Question 1

\(a\) \[ h(p, v(p, w)) = x(p, w) \]

\(b\) \[ ph(p, v(p, w)) = w \]

\(c\) \[ px(p, e(p, u)) = e(p, u) \]

Question 2  Note that \(x_1(p, w) = h_1(p, v(p, w))\). Thus, \(x_1(p_1, p_2, w) = \frac{p_2^2}{p_1^2}\). We can use Walras’ law to determine the demand of good 2. In particular, \(p_1 \frac{p_2^2}{p_1^2} + p_2 x_2 = w\).

Thus, \(p_2 x_2 = w - \frac{p_2^2}{p_1}\), which implies

\[ x_1(p, w) = \frac{p_2^2}{p_1}, x_2(p, w) = \frac{w}{p_2} - \frac{p_2}{p_1} \]

Question 3  Proof: Let \(x = x(p, w)\) and \(x' = x(p', w')\). Then

\[ x = (w/2, w/2), \text{ and } x' = (w'/2, w'/4) \]

Thus, \(p'x = 3w/2\) and \(px' = 3w'/4\). If \(p'x \leq w'\) then \(w' \geq 3w/2\). Thus, \(3w'/4 > 9w/8 > w\).

Next, suppose that \(px' \leq w\). Then \(3w'/4 \leq w\). Then \(p'x = 3w/2 \geq 9w/8 > w\).

Question 4  Note that \(w = px = 36\) and \(w' = p'x' = 4 + x_2'\). If \(px' \leq w\) then \(16 + 2x_2' \leq 36\) (i.e., \(x_2' \leq 10\)). But then \(p'x > w'\), i.e., \(13 > 4 + x_2'\), i.e., \(x_2' < 9\). If, instead, \(p'x \leq w'\), i.e., \(13 \leq 4 + x_2'\) implying \(x_2 \geq 9\), then \(px' > w\), i.e., \(16 + 2x_2' > 36\), i.e., \(x_2' > 10\). Thus,

\[ x_2' < 9 \text{ or } x_2' > 10. \]

Question 5  Note that \(x_2 = x_3\). Let \(y = x_2 = x_3\). Then the consumer maximizes \(u(x_1, y) = x_1y\) subject to prices \(q_1 = p_1, q_2 = p_2 + p_3\) and wealth \(w\). The MRS is given by \(y/x_1\), the wealth expansion path by \(x_1 = q_2y\). To get a utility of \(u\) we need \(x_1y = u\). Thus, \(x_1q_2y = q_2u\) which implies \(q_1x_1^2 = q_2u\). Thus, \(x_1 = \sqrt{(q_2/q_1)u}\) and similarly, \(y = \sqrt{(q_1/q_2)u}\). Substituting \(q_1\) and \(q_2\) yields

\[ h_2(p_1, p_2, p_3, u) = \sqrt{\frac{p_1}{p_2 + p_3}}u \]
As a consequence, goods 2 and 3 are \textbf{complements} because \( \frac{\partial h_2(p_1, p_2, p_3, u)}{\partial p_3} < 0 \).

**Question 6** Suppose a utility function is given by \( u(x_1, x_2) = x_1^2 - 2x_2 \). Then \( x_2 = 0 \). To obtain utility \( u, x_1 = \sqrt{u} \). Thus, \( h_1(p, u) = \sqrt{u} \). Therefore,

\[
e(p_1, p_2, u) = p_1 \sqrt{u}
\]

\( w = p_1 \sqrt{u} \) implies \( u = w^2 / p_1^2 \).

\[
v(p_1, p_2, w) = \frac{w^2}{p_1^2}
\]

**Question 7**

\[
\begin{pmatrix}
-6 & 3 & 9 \\
3 & -10 & 4 \\
9 & 4 & -22
\end{pmatrix}
\]

**Question 8** We must show that \( x'' = \alpha x + (1 - \alpha)x' \in h(p, u) \). Note that \( u(x) \geq u \) and \( u(x') \geq u \). Thus, quasi concavity implies \( u(x'') \geq \min\{u(x), u(x')\} \geq u \). As a consequence, \( x'' \) fulfills the constraint of the expenditure minimization problem, when the required utility level is \( u \). Further \( px'' = \alpha px + (1 - \alpha)px' = \alpha e(p, u) + (1 - \alpha)e(p, u) = e(p, u) \). Hence, \( x'' \) minimizes expenditure, which implies \( x'' \in h(p, u) \).