alpha))\) is obviously incomplete. It may differ from the composition of \((\hat{e}^n(v), \hat{t}^n(v))\) with the valuation mapping \(v\) at states \(\alpha\) such that \(v(\alpha) = \overline{x}_i\) for some \(i = 1, \ldots, m_n\) and so the incentive compatibility and interim individual rationality of \((\hat{e}(\alpha), t^n(\alpha))\) does not follow immediately from Theorem 2. The incentive compatibility and individual rationality of \((\hat{e}^n(v), \hat{t}^n(v))\), however, together with the fact that \(\hat{e}^n(v)\) and \(\hat{t}^n(v)\) are constant on each interval \([\overline{x}_{i−1}, \overline{x}_i)\) imply that

\[
t_i(v′) − \overline{x}_i \cdot \hat{e}^n(v′) = t_i(v″) − \overline{x}_i \cdot \hat{e}^n(v″) \geq r
\]

for \(v′ ∈ (\overline{x}_{i−1}, \overline{x}_i)\) and \(v″ ∈ (\overline{x}_i, \overline{x}_{i+1})\). It follows that \((e^n(\alpha), t^n(\alpha))\) satisfies IC and IIR. Statement (35) together with the uniform convergence of \(\hat{e}^n(v), \hat{t}^n(v)\) to \((\hat{e}(v), \hat{t}(v))\) imply that \(((e^n(\alpha), t^n(\alpha)))_{n \in \mathbb{N}}\) converges uniformly to \((e, t)\).

9.4 The Model of Bilateral Trade

**Proof of Theorem 7.** Our notation in the following argument addresses the case of \(b, s > 1\). The degenerate case in which either \(b = 1\) or \(s = 1\) is addressed by properly interpreting the notation in this argument: all expressions involving degenerate contingencies, strings, or sets (such as \(\{0, 1\}^0\)) should simply be omitted, and the argument then goes through correctly.

Let \(A_B^-\) denote the set of initial strings of buyers’ types of length \(b − 1\) let \(A_B^+\) denote the set of all tails from the \(b\)th contingency on:

\[
A_B^- = \left\{(a_{B,i})_{1 \leq i \leq b−1} \mid a_{B,i} \in \{0, 1\}, 1 \leq i \leq b−1 \right\},
\]

\[
A_B^+ = \left\{(a_{B,i})_{b \leq i \leq i} \mid a_{B,i} \in \{0, 1\}, \forall i \geq b \right\},
\]

\[
A_B = A_B^- \times A_B^+.
\]

Let \(\rho_B : A_B → A_B^-\) denote the projection mapping

\[
\rho_B \left((a_{B,i})_{i \in \mathbb{N}}\right) = \left(a_{B,i}\right)_{1 \leq i \leq b−1}.
\]

The sets \(A_S^-\) and \(A_S^+\) and the projection mapping \(\rho_S\) are defined similarly using the number \(s\).

Let \(((p_m, t_m))_{m \in \mathbb{N}}\) be a sequence of incomplete contracts that demonstrates the recordability of \((p, t)\). While each contract \((p_m, t_m)\) is incomplete, its finite length is a priori unspecified. The first step in the proof is to construct an incentive compatible and interim individually rational contract \((p_B^m, t_B^m)\) that is interim payoff
equivalent to \((p_m, t_m)\) and whose value at \(\alpha\) is determined by \(\rho_B(\alpha_B)\) and \(\rho_S(\alpha_S)\). Define \((p_m^*(\alpha'), t_m^*(\alpha'))\) by averaging \((p_m, t_m)\) over all states \(\alpha\) that begin with same initial strings of \(b - 1\) contingencies observed by the buyer and \(s - 1\) contingencies observed by the seller: for \(j = 1, 2\) and \(\alpha' \in A\),

\[
p_m^*(\alpha') = \int_{A^+_b \times A^+_s} p_m(\alpha) d\pi(\alpha \mid \rho_B(\alpha_B) = \rho_B(\alpha'_B), \rho_S(\alpha_S) = \rho_S(\alpha'_S)),
\]

and

\[
t_m^*(\alpha') = \int_{A^+_b \times A^+_s} t_m(\alpha) d\pi(\alpha \mid \rho_B(\alpha_B) = \rho_B(\alpha'_B), \rho_S(\alpha_S) = \rho_S(\alpha'_S)).
\]

Now consider the buyer: for any \(\alpha'_B \in A_B\),

\[
\begin{align*}
\int_{A_b} p_m(\alpha'_B, \alpha_S) d\pi_S(\alpha_S) & = \int_{A_b} \int_{A_S} p_m(\alpha_B, \alpha_S) d\pi_S(\alpha_S) d\pi_B(\alpha_B \mid \rho_B(\alpha_B) = \rho_B(\alpha'_B)) \\
& = \int_{A_b} \int_{A_B} p_m(\alpha_B, \alpha'_S) d\pi_B(\alpha_B) d\pi_S(\rho_B(\alpha_B) = \rho_B(\alpha'_B), \rho_S(\alpha'_S) = \rho_S(\alpha_S)) \\
& = \int_{A_b} p_m^*(\alpha'_B, \alpha_S) d\pi_S(\rho_S(\alpha_S)) \\
& = \int_{A_b} p_m^*(\alpha'_B, \alpha_S) d\pi_S(\alpha_S).
\end{align*}
\]

Theorem 4 implies that \(\int_{A_b} p_m(\alpha'_B, \alpha_S) d\pi_S(\alpha_S)\) is constant over the set of all \(\alpha_B\) such that \(\rho_B(\alpha_B) = \rho_B(\alpha'_B)\), which implies the first equality above. The second changes the order of integration, the third uses the definition of \(p_m^*(\alpha'_B, \alpha_S)\), and the fourth is true because \(p_m^*(\alpha'_B, \alpha_S)\) is determined by the value of \(\rho_S(\alpha_S)\). Similar arguments prove that for all \(\alpha'_B \in A_B\) and \(\alpha'_S \in A_S\),

\[
\begin{align*}
\int_{A_b} t_m(\alpha'_B, \alpha_S) d\pi_S(\alpha_S) & = \int_{A_b} t_m^*(\alpha'_B, \alpha_S) d\pi_S(\alpha_S), \\
\int_{A_b} p_m(\alpha_B, \alpha'_S) d\pi_B(\alpha_B) & = \int_{A_b} p_m^*(\alpha_B, \alpha'_S) d\pi_B(\alpha_B), \\
\int_{A_b} t_m(\alpha_B, \alpha'_S) d\pi_B(\alpha_B) & = \int_{A_b} t_m^*(\alpha_B, \alpha'_S) d\pi_B(\alpha_B).
\end{align*}
\]
The interim individual rationality and the incentive compatibility of \((p_m^*, t_m^*)\) thus follows from the corresponding properties of \((p_m, t_m)\). The interim expected utility function of each trader is also the same in \((p_m^*, t_m^*)\) as in \((p_m, t_m)\).

The value of each contract \((p_m^*, t_m^*)\) at \(\alpha\) is determined by \(\rho_B(\alpha_B)\) and \(\rho_S(\alpha_S)\), and the set \(A_B^* \times A_S^*\) of all pairs \((\rho_B(\alpha_B), \rho_S(\alpha_S))\) of such initial strings is finite. As a probability, \(p_m^*(\alpha) \in [0, 1]\); interim individual rationality and the boundedness of the sets \(v_B(A_B)\) and \(v_S(A_S)\) imply that \(t_m^*(A)\) is a bounded set. By taking a subsequence (if necessary), it can thus be assumed without loss of generality that \((p_m^*(\alpha), t_m^*(\alpha))_{m \in \mathbb{N}}\) converges for all \(\alpha \in A\). The contract \((p^*, t^*)\) defined by

\[
(p^*(\alpha), t^*(\alpha)) = \lim_{m \to \infty} (p_m^*(\alpha), t_m^*(\alpha)), \text{ for } \alpha \in A
\]

is thus well defined. The contract \((p^*, t^*)\) inherits \(IIC\) and \(IIR\) from the contracts in the sequence, and it clearly satisfies property 2 of the statement of the theorem. Let \(p_m^*(\rho_B(\alpha_B'), \rho_S(\alpha_S'))\) denote the value of \(p_m^*(\alpha)\) at any state \(\alpha\) such that \(\rho_B(\alpha_B) = \rho_B(\alpha_B')\) and \(\rho_S(\alpha_S) = \rho_S(\alpha_S')\). Let \(\pi_{A_B^* \times A_S^*}\) the marginal distribution on \(A_B^* \times A_S^*\).

To verify statement 2 in the theorem, we have

\[
\int_A [(v_B(\alpha_B) - v_S(\alpha_S)] p^*(\alpha) d\pi(\alpha) = \lim_{m \to \infty} \int_A [(v_B(\alpha_B) - v_S(\alpha_S)] p_m^*(\alpha) d\pi(\alpha)
\]

\[
= \lim_{m \to \infty} \sum_{\alpha_B' \in A_B^*, \alpha_S' \in A_S^*} p_m^* (\alpha_B', \alpha_S') \pi_{A_B^* \times A_S^*} (\alpha_B', \alpha_S') 
\]

\[
\cdot \int_{A_B^* \times A_S^*} [v_B(\alpha_B') - v_S(\alpha_S')] \pi (\alpha' \mid \rho_B(\alpha_B) = \alpha_B', \rho_S(\alpha_S) = \alpha_S')
\]

\[
= \lim_{m \to \infty} \int_{A_B^* \times A_S^*} [v_B(\alpha_B) - v_S(\alpha_S)] p_m^*(\alpha) d\pi(\alpha)
\]

\[
\geq \int_A [v_B(\alpha_B) - v_S(\alpha_S)] p(\alpha) d\pi(\alpha),
\]

where the last two lines are the recordability inequality. ■

**Proof of Theorem 9.** Lemma 1 implies that for \(j = B, S\) the sets \(D_{a,j,n}\) are intervals \([\bar{x}_{j,i}, \bar{x}_{j,i+1}]\), \(i = 1, \ldots, m_n\). Asymptotic decreasing importance implies that for every \(\varepsilon > 0\) there exists \(\bar{n} \in \mathbb{N}\) such that \(|\bar{x}_{j,i+1} - \bar{x}_{j,i}| < \varepsilon\) for all \(n \geq \bar{n}, i = 1, \ldots, m_n\) and \(j = B, S\). Define the functions \(\xi_B^n\) and \(\xi_S^n\) as follows:

\[
\xi_B^n (v_B) = \sup \{\bar{x}_{B,i} \mid \bar{x}_{B,i} \leq v_B, i \in \{1, \ldots, m_n\}\},
\]

\[
\xi_S^n (v_S) = \inf \{\bar{x}_{S,i} \mid v_S \leq \bar{x}_{S,i}, i \in \{1, \ldots, m_n\}\}.
\]

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The function $\xi^n_B$ rounds a buyer’s valuation downward while $\xi^n_S$ rounds a seller’s valuation upward, in each case to the nearest boundary of one of the intervals $D_{\alpha B, n}$ and $D_{\alpha S, n}$, respectively.

Theorem 3 implies that an optimal contract has the form $(\hat{p}^*(v(\alpha)), \hat{t}^*(v(\alpha)))$, where $p^*$ and $v^*$ solve the optimal mechanism problem

$$\max_{(p, t)} \int \int [v_B - v_S] p(v_B, v_S) d\mu_B d\mu_S \text{ s.t. IC and IIR.} \quad (36)$$

Given the regularity condition (iv), Theorem 2 of Myerson and Satterthwaite (1985) characterizes a constant $k^* \in [0, 1]$ such that $\hat{p}^*(v_B, v_S)$ has the form

$$\hat{p}^*(v_B, v_S) = \begin{cases} 1 & \text{if } H_B(v_B, k^*) \geq H_S(v_S, k^*); \\ 0 & \text{otherwise.} \end{cases}$$

Define the probability function $\hat{p}_n(v_B, v_S)$ as

$$\hat{p}_n(v_B, v_S) = \begin{cases} 1 & \text{if } H_B(\xi^n_B(v_B), k^*) \geq H_S(\xi^n_S(v_S), k^*); \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $(\hat{p}_n)_{n \in \mathbb{N}}$ converges pointwise to $\hat{p}^*$. A comparison of $\hat{p}_n(v_B, v_S)$ to $\hat{p}^*(v_B, v_S)$ shows that $\hat{p}_n(v_B, v_S)$ satisfies inequality (2) of Myerson and Satterthwaite (1985) because $\hat{p}^*(v_B, v_S)$ satisfies it. It also inherits from $\hat{p}^*(v_B, v_S)$ the monotonicity properties required by Theorem 1 in their paper because $\xi^n_B$ and $\xi^n_S$ are nondecreasing. Formula (6) of their paper thus defines a transfer function $\hat{t}_n(v_B, v_S)$ such that the revelation mechanism $(\hat{p}_n, \hat{t}_n)$ satisfies IC and IIR. Like $\hat{p}_n(v_B, v_S)$, $\hat{t}_n(v_B, v_S)$ is constant on the interior of all sets $D_{\alpha B, n} \times D_{\alpha S, n}$.

The sequence $(p_n(\alpha), t_n(\alpha))_{n \in \mathbb{N}}$ that demonstrates the recordability of the optimal contract $(\hat{p}^*(v(\alpha)), \hat{t}^*(v(\alpha)))$ is defined as follows: for $\alpha = (\alpha B, \alpha S) \in A$, $(p_n(\alpha), t_n(\alpha))$ equals the value of $(\hat{p}_n(v), \hat{t}_n(v))$ in the interior of $D_{\alpha B, n} \times D_{\alpha S, n}$. As in the proof of Theorem 6, it is straightforward to show that $(p_n(\alpha), t_n(\alpha))$ satisfies IC and IIR because $(\hat{p}_n(v), \hat{t}_n(v))$ has these properties. It is clear that $\lim_{n \to \infty} p_n(\alpha) = \hat{p}^*(v(\alpha)), \pi$-a.e because $\hat{p}_n$ converges pointwise to $\hat{p}^*$ and $p_n(\alpha) = \hat{p}_n(v(\alpha)), \pi$-a.e.

We conclude the proof by showing that the recordability inequality holds.

$$\int [v_B(\alpha B) - v_S(\alpha S)] \hat{p}^*(v_B(\alpha B), v_S(\alpha S)) d\pi(\alpha)$$

$$= \int \int [v_B - v_S] \left( \lim_{n \to \infty} \hat{p}_n(v_B, v_S) \right) d\mu_B(v_B) d\mu_S(v_S)$$

$$\leq \liminf \int \int [v_B - v_S] \hat{p}_n(v_B, v_S) d\mu_B(v_B) d\mu_S(v_S)$$

$$= \liminf \int [v_B(\alpha B) - v_S(\alpha S)] p_n(\alpha B, \alpha S) d\pi(\alpha).$$

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The inequality is an application of Fatou’s Lemma. The last equality follows because $p_n(\alpha) = \hat{p}_n(v(\alpha))$ for $\pi$-a.e $\alpha$. By selecting a subsequence of $(p_n, t_n)_{n \in \mathbb{N}}$, we can conclude that

$$\int [v_B(\alpha_B) - v_S(\alpha_S)] \hat{p}^*(v_B(\alpha_B), v_S(\alpha_S)) d\pi(\alpha)$$

$$\leq \lim_{m \to \infty} \int [v_B(\alpha_B) - v_S(\alpha_S)] p_n(\alpha_B, \alpha_S) d\pi(\alpha),$$

which proves recordability. ■
References


