Cournot Oligopoly with Network Effects

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Abstract

The nature of competition is quite different in network industries as compared to the other, more traditional, ones. The purpose of the present paper is to thoroughly examine the implications of these differences as reflected in the dependence of equilibrium outputs, price and profits on industry concentration. We restrict the analysis to oligopolistic competition amongst firms in a market characterized by positive (direct) network effects when the relevant network is industry-wide.

The proofs rely on lattice-theoretic methods; this approach allows us to unify in a common setting the general results in the literature on network goods, weakening considerably the assumptions. As a by-product we offer an alternative explanation of the start-off phenomenon in network industries.

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1 Introduction

"From computer software and hardware, to fax machines and video game systems, to compact discs and digital video discs, to communications networks and the Internet, technology is the driver and compatibility the navigator."

Carl Shapiro

Antitrust in Network Industries

The nature of competition is quite different in network industries as compared to the other, more traditional, ones. The presence of adoption effects gives rise to demand-side economies of scale that highly affect market behavior and performance. When these effects prevail consumers and firms must form expectations about the size of the network to make their decisions; since demand-side economies of scale vary with these expectations, even restricting attention to rational expectations equilibrium, multiple equilibria easily arise [Katz and Shapiro (1985) and (1994), Liebowitz and Margolis (1994), Economides (1996) and Varian (2001)]. From the supply-side of the economy, some proponents of network externalities models suggest that these newer technologies often display increasing returns to scale. These distinctive features raise new questions and impose some challenges from a methodological perspective.

The purpose of the present paper is to thoroughly examine the implications of these differences as reflected in the dependence of equilibrium outputs, price and profits on industry concentration. Since antitrust policy depends upon a solid understanding of business strategy and economics, the interest on this topic is not purely academic [Shapiro (1996)].

We restrict the analysis to oligopolistic competition amongst firms in a market characterized by positive (direct) network effects when the products of the firms are perfectly compatible with each other, so the relevant network is industry-wide. Although most papers in the literature on network goods focus on the opposite case –perfect incompatibility– and analyze the incentives of the incumbent firms to make their products compatible, we think that many important industries fit the perfect compatible framework. The market of telecommunications, *e.g.* fax machines and phones, is a good example; here, the value that people derive from consumption increases when other people acquire the good, irrespective of which firms the other people choose.

A key feature of this paper is that it relies on lattice-theoretic methods for proving existence and stability of equilibria and doing comparative static analysis [Topkis (1978), Milgron and Roberts (1990) and (1994) and Milgron and Shannnon (1994)]. This approach allows us to unify in a common setting the general results in the literature on network goods, weakening considerably the required assumptions. Isolating minimal constraints, we provide more transparent economic intuition behind the cause-effect relationships we analyze.

In order to provide a summary of our findings let Z, y and S denote aggregate output, total output of the other firms in the market and the expected size of the network, respectively. We prove existence based on only two natural conditions on the inverse demand and common cost functions that guarantee that the profit functions of the firms under consideration have increasing differences in Z and y and the single-crossing property in Z and S. Our proof allows for increasing returns to scale and cross-effects in the inverse demand function. It can be thought of as an extension of the proof of existence in Amir and Lambson (2000) and Kwon (2006); the first one does not include network effects and the second one assumes that the inverse demand function is additively separable in output and network size.

Although our model is static by nature, we construct a dynamic argument to analyze the stability of the equilibria. There are several antecedents on this issue in the market of telecommunications [Rohlfs (1974), Economides and Himmelberg (1995) and Varian (2001)]. These papers suggest that this kind of industries usually have three equilibria. Under natural dynamics, the two extreme equilibria are stable and the middle equilibrium (usually called "critical mass") is unstable. The argument behind the proof is quite simple; if all the consumers expect that none will acquire the good, then the good has no value and none will buy it, resulting in a no-trade equilibrium. However, if expectations are higher to start with, other non-trivial equilibria are also possible. They use this framework to explain the start-up problem in network industries, that is, the difficulties of the incumbent firms to generate enough expectations to achieve "critical mass." Fax machines illustrate nicely this common pattern; the Scottish inventor Alexander Bain patented the basic technology for fax machines in 1843, but faxes remained a niche product until the mid-1980s [Shapiro and Varian (1998)].

The current explanation for the take-off is based on the positive feedback argument: As the installed base of users grows, more and more users find adoption worthwhile, eventually, the product achieves "critical mass" and the market explodes. We complement this explanation by showing the role of market structure. We prove that, under specific conditions, the no-trade equilibrium is stable when there are just a few firms in the market but becomes unstable when more firms decide to enter. In these cases, the take-off only occurs if the number of firms in the market is big enough.

Regarding market behavior, the extremal equilibria (*i.e.* maximal and minimal) call for an aggregate output that increases in the number of firms. As this also means that expectations are higher at the extremal equilibria, this result does not imply that the extremal equilibria prices increase in n. Thus, the so-called property of quasi-competitiveness[1], which under similar assumptions holds in the standard Cournot game, does not hold here. In addition to these, when n increases per-firm equilibrium output increases if the demand is not too log-concave and decreases otherwise.

As far as per-firm profits are concerned if both the extremal equilibrium prices and per-firm outputs increase as a function of n, per-firm profits also increase with the number of firms in the market. This last statement is quite surprising; it means that under some feasible conditions the incumbent firms in the market prefer to see further entry by new firms. This possibility is discussed but not proved in Katz and Shapiro (1985) and formalized under different assumptions in Economides (1995).

The effects of entry on industry performance as reflected in social welfare, consumer surplus and aggregate profits also display some distinct features as compared to standard Cournot competition. The demand-side economies of scale weaken the conditions under

[1] A Cournot competition is said to be quasi-competitive if equilibrium industry output increases and the equilibrium market price decreases with an increase in the number of firms in the industry.

which social welfare and aggregate profits increase with new entries. Alternatively, if the cross-effect on the inverse demand function is negative, it is possible that consumer surplus decreases with n. Katz and Shapiro (1985) explain the intuition behind this result: "If the network externality is strong for the marginal consumer, then the increase in the expected network caused by the change in the number of firms will raise his or her willingness to pay for the good by more than that of the average consumer. As a consequence, the firms will be able to raise the price by more than the increase in the average consumer's willingness to pay for the product and consumer's surplus will fall."

In the context of the standard Cournot oligopoly, the two extremal equilibria enjoy particular welfare properties. The largest [smallest] equilibrium output is preferred most [least] by consumers, but preferred least [most] by firms [Amir (2003)]. When network effects are present, these properties are no longer true; in particular, we give sufficient conditions under which the largest equilibrium is the most preferred by both firms and consumers.

The paper is organized as follows. Section 2 presents the model, introduces our equilibrium concept and defines the notation. Section 3 proves existence of equilibria and studies stability and uniqueness. Section 4 analyzes output, price and per-firm profits as a function of the number of firms in the market. Section 5 deals with market performance as reflected in social welfare, consumer surplus and aggregate profits, again, as a function of n. Section 6 assumes that the game has multiple equilibiria, and compares the different outputs from the perspective of firms and consumers. All these sections contain several examples[2]. Section 7 shows a discussion of the main results and suggests further extensions, and Section 8 contains the proofs of this paper, when not included in the corresponding sections. Finally, a very simple and self-contained review of the lattice-theoretic notions and results needed here form the Appendix.

[2] Some of the examples we use do not satisfy all the assumptions in this paper. Anyway, we include them because they are simple and capture the features we want to highlight.

2 Analytical Framework

In this section we present a simple model of oligopoly with network effects and introduce our equilibrium concept. In addition to these, we define the notation and enumerate all the assumptions we use in the paper.

2.1 Firms, Consumers and Equilibrium

We consider a simple, static, model to analyze the oligopolistic competition for goods with network effects. For us, this means that the willingness to pay of consumers is increasing in the number of other agents buying the same kind of good.

We assume that the products of the firms are homogeneous and perfectly compatible with each other, so the relevant network comprises the outputs of all firms producing the good.

The market is fully described by the number of (identical) firms n and the inverse demand function $P(Z, S)$, where Z is aggregate output in the market and S is the expected size of the network, *i.e.* the expected number of people buying the good. We consider that each consumer buys at most one unit of the good; thus, S also denotes the expected number of units to be sold.

The profit function of the firm under consideration is

$$\pi(x, y, S) = xP(x + y, S) - C(x) \tag{2.1}$$

where x is the level of output chosen by the firm and y the output of the other $(n-1)$ firms in the market. Sometimes, it will be useful to express total cost of production in terms of average cost $A(x)$[3]. At equilibrium, all relevant quantities -x, y and Z- and π will be indexed by the underlying number of firms n.

Each firm chooses its output level to maximize (2.1) under the assumptions that: (a) consumers' expectations about the size of the network S is given; and (b) the output

[3] $C(x)$ and $A(x)$ are normalized to be zero at $x = 0$.

level of the other firms, $\sum_{i\neq j} x_{jn} = y$, is fixed[4]. An equilibrium in this game is a vector $(x_{1n}, x_{2n}, ..., x_{nn})$ that satisfies the following conditions:

1. $\sum_{i=1}^{n} x_{in} = Z_n$
2. $x_{in} = \arg\max \left\{ xP\left(x + \sum_{j\neq i} x_{jn}, Z_n\right) - C\left(x\right)\right\}.$

In the literature of network effects this notion of equilibrium is well known as "Fulfilled Expectations Cournot Equilibrium (FECE)" [Katz and Shapiro (1985)]. Note that this equilibrium requires that consumers and firms predict the market outcome correctly, so that their beliefs are confirmed in equilibrium; in other terms, expectations are rational. As Katz and Shapiro (1985) assert: "Although it is possible that (in the short run, at least) consumers could be mistaken about network sizes, it is useful to limit the set of possible equilibria by imposing the restriction that expected sales be equal to actual sales in equilibrium." We will see later that even restricting attention to rational expectations equilibrium, FECE still allows multiple equilibria to occur.

Alternatively, we may think of the firm as choosing total output $Z = x + y$, given the other firm's cumulative output y and the expected size of the network S, in which case its profit is given by

$$\widetilde{\pi}\left(Z, y, S\right) = \left(Z - y\right) P\left(Z, S\right) - C\left(Z - y\right). \tag{2.2}$$

This second representation will result to be very useful in some of the proofs.

Related to these two specifications of the profit function, $\widetilde{x}\left(y, S\right)$ denotes the arg max of (2.1) and $\widetilde{Z}\left(y, S\right)$ the arg max of (2.2); note that $\widetilde{Z}\left(y, S\right) = \widetilde{x}\left(y, S\right) + y$. In addition to these, $Z^*\left(S, n\right)$ is defined as the value of $\widetilde{Z}\left(y, S\right)$ that satisfies $\frac{n}{(n-1)} y = \widetilde{Z}\left(y, S\right)$. We can think of $Z^*\left(S, n\right)$ as an equilibrium in the standard Cournot competition, considering S as a simple inverse demand shifter. Under this alternative notion of equilibrium we allow firms and consumers to make mistakes about the size of the network; this idea plays an important role in the theorems related to stability and uniqueness.

[4] The first sub-index in x_{in} indicates that this quantity corresponds to firm i, the second one denotes the number of firms in the market.

Whenever well-defined, we denote the maximal and minimal points of a set by an upper and a lower bar, respectively. Thus, for instance, $\overline{Z}_n$ and $\underline{Z}_n$ are the highest and lowest aggregate equilibrium outputs. Performing comparative static on the equilibrium sets will consist of predicting the direction of change of these extremal elements as the exogenous parameter varies. In our game, the exogenous parameter is the number of firms in the market.

To facilitate the reading, the symbol ■ denotes the end of a proof, ▲ the end of an example and $\triangleq$ means "by definition."

We close this subsection with two definitions. The Marshallian social welfare when aggregate output is Z, all firms produce the same quantity and the expected size of the network is S is $W(Z,S) \triangleq \int_0^Z P(t,S)\,dt - ZA(Z/n)$. Similarly, the Marshallian consumer surplus is given by $CS(Z,S) \triangleq \int_0^Z P(t,S)\,dt - ZP(Z,S)$.

2.2 The Assumptions

In this subsection we describe all the assumptions we use later. We distinguish two different groups of constraints: The first one contains standard assumptions, or those that are usually imposed, the second one is more specific to the methodology we use.

The standard assumptions are:

(A1) $P(.,.)$ is twice continuously differentiable, $P_1(Z,S) < 0$ and $P_2(Z,S) \geq 0$;

(A2) $C(.)$ is twice continuously differentiable and increasing;

(A3) $x_i \leq K$ $\forall i$.

Assumption 1 imposes two natural conditions on the inverse demand function: It is decreasing in aggregate output, meaning that if firms want to sell more they have to choose a lower price, and is increasing in the expected size of the network, reflecting the fact that willingness to pay of consumers is increasing in the expected number of people buying the good. Assumption 2 implies that cost increases with production; and assumption 3 sets capacity constraints in the production process of each firm. Although convenient, the smoothness assumptions are by no means necessary for the main results, we include them just to simplify the proofs.

Before describing the second set of conditions we need to introduce two functions that play a key role in our results. Let $\Delta_1(Z, y)$ denote the cross-partial derivative of $\widetilde{\pi}(Z, y, S)$ with respect to Z and y, and $\Delta_2(Z, S)$ the cross-partial derivative of $\ln P(Z, S)$ with respect to Z and S [multiplied by $P(Z, S)^2$]. Then,

$$\Delta_1(Z, y) = P_1(Z, S) + C''(Z - y)$$
$$\Delta_2(Z, S) = P(Z, S) P_{12}(Z, S) - P_1(Z, S) P_2(Z, S)$$

Note that $\Delta_1(Z, y)$ and $\Delta_2(Z, S)$ are defined on the lattices $\varphi_1 \triangleq \{(Z, y) : y \geq 0, Z \geq y\}$ and $\varphi_2 \triangleq \{(Z, S) : Z \geq 0, S \geq 0\}$, respectively. The second set of constraints is:

(A4) $\Delta_1(Z, y) = -P_1(Z, S) + C''(Z - y) > 0$ globally on φ_1;

(A5) $\Delta_2(Z, S) = P(Z, S) P_{12}(Z, S) - P_1(Z, S) P_2(Z, S) \geq 0$ globally on φ_2;

(A6) $P(Z, S)$ is log-concave in Z.

Assumptions 4 and 5 guarantee that the profit function $\widetilde{\pi}(Z, y, S)$ of the firm under consideration has increasing differences and the single-crossing property on the lattices φ_1 and φ_2, respectively. It is interesting to note that A5 is equivalent to assume that the elasticity of demand is increasing in S; this means that the higher S is the less responsive are prices to changes in aggregate output[5]. Assumption 6 is needed for two results: uniqueness and stability; when added to the others, A6 guarantees that for all S there exists only one arg max, $\widetilde{Z}(y, S)$, of (2.2) that satisfies $\frac{n}{(n-1)} y = \widetilde{Z}(y, S)$, in other words, $Z^*(S, n)$ is uniquely defined as a function of S.

[5]The price elasticity of demand is $-\left(\frac{\partial P(Z,S)}{\partial Z} \frac{Z}{P(Z,S)}\right)^{-1} = -\left(Z \frac{\partial \ln P(Z,S)}{\partial Z}\right)^{-1}$, which is increasing in S if and only if $\ln P(Z, S)$ has increasing differences in (Z, S) [Topkis (1998), pp. 66].

3 Existence, Stability and Uniqueness of Equilibria

In this section we prove existence and analyze stability and uniqueness of equilibria. The fundamental questions under consideration are: Under what assumptions the equilibrium exists?, What conditions guarantee stability and how are these conditions affected by the entry of new firms in the market?, and What extra requirements do we need for uniqueness?.

First we provide conditions on $P(.,.)$ and $C(.)$ that guarantee the existence of the equilibrium. As Theorem 3.1 says, the same conditions that assure existence preclude the possibility of asymmetric equilibria.

Theorem 3.1. *Under assumptions A1-A5, for each $n \in N$, the Cournot oligopoly with network effects has at least one symmetric equilibrium and no asymmetric equilibria.*

Let $\widetilde{x}$ and $\widetilde{Z}$ denote the best response output level and total output, respectively, by a firm to a joint output y by the other $(n-1)$ firms when the expected size of the network is S. The following mapping, which can be thought of as a normalized cumulative best-response correspondence, is the key element in dealing with symmetric equilibria for any number of firms n

$$\begin{aligned} B_n : [0,(n-1)K] \times [0,nK] &\longrightarrow 2^{[0,(n-1)K]\times[0,nK]} \\ (y,S) &\longrightarrow \left[\tfrac{n-1}{n}\left(\widetilde{x}+y\right), \widetilde{x}+y\right]. \end{aligned}$$

It is readily verified that the (set-valued) range of B_n is as given, *i.e.* if $\widetilde{x} \in [0,k]$ and $y \in [0,(n-1)k]$, then $\frac{n-1}{n}(\widetilde{x}+y) \in [0,(n-1)k]$ and $\widetilde{x}+y \in [0,nK]$. Also, a fixed-point of B_n is easily seen as a symmetric equilibrium, for it must satisfy both, $y^* = \frac{n-1}{n}(\widetilde{x}^*+y^*)$, or $\widetilde{x}^* = \frac{1}{n-1}y^*$, and $S^* = \widetilde{x}^* + y^*$, which says that the responding firm produces as much as each of the other $(n-1)$ firms and the expected size of the network equals aggregate output at equilibrium.

Assumptions A4 and A5 guarantee that $\widetilde{Z} = \widetilde{x}+y$ increases with y and S, respectively. As a consequence, B_n is an increasing mapping and the proof of existence follows as a simple application of Tarski's fixed point theorem.

Alternatively, the absence of asymmetric equilibria obeys the fact that the best response mapping $(y,S) \longrightarrow \widetilde{Z}$ [the arg max in (2.2)] is strictly increasing (in the sense that all its selections are strictly increasing) in y for all S. Thus, if we keep S fixed, to each $\widetilde{Z}$ corresponds (at most) one y such that $\widetilde{Z} = \widetilde{x} + y$ with $\widetilde{Z}$ being a best-response to y and S. In other words, for the arg max $\widetilde{Z}$, each firm must be producing the same $\widetilde{x} = \widetilde{Z} - y$, with $y = (n-1)\,\widetilde{x}$.

Theorem 3.1 extends the existence results in the literature on network goods to very general settings. It places all the restrictions on a general inverse demand function, and thus it does not assume any kind of homogeneity among consumers. Additionally, it dispenses with the assumptions of no cross-effects in the demand-side and constant marginal costs of production in the supply-side of the economy.

It is important to note that differentiability of the demand and cost functions is assumed here purely for convenience. As will become clear in the proofs, the fundamentally needed assumptions are the supermodularity of $\widetilde{\pi}\,(Z, y, S)$ with respect to (Z, y) on φ_1, and the single-crossing property with respect to (Z, S) on φ_2.

If we compare the conditions we impose here with the ones required for the standard Cournot competition, assuming that $P_2\,(Z, S) \geq 0$, which holds by construction in network goods, the only extra constraint is a cross-effect on the inverse demand function $[P_{12}\,(Z, S)]$ that is not too negative. All other assumptions are also needed for proving existence in the standard analysis [Theorem 2.1, Amir and Lambson (2000)].

Theorem 3.1 guarantees existence but it does not eliminate the possibility of multiple equilibria. As many network industries display this feature, the predictions that we obtain are often incomplete. Since one possible criterion to sharpen predictions is stability, Theorem 3.2 gives sufficient conditions to characterize the interior equilibria (including the no-trade equilibrium) in terms of this property. For this purpose let's define a new function

$$g\,(z, n) = \frac{-n\,\{P_1(z,z)P_2(z,z) + [C'(z/n) - P(z,z)]\,P_{12}(z,z)\}}{(n+1)\,P_1^2(z,z) + n\,[C'(z/n) - P(z,z)]\,P_{11}(z,z) - P_1(z,z)C''(z/n)}$$

Theorem 3.2. *Under assumptions A1-A6 , we have:*

(a) If $g\,(z, n) < 1$ *at some interior equilibrium* $z = Z_n$*, then the equilibrium is stable;*

(b) If $g\,(z, n) > 1$ *at some interior equilibrium* $z = Z_n$*, then the equilibrium is unstable.*

Figure 1 clarifies the idea behind the proof of this theorem. Remember that $Z^*\,(S, n)$

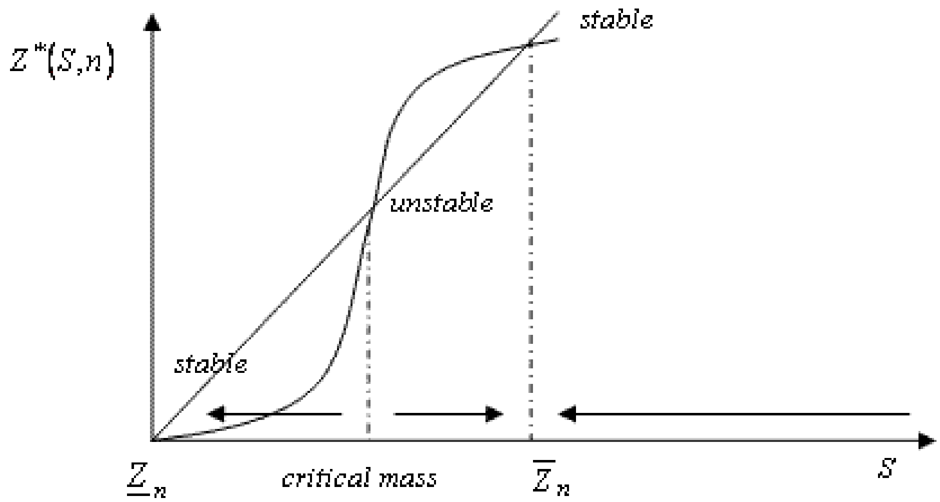

Figure 1: Stability of equilibria.

denotes the best response aggregate output of the firms to a certain value of S, subject to the constraint $\frac{n}{(n-1)}y = \widetilde{Z}(S,y)$. Together with assumptions A1-A4, A6 guarantees that $Z^*(S,n)$ is uniquely defined as a function of S. The (fulfilled expectations) equilibria of the network game are the points where $Z^*(S,n)$ crosses the 45^0 line. "Stability" has to do with the following thought experiment: Suppose that, starting at some equilibrium, we change S slightly and let firms react by choosing an action that maximizes their current profits. This will generate a new $Z^*(S,n)$ that will in turn affect people expectations, and so forth. This yields a sequential adjustment process in which consumers and firms behave myopically (*i.e.* firms ignore the effect that their adjustment has on their rivals). The equilibrium is considered "stable" if this process converges to the initial position. The function $g(z,n)$, recently defined, reflects the slope of $Z^*(S,n)$ with respect to S at the diagonal. Figure 1 shows that Z_n is stable [unstable] when the slope of this function is lower [higher] than one, *i.e.* $g(Z_n,n) < 1$ $[g(Z_n,n) > 1]$ [6].

Figure 1 was not randomly chosen; many papers suggest that it reflects the structure

[6] This argument is based on the fact that the slope of $\widetilde{Z}(S,n)$ with respect to S is always positive; this result holds by A5.

of the telecommunications industries. The implicit game there displays three possible equilibria: The no-trade equilibrium, a middle unstable equilibrium (usually called "critical mass"[7]), and a third equilibrium that is stable and displays strictly positive sales. The justification of this structure is quite simple: If all the consumers expect that none will acquire the good, then the good has no value and none will buy it, resulting in a no-trade equilibrium. However, if expectations are higher to start with, other non-trivial equilibria can also occur.

When the no-trade equilibrium is stable, as in Figure 1, it is possible that the market never emerges; the reason is that if the "critical mass" is very large it could be too risky for the incumbent firms to produce enough to reach it, and any initial level of output lower than the "critical mass" generates a process that converges to the trivial, no-trade, equilibrium. This argument is commonly used to justify the start-up problem in network goods, that is, the difficulties of the incumbent firms to generate enough expectations to achieve "critical mass."[8]

The current explanation for the take-off is based on the positive feedback idea: As the installed base of users grows, more and more users find adoption worthwhile, eventually, the product achieves "critical mass," and the market explodes to the following stable equilibrium. This argument is intrinsically dynamic. We complement this explanation by showing the role of market structure. This analysis is based on the dependence of the stability condition in Theorem 3.2 on the number of firms in the market, at the no-trade equilibrium.

Lemma 3.1 gives sufficient conditions that guarantee the existence of the no-trade equilibrium; after it, Theorem 3.3 examines how the stability of this equilibrium is affected by the entry of new firms in the market.

Lemma 3.1. *Suppose that $xP(x,0) \leq C(x)$ $\forall x$, then, for each $n \in N$, $Z_n = 0$ is an equilibrium for the Cournot Oligopoly with networks effects.*

[7] "Critical mass" is defined as the minimal non-zero equilibrium size of a network good or service.

[8] For an interesting discussion of this phenomenon see Shapiro and Varian (1998).

Proof. Let's assume that $y = S = 0$, then $\widetilde{x}(0,0) = 0$ is an arg max of (2.1) if and only if the following hold

$$\begin{aligned} \pi(0,0,0) &\geq \pi(x,0,0) && \forall x \\ 0 &\geq xP(x,0) - C(x) && \forall x \\ C(x) &\geq xP(x,0) && \forall x \end{aligned}$$

Thus, if $xP(x,0) \leq C(x)\ \forall x$, $\widetilde{x}(0,0) = 0$ is an arg max of (2.1) and $Z_n = 0$ is an equilibrium for each $n \in N$. ■

The necessary and sufficient condition in Lemma 3.1 is often satisfied. The reason is that in many network markets if all consumers expect that none else will buy the good, then the good has no value; in other terms, $P(x,0) = 0\ \forall x$.

Theorem 3.3. *In addition to A1-A6, assume that* $xP(x,0) \leq C(x)\ \forall x$ *and* $C'(x) = c \geq 0$. *Then,*

(a) The slope of $Z^*(S,n)$ *with respect to* S *at the no-trade equilibrium increases when an extra firm enters the market, i.e.* $\dfrac{\partial Z^*(0,n+1)}{\partial S} \geq \dfrac{\partial Z^*(0,n)}{\partial S}$;

(b) $\dfrac{\partial Z^*(0,n)}{\partial S} \gtreqless 1$ *according to* $n \gtreqless -(1+n)\dfrac{P_1(0,0)}{P_2(0,0)}$.

The first part of Theorem 3.3 shows that when the no-trade equilibrium exists, it becomes more unstable when other firms enter the market. The second part points out the values of n for which the equilibrium is stable or unstable.

As was mentioned before, it suggests an alternative explanation for the start-off phenomenon in the network industries: It is possible that the no-trade equilibrium is stable for a small number of firms in the market, but it becomes unstable when more firms decide to enter. Example 2, at the end of this section, illustrates this idea. It describes an extreme case where if the number of firms is small, the no-trade equilibrium is stable and the market never emerges; but if the number of firms increases enough, the no-trade equilibrium becomes unstable and the market easily converges to the highest equilibrium.

The last theorem deals with uniqueness. Its proof is quite similar to the one of Theorem 3.4. In fact, all we need for uniqueness is that the condition for stability holds globally at the equilibrium path, *i.e.* at $z = Z = S$.

Theorem 3.4. *In addition to A1-A6, suppose that condition (3.1) is satisfied* $\forall z$

$$\left[P(z,z) - C'(z/n)\right]\left[P_{11}(z,z) + P_{12}(z,z)\right] - P_1(z,z)\left[P_1(z,z) + P_2(z,z) + \frac{P_1(z,z) - C''(z/n)}{n}\right] < 0. \tag{3.1}$$

Then, there exists a unique and symmetric equilibrium.

We end this section with two examples. The first one clarifies the idea of Theorem 3.2 and the second one illustrates the relevance of Theorem 3.3.

Example 1. Consider the symmetric Cournot oligopoly with no production costs, and inverse demand function given by

$$P(Z,S) = e^{-\frac{Z}{S^\alpha}}, \qquad nK \geq Z \geq 0, S \geq 0, \alpha > 0\,[\neq 1].$$

The reaction function is $x(y,S) = S^\alpha$, for $S \geq 0$. Adding for all firms in the market, we obtain $Z^*(S,n) = nS^\alpha$. At equilibrium, $Z^*(S,n) = S$. Thus,

$$Z_n = \left\{0, n^{\frac{1}{1-\alpha}}\right\} \;\; if \;\; 0 < \alpha < 1, \quad Z_n = \left\{0, n^{\frac{1}{1-\alpha}}, nK\right\} \;\; if \;\; \alpha > 1.$$

From the previous computations, $\dfrac{\partial Z^*(S,n)}{\partial S} = n\alpha S^{\alpha-1}$ for $Z^* \leq nK$, and 0 otherwise.

As a consequence, it is readily verified that when $0 < \alpha < 1$, the no-trade equilibrium is unstable and the highest one is stable. However, when $\alpha > 1$, the no-trade equilibrium and the highest one are stable but the middle equilibrium becomes unstable. ▲

Example 2. Let's consider now this alternative inverse demand function

$$P(Z,S) = e^{-\frac{Z}{bS}}, \qquad nK \geq Z \geq 0, S \geq 0, b > 0.$$

The reaction function is $x(y,S) = bS$. Adding for all firms in the market, we obtain $Z^*(S,n) = nbS$. At equilibrium, $Z^*(S,n) = S$. Thus, we have three possible equilibrium configurations

$$Z_n = 0 \;\; if \;\; nb < 1, \quad Z_n \in [0, nK] \;\; if \;\; nb = 1, \quad Z_n = \{0, nK\} \;\; if \;\; nb > 1.$$

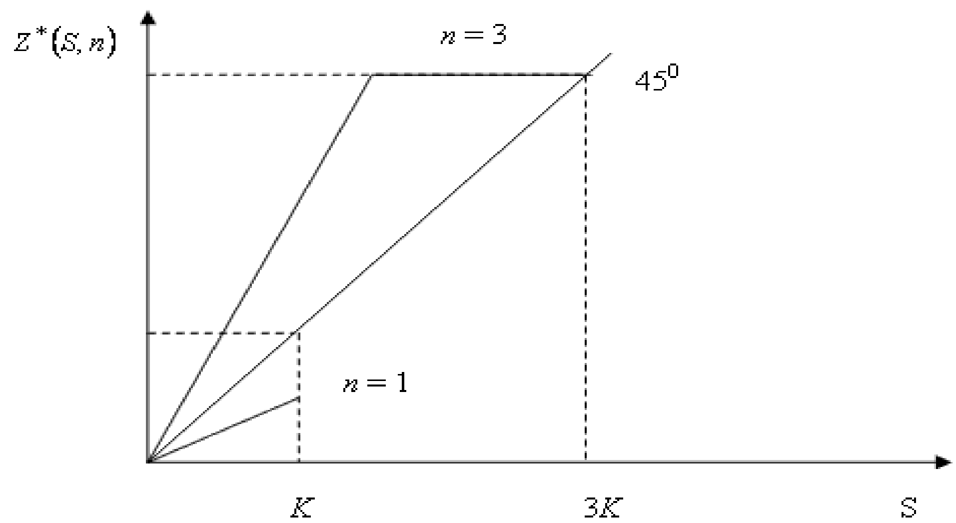

Figure 2: No-trade equilibrium, stability and n.

As Theorem 3.3 predicts, $\dfrac{\partial Z^*(S,n)}{\partial S} = nb$ increases with n. In particular, it can be verified that if $b < n^{-1}$ the no-trade equilibrium is stable, but if $b > n^{-1}$ it becomes unstable.

Figure 2 illustrates these features for $b = \frac{1}{2}$. We observe in the graph that when $n \leq 2$ the slope of $Z^*(S,n)$ with respect to S is less than 1, and the no-trade equilibrium is always stable; but when $n > 2$ the slope is higher than 1 (for small values of S), the no-trade equilibrium becomes unstable and a new stable equilibrium appears in the capacity constraints of the firms. ▲

The last example also has an interesting implication for patent law in new network industries: If one firm gets the rights for the good in that context, it would rather prefer to give it for free to other potential firms than to keep it for itself; alone, the firm could never generate enough expectations on consumers to start the market off.

4 Equilibrium Price, Output and Profits

In this section we start the analysis of market behavior and performance. The main questions are: How do total output, per-firm output, and industry price vary with the number of firms in the market? and How are per-firm profits affected by the entry of a new firm?.

The approach we follow to answer these questions is similar to the one used by Amir and Lambson (2000) and Amir (2003) to analyze the standard Cournot competition; we show that their results change significantly when network effects are included.

The mapping B_n, introduced in Section 3, can also be used for comparative static purposes; the results in Theorem 4.1 are based on the fact that B_n is increasing in n.

Theorem 4.1. *Under assumptions A1-A5, we have:*
(a) The extremal equilibrium cumulative outputs of $(n-1)$ *firms* $\overline{y}_n$ *and* $\underline{y}_n$ *are increasing in* n*;*
(b) The extremal equilibrium total outputs $\overline{Z}_n$ *and* $\underline{Z}_n$ *are increasing in* n*.*

Note: Theorem 4.1 part (b) does not imply that the maximal and minimal selections of the equilibrium price $P\left(\overline{Z}_n, \overline{Z}_n\right)$ and $P\left(\underline{Z}_n, \underline{Z}_n\right)$ are decreasing in n, in contrast to standard Cournot competition [Theorem 2.2, Amir and Lambson (2000)]. The reason is that in our case the increase in the extremal equilibrium outputs is accompanied by the same increase in the size of networks, and these two variables have opposite effects on market price. In fact, if the network effect dominates the output effect at the equilibrium path, *i.e.* $P_1(z,z) + P_2(z,z) > 0\ \forall z \in \left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ and $\left[\underline{Z}_n,\ \underline{Z}_{n+1}\right]$, then $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) > P\left(\overline{Z}_n, \overline{Z}_n\right)$ and $P\left(\underline{Z}_{n+1}, \underline{Z}_{n+1}\right) > P\left(\underline{Z}_n, \underline{Z}_n\right)$; the opposite occurs if the output effect dominates the network one.

The previous note shows that we can not globally define the direction of the extremal equilibrium prices as a function of n; for similar reasons we can neither conclude about the direction of the change in the maximal and minimal selections of per-firm equilibrium outputs $\overline{x}_n$ and $\underline{x}_n$ nor profits $\overline{\pi}_n$ and $\underline{\pi}_n$. To make inferences about them we need to add some extra conditions. Doing this, Theorems 4.2 and 4.3 extend the results; but before introducing them we need to define a new function

$$f(z) = \left[P(z,z) - C'(z/n)\right]\left[P_{11}(z,z) + P_{12}(z,z)\right] - P_1(z,z)\left[P_1(z,z) + P_2(z,z)\right].$$

Theorem 4.2. *Under assumptions A1-A5, we have:*

(a) If $f(z) \geq 0$ *over* $[\overline{Z}_n, \overline{Z}_{n+1}]$ *and* $[\underline{Z}_n,\ \underline{Z}_{n+1}]$*, then* $\overline{x}_{n+1} \geq \overline{x}_n$ *and* $\underline{x}_{n+1} \geq \underline{x}_n$*;*
(b) If $f(z) \leq 0$ *over* $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ *and* $\left[\underline{Z}_n, \underline{Z}_{n+1}\right]$*, then* $\overline{x}_{n+1} \leq \overline{x}_n$ *and* $\underline{x}_{n+1} \leq \underline{x}_n$*.*

The implication of part (b) is well-known as "business-stealing effect," since the strategic response of existing firms to new entry results in lowering their volume of sales. In other words, the new entrant "steals business" from incumbent firms.

Alternatively, we can call the phenomenon in part (a) "business-enhancing effect," because in this case the new entry encourages the incumbent firms to produce more. When $f(z) \geq 0$ the inverse demand function decreases at a rapidly decreasing rate as aggregate output increases at the equilibrium path, *i.e.* at $z = Z = S$. This leads to a surprising situation where a given increase in own output generates more (extra) revenue for a firm when the rival firms are producing at a higher joint output.

It can be shown that if the inverse demand function is log-convex in aggregate output, then $f(z) \geq 0$. As a consequence, log-convexity is a sufficient but not necessary condition for the extremal per-firm equilibrium outputs to increase after new entry. Theorem 2.3 in Amir and Lambson (2000) requires log-convexity and $C(.) \equiv 0$ to guarantee the same result for the standard analysis. This means that the "business-enhancing effect" arises easier when network effects are present.

It is important to note that these results do not hold globally. They depend on the sign of the function $f(z)$ over the extremal equilibrium levels of aggregate output and expected networks. This means that the extremal equilibrium per-firm outputs $\overline{x}_n$ and $\underline{x}_n$ can be increasing for some values of n and decreasing for some others.

Based on Theorem 4.2, Theorem 4.3 displays a quite interesting result. It concludes about the direction of change of the extremal per-firm equilibrium profits $\overline{\pi}_n$ and $\underline{\pi}_n$ when an extra firm enters the market.

Theorem 4.3. *Under assumptions A1-A5, we have:*
(a) If $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \geq P\left(\overline{Z}_n, \overline{Z}_n\right)$, $P\left(\underline{Z}_{n+1}, \underline{Z}_{n+1}\right) \geq P\left(\underline{Z}_n, \underline{Z}_n\right)$ *and* $f(z) \geq 0$ *over* $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ *and* $\left[\underline{Z}_n,\ \underline{Z}_{n+1}\right]$*, the extremal per-firm equilibrium profits* $\overline{\pi}_n$ and $\underline{\pi}_n$ *increase when an extra firm enters the market*;

(b) If $P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right) \leq P\left(\overline{Z}_{n},\overline{Z}_{n}\right)$, $P\left(\underline{Z}_{n+1},\underline{Z}_{n+1}\right) \leq P\left(\underline{Z}_{n},\underline{Z}_{n}\right)$ *and* $f(z) \leq 0$ *over* $\left[\overline{Z}_{n},\overline{Z}_{n+1}\right]$ *and* $\left[\underline{Z}_{n},\ \underline{Z}_{n+1}\right]$, *the extremal per-firm equilibrium profits* $\overline{\pi}_n$ and $\underline{\pi}_n$ *decrease when an extra firm enters the market.*

Theorem 4.3 has a surprising implication; it gives sufficient conditions under which the incumbent firms in the market prefer to see further entry by new firms. Although unusual, this result follows directly from the fulfilled expectations condition: When one extra firm enters the market not only aggregate output increases but also expectations are higher, thus if the network effect is important enough and the inverse demand function is not too log-concave (at equilibrium) it is possible that the extremal equilibrium prices and per-firm outputs go up and per-firm profits increase. This possibility is discussed but not proved in Katz and Shapiro (1985) and formalized under different assumptions in Economides (1995).

The proof of Theorem 4.3 is quite simple; it involves only a few steps based on the comparison of the profit function under different levels of its arguments. It is important to remark that, as it happens with Theorem 4.2, these results do not hold globally, they depend on the value of n.

We end this section with an interesting example that highlights the implications of Theorem 4.3.

Example 3. Consider the symmetric Cournot oligopoly with no production costs, and inverse demand function given by

$$P(Z,S) = \max\{a + bS^{\alpha} - Z, 0\} \quad Z \geq 0, S \geq 0, a \geq 0, b > 0, 0 < \alpha < 1.$$

The reaction function is $x(y,S) = \max\left\{\dfrac{a + bS^{\alpha} - y}{2}, 0\right\}$, for $y, S \geq 0$. Following a simple computation, we get that the (unique) aggregate equilibrium output is implicitly defined by the equality

$$-Z_n(1+n) + na + nbZ_n^{\alpha} = 0, \quad Z_n \geq 0.$$

It is readily verified that when $a = 10$, $b = 5$ and $\alpha = \frac{4}{5}$, per-firm equilibrium profits for $1, 2, 3, 4$ and 5 firms in the market are

$$\pi_1 \approx 14,437, \pi_2 \approx 49,123, \pi_3 \approx 67,280, \pi_4 \approx 70,651, \pi_5 \approx 67,288.$$

Since $\pi_1 < \pi_2 < \pi_3 < \pi_4$, we note that when the number of firms is very small, $n = 1, 2$ or 3, they are better off if an extra firm joins them, but the opposite is true when $n \geq 4$ ($\pi_4 > \pi_5$). ▲

5 Social Welfare, Consumer Surplus and Industry Profits

In this section we analyze the effects of entry on industry performance as reflected in social welfare, consumer surplus and industry profits. Our purpose is to give sufficient conditions that guarantee the validity of the following inequalities at the extremal equilibria: $W_{n+1} \geq W_n$, $CS_{n+1} \geq CS_n$ and $(n+1)\,\pi_{n+1} \geq n\pi_n$.

Since network effects introduce new features into the analysis, we first explore the general consequences of new entry. For this purpose we decompose the effects of entry in three key terms:

1. $\displaystyle\int_0^{Z_{n+1}} P\left(t, Z_{n+1}\right) dt - \int_0^{Z_n} P\left(t, Z_n\right) dt;$
2. $Z_{n+1} P\left(Z_{n+1}, Z_{n+1}\right) - Z_n P\left(Z_n, Z_n\right);$
3. $Z_{n+1} A\left(x_{n+1}\right) - Z_n A\left(x_n\right).$

The first term reflects the change of the area below the inverse demand function, until the equilibrium aggregate output corresponding to $(n+1)$ firms. This area increases with n via two different channels: First, the change in expectations causes an upward displacement of the whole inverse demand function; second, since $Z_{n+1} \geq Z_n$ (at the extremal equilibria), the area also increases because more people buy the good. Figure 3 shows these two effects (the first effect is in light grey and the second one in dark grey). Based on this analysis, Term 1 is always positive.

The second and the third terms reflect the changes in industry costs of production and total expenditures, respectively. These two terms can not be signed a priori because the direction of change of the extremal equilibrium per-firm outputs and prices are not globally defined.

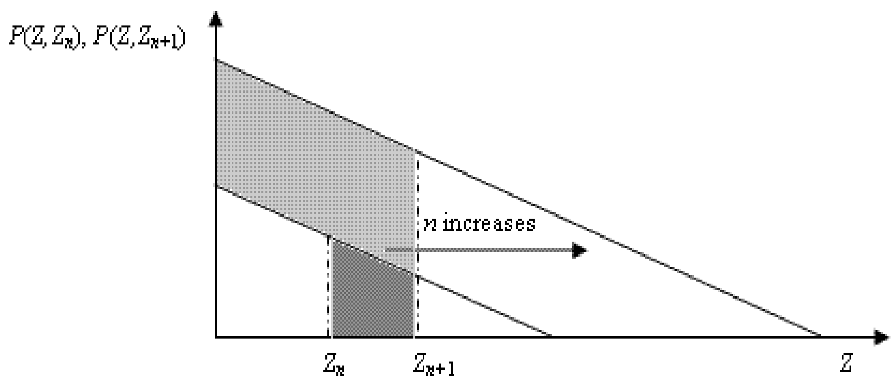

Figure 3: Inverse demand function, expectations and n.

We now present the main results of this section. The three theorems that follow are based on different comparisons of the three terms just described.

Theorem 5.1 gives sufficient conditions for social welfare to increase after the entry of a new firm.

Theorem 5.1. *In addition to A1-A5, suppose that condition (5.1) is satisfied at the extremal equilibria*

$$\int_0^{Z_n} \left[P\left(t, Z_{n+1}\right) - A\left(x_{n+1}\right)\right] dt \geq \int_0^{Z_n} \left[P\left(t, Z_n\right) - A\left(x_n\right)\right] dt. \tag{5.1}$$

Then, $W_{n+1} \geq W_n$ *for any* $n \in N$ *(at the extremal equilibria).*

Proof. For an extremal equilibrium, consider

$$\begin{aligned} W_{n+1} - W_n &= \int_0^{Z_{n+1}} \left[P\left(t, Z_{n+1}\right) - A\left(x_{n+1}\right)\right] dt - \int_0^{Z_n} \left[P\left(t, Z_n\right) - A\left(x_n\right)\right] dt \\ &\geq \int_0^{Z_n} \left[P\left(t, Z_{n+1}\right) - A\left(x_{n+1}\right)\right] dt - \int_0^{Z_n} \left[P\left(t, Z_n\right) - A\left(x_n\right)\right] dt \end{aligned}$$

The inequality follows directly from Theorem 4.1. Theorem 5.1 just says that if the RHS of this inequality is positive, then $W_{n+1} \geq W_n$ at the extremal equilibria. ■

Note: Since, by A1, $\int_0^{Z_n} \left[P\left(t, Z_{n+1}\right) - P\left(t, Z_n\right)\right] dt \geq 0$, $A\left(x_{n+1}\right) \leq A\left(x_n\right)$ is also a sufficient condition for $W_{n+1} \geq W_n$.

Theorem 5.1 suggests that if firms display economies of scale, or if the diseconomies are lower than the increase in willingness to pay of consumers due to the higher network size, then social welfare increases after new entry.

To guarantee the same result in the standard Cournot competition we need to assume $A(x_n) \geq A(x_{n+1})$, that is, the sufficient condition mentioned in the previous note [Proposition 6, Amir (2003)]. Since this condition is stronger than condition (5.1) we conclude that, keeping everything else equal, the number of firms that maximizes social welfare is higher when network effects are present.

Theorem 5.2 gives sufficient conditions for consumer surplus to increase after the entry of a new firm.

Theorem 5.2. *In addition to A1-A5 suppose that either (a) $P(Z_{n+1}, Z_{n+1}) \leq P(Z_n, Z_n)$ or (b) $P_{12}(Z, S) \leq 0 \,\forall Z, S$ is satisfied. Then, $CS_{n+1} \geq CS_n$ for any $n \in N$ (at the extremal equilibria).*

The proof of part (a) is obvious, so we omit it. The following steps prove part (b).

Proof. For an extremal equilibrium, consider

$$\begin{aligned}
CS_{n+1} - CS_n &= \int_0^{Z_{n+1}} [P(t, Z_{n+1}) - P(Z_{n+1}, Z_{n+1})]\, dt - \int_0^{Z_n} [P(t, Z_n) - P(Z_n, Z_n)]\, dt \\
&\geq \int_0^{Z_n} [P(t, Z_{n+1}) - P(Z_{n+1}, Z_{n+1})]\, dt - \int_0^{Z_n} [P(t, Z_n) - P(Z_n, Z_n)]\, dt \\
&= \int_0^{Z_n} [P(Z_n, Z_n) - P(Z_{n+1}, Z_n)]\, dt + \\
&\quad \int_0^{Z_n} [P(t, Z_{n+1}) - P(t, Z_n)]\, dt - \int_0^{Z_n} [P(Z_{n+1}, Z_{n+1}) - P(Z_{n+1}, Z_n)]\, dt \\
&\geq \int_0^{Z_n} [P(Z_n, Z_n) - P(Z_{n+1}, Z_n)]\, dt \\
&\geq 0
\end{aligned}$$

The first inequality follows directly from A1 and Theorem 4.1. The third expression is obtained from the second one by adding and subtracting $\int_0^{Z_n} P(Z_{n+1}, Z_n)\, dt$, and rearranging terms. The second inequality follows because $P_{12}(Z, S) \leq 0$ is sufficient for

$\int_0^{Z_n} \left[P\left(t, Z_{n+1}\right) - P\left(t, Z_n\right)\right] dt \geq \int_0^{Z_n} \left[P\left(Z_{n+1}, Z_{n+1}\right) - P\left(Z_{n+1}, Z_n\right)\right] dt$ and the last one is true by A1.

Thus, under A1-A5, $P_{12}\left(Z, S\right) \leq 0 \; \forall Z, S$ is sufficient for $CS_{n+1} - CS_n \geq 0$, or $CS_{n+1} \geq CS_n$, at the extremal equilibria. ■

It is important to note that condition (a) in Theorem 5.2 is always satisfied in the standard Cournot competition. As a consequence, consumer surplus always increases with n in the standard analysis. Example 4, at the end of this section, shows that the opposite sometimes happens in network industries.

Katz and Shapiro (1985) clearly explain why this surprising result can occur: "If the network externality is strong for the marginal consumer, then the increase in the expected network caused by the change in the number of firms will raise his or her willingness to pay for the good by more than that of the average consumer. As a consequence, the firms will be able to raise the price by more than the increase in the average consumer's willingness to pay for the product and consumer's surplus will fall." Note that this explanation is fully related to the violation of conditions (a) and (b) in Theorem 5.2.

The last theorem deals with industry profits; as the previous two theorems, it gives sufficient conditions for aggregate profits to increase after the entry of a new firm.

Theorem 5.3. *In addition to A1-A5, assume that condition (5.2) is satisfied at the extremal equilibria*

$$P\left(Z_{n+1}, Z_{n+1}\right) - P\left(Z_n, Z_n\right) > A\left(x_{n+1}\right) - A\left(x_n\right). \tag{5.2}$$

Then, $\left(n+1\right)\pi_{n+1} \geq n\pi_n$ *for any* $n \in N$ *(at the extremal equilibria).*

Proof. For an extremal equilibrium, consider

$$\begin{aligned}
\left(n+1\right)\pi_{n+1} - n\pi_n &= Z_{n+1}\left[P\left(Z_{n+1}, Z_{n+1}\right) - A\left(x_{n+1}\right)\right] - Z_n\left[P\left(Z_n, Z_n\right) - A\left(x_n\right)\right] \\
&\geq Z_n\left[P\left(Z_{n+1}, Z_{n+1}\right) - A\left(x_{n+1}\right)\right] - Z_n\left[P\left(Z_n, Z_n\right) - A\left(x_n\right)\right] \\
&= Z_n\left\{\left[P\left(Z_{n+1}, Z_{n+1}\right) - P\left(Z_n, Z_n\right)\right] - \left[A\left(x_{n+1}\right) - A\left(x_n\right)\right]\right\}
\end{aligned}$$

By the optimization principle, $P(Z_{n+1}, Z_{n+1}) - A(x_{n+1}) \geq 0$. Then, the second step follows by Theorem 4.1. Theorem 5.3 says that if the last term is positive, then $(n+1)\,\pi_{n+1} \geq n\pi_n$ at the extremal equilibria. ■

Theorem 5.3 can be easily interpreted: If market price increases more (or decreases less) than average cost of production, then industry profits increase after new entry.

Example 4 ends this section. It shows an appealing situation where consumer surplus decreases after new entry but industry profits are higher.

Example 4. Consider the symmetric Cournot oligopoly with no production costs and inverse demand function given by

$$P(Z, S) = \max\{a - ZS^{-3}, 0\} \quad nK \geq Z \geq 0, S \geq 0, \alpha > 0.$$

The reaction function is $x(y, S) = \max\left\{\dfrac{aS^3 - y}{2}, 0\right\}$ for $y, S \geq 0$. Thus, we have three possible equilibria

$$Z_n = \left\{0, \sqrt{(n+1)\,(na)^{-1}}, nK\right\}.$$

From a simple computation, consumer surplus is 0 at the smallest equilibrium and, assuming $a \geq (nK)^{-2}$, it equals the following expression at the highest one

$$\overline{CS}_n = (2nK)^{-1}.$$

Since this expression is decreasing in n, consumer surplus decreases with new entry at the highest equilibrium. This result holds because conditions (a) and (b) in Theorem 5.2 are not satisfied, *i.e.* the highest equilibrium price is increasing in n and $P_{12}(Z, S) = 3S^{-4} > 0\ \forall Z, S$.

Note that the opposite is true for aggregate profits. The following expression shows that they increase with n at the highest equilibrium

$$n\overline{\pi}_n = nK\left[a - \frac{1}{(nK)^2}\right].$$

These two results point out some of the relevant differences between the standard analysis and Cournot competition with network effects. ▲

6 Equilibria Comparisons

We observed before that even restricting attention to rational expectations equilibrium, FECE still allows multiple equilibria to occur. In this section we fix n and compare the extremal equilibria from the perspective of firms and consumers. Our main findings are that social welfare is always maximized at the highest equilibrium aggregate output and that this equilibrium is also preferred by the firms when demand-side economies of scale are important enough. In addition to these, if the cross-effect on the inverse demand function is negative or the market price is minimized at the highest equilibrium, consumer surplus is also maximized at $\overline{Z}_n$.

Theorem 6.1 shows that social welfare is always maximized at the highest equilibrium.

Theorem 6.1. *Let Z_n and Z_n' denote two distinct equilibrium aggregate output with corresponding social welfare levels W_n and W_n'. Under assumptions A1-A5, if $Z_n < Z_n'$, then $W_n < W_n'$. Hence, $\overline{Z}_n$ is the social welfare maximizer among all equilibrium aggregate output.*

This result is also true for the standard Cournot competition. However, in our case, the network effects act increasing the difference between W_n' and W_n. Therefore, the loss in social welfare by lack of coordination is much higher in these industries than in the regular ones.

Theorem 6.2 compares the extremal equilibria from the perspective of consumers. It gives sufficient conditions such that consumer surplus is also maximized at the highest equilibrium. This second result clearly differs with the standard analysis.

Theorem 6.2. *Let Z_n and Z_n' denote two distinct equilibrium aggregate outputs with corresponding consumer surplus levels CS_n and CS_n', such that $Z_n < Z_n'$. If A1-A5 hold and either (a) $P\left(Z_n', Z_n'\right) \leq P\left(Z_n, Z_n\right)$ or (b) $P_{12}\left(Z, S\right) \leq 0$ is satisfied, then $CS_n' \geq CS_n$ for any $n \in N$.*

Condition (a) is always true in the standard Cournot competition, and then consumer surplus is always maximized at the highest equilibrium in the regular case. As it was mentioned before, this result changes when network effects are present. Theorem 6.2 suggests that if $P_{12}(Z,S) > 0$ for some Z and S, and the price is significantly larger at the highest equilibrium, consumer surplus could be maximized at a smaller one.

The last theorem compares industry profits. Again, the results we obtain differ with the standard analysis. In the regular Cournot competition, industry profits are always maximized at the minimal equilibrium; Theorem 6.3 shows that when network effects are present it is possible that firms prefer the highest equilibrium.

Theorem 6.3. *Let Z_n and $Z_n^{'}$ denote two distinct equilibrium aggregate outputs with corresponding industry profits levels $n\pi_n$ and $n\pi_n^{'}$. Under A1-A5, if $Z_n < Z_n^{'}$, we have:*
(a) If $P\left(\frac{Z_n - Z_n^{'}}{n} + Z_n^{'}, Z_n^{'}\right) \geq P(Z_n, Z_n)$, then $n\pi_n^{'} \geq n\pi_n$;
(b) If $P\left(\frac{Z_n^{'} - Z_n}{n} + Z_n, Z_n\right) \geq P\left(Z_n^{'}, Z_n^{'}\right)$, then $n\pi_n \geq n\pi_n^{'}$.

We end this section with an interesting example that highlights the main results in this section and has relevant public policy implications. In Example 5 industry profits and consumer surplus are maximized at the highest equilibrium.

Example 5. Consider the symmetric Cournot oligopoly with no production costs and inverse demand function given by

$$P(Z,S) = Se^{-Z} \quad S \geq 0, Z \geq 0.$$

The first order condition of the optimization problem is $Se^{-(x+y)}(1-x) = 0$. Thus, this example has two possible equilibria: An interior equilibrium (in dominant strategies) and the trivial no-trade equilibrium. After a simple computation, the (symmetric) Cournot equilibria aggregate output, total profits and consumer surplus are, respectively

$$Z_n = \{0, n\} \quad n\pi_n = \left\{0, n^2 e^{-n}\right\}, \quad CS_n = \{0, n\left[1 - e^{-n}(1+n)\right]\}.$$

We observe that in this game the highest equilibrium, which is also the stable one, maximizes both consumer surplus and industry profits. Katz and Shapiro (1994) has an interesting analysis of this issue; they discuss how different public and private institutions can help to facilitate coordination when, as in this case, one equilibrium is clearly preferred with respect to the other one. ▲

7 Conclusion and Further Extensions

The network industries clearly differ from the other, more traditional, ones. Recall that the main aim of the present paper is to thoroughly examine the implications of these differences as reflected in market behavior and performance. The general methodology of lattice-theoretic comparative statics allows us to unify in a common setting all the results in the literature on network goods (related to the perfect compatible case) weakening considerably the required assumptions. As a consequence, we provide clear economic interpretations of the forces behind them.

Many historical events, *e.g.* fax machines and phones [Shapiro and Varian (1998)], point out the relevance of the start-up difficulty in network goods. The current explanation for the start-off phenomenon is based on the positive feedback idea. We complement this explanation by showing the role of market structure; this extension is based on the dependence of the stability condition of the no-trade equilibrium on the number of firms in the market.

We show that, assuming specific conditions, it is possible that the no-trade equilibrium is stable for a small number of firms in the market but it becomes unstable when more firms decide to enter. This means that the market could never emerge with a few firms, but it could easily converge to a higher equilibrium for a larger n. This finding has an interesting implication for patent law in new network industries: If one firm gets the rights for the good in that context, it would rather prefer to give it for free to other potential firms than to keep it for itself.

In addition to the previous outcome, we confirm other interesting results. For example, the possibility that per-firm profits increase with the number of firms in the market and consumer surplus decreases with the same parameter. In the same direction, we observe that when the game has multiple equilibria, it is possible that both consumers and firms strictly prefer the highest equilibrium. This result is particularly useful for policy-makers: When the highest equilibrium is most preferred by both sides of the market, *i.e.* firms and consumers, specific policies should be implemented to facilitate coordination.

The perfect compatible case is just the start of a continuum that ends at the opposite extreme with the perfect incompatible analysis. Given that we restricted the analysis to the former, many questions are still open. We believe that the same methodology can be used to study them. Another promising extension to this paper is the study of the negative network case. Although many interesting examples fit this framework, *e.g.* vaccine market, only a few papers have studied it.

8 Proofs

Notation

We begin by formalizing some of the concepts previously mentioned. A firm's best-response correspondence is defined [for $0 \leq y \leq (n-1)\,k$ and $0 \leq S \leq nk$] by

$$\widetilde{x}\,(y,S) = \arg\max\left\{xP\,(x+y,S) - C\,(x) : 0 \leq x \leq k\right\}. \tag{8.1}$$

It will often be convenient to think of a firm as choosing cumulative output Z, given the other $(n-1)$ firm's total output y and the expected size of the network S, instead of simply choosing its own output x. With $Z \triangleq x+y$, a best-response correspondence is now defined by

$$\widetilde{Z}\,(y,S) = \arg\max\left\{(Z-y)\,P\,(Z,S) - C\,(Z-y) : y \leq Z \leq y+k\right\}. \tag{8.2}$$

The following mapping, which can be thought of as a normalized cumulative best-response correspondence, is the key element in dealing with symmetric equilibria for any n

$$\begin{aligned} B_n : [0,(n-1)\,K] \times [0,nK] &\longrightarrow 2^{[0,(n-1)K]\times[0,nK]} \\ (y,S) &\longrightarrow \left[\tfrac{n-1}{n}\,(\widetilde{x}+y)\,, \widetilde{x}+y\right]. \end{aligned}$$

It is readily verified that the (set-valued) range of B_n is as given, *i.e.* if $\widetilde{x} \in [0,k]$ and $y \in [0,(n-1)\,k]$, then $\frac{n-1}{n}\,(\widetilde{x}+y) \in [0,(n-1)\,k]$ and $\widetilde{x}+y \in [0,nK]$. Also, a fixed-point of B_n is easily seen as a symmetric equilibrium, for it must satisfy both $y^* = \frac{n-1}{n}\left(\widetilde{x}^* + y^*\right)$, or $\widetilde{x}^* = \frac{1}{n-1} y^*$, and $S^* = \widetilde{x}^* + y^*$, which says that the responding firm produces as much as each of the other $(n-1)$ firms and the expected size of the network is fulfilled at equilibrium.

The proof of Theorem 3.1 requires an intermediate result. Lemma 3.2 shows that, under A5, $\widetilde{\pi}(Z, y, S)$, as defined in equation (2.2), has the single-crossing property on the lattice $\varphi_2 \triangleq \{(Z, S) : Z \geq 0, S \geq 0\}$. This condition is needed for B_n to be an increasing mapping.

Lemma 3.2. *If* $\Delta_2(Z, S) \geq 0$, $\widetilde{\pi}(Z, y, S)$ *has the single-crossing property on the lattice* $\varphi_2 \triangleq \{(Z, S) : Z \geq 0, S \geq 0\}$.

Proof of Lemma 3.2. By definition,

$$\Delta_2(Z, S) = P(Z, S) P_{12}(Z, S) - P_1(Z, S) P_2(Z, S) = \frac{\partial^2 \ln P(Z, S)}{\partial Z \partial S} P(Z, S)^2 .$$

Thus, $\Delta_2(Z, S) \geq 0$ is equivalent to assume that $\dfrac{\partial^2 \ln P(Z, S)}{\partial Z \partial S} \geq 0$. We need to show that this last condition is sufficient for $\widetilde{\pi}(Z, y, S)$ to have the single-crossing property on the lattice φ_2; in formal terms, considering $Z' > Z$ and $S' > S$, this means

$$\widetilde{\pi}\left(Z', y, S\right) \geq \widetilde{\pi}(Z, y, S) \Longrightarrow \widetilde{\pi}\left(Z', y, S'\right) \geq \widetilde{\pi}\left(Z, y, S'\right). \tag{8.3}$$

Since $\dfrac{\partial^2 \ln P(Z, S)}{\partial Z \partial S} \geq 0$, the following hold

$$\begin{aligned} \ln P\left(Z', S'\right) - \ln P\left(Z, S'\right) &\geq \ln P\left(Z', S\right) - \ln P(Z, S) && (8.4) \\ \frac{P(Z', S')}{P(Z, S')} &\geq \frac{P(Z', S)}{P(Z, S)}. && (8.5) \end{aligned}$$

Additionally, the LHS of (8.3) can be rewritten as

$$\left(Z' - y\right) P\left(Z', S\right) - C\left(Z' - y\right) \geq (Z - y) P(Z, S) - C(Z - y). \tag{8.6}$$

Substituting (8.5) in the LHS of (8.6) we obtain

$$\left(Z' - y\right) P\left(Z', S'\right) \frac{P(Z, S)}{P(Z, S')} - C\left(Z' - y\right) \geq (Z - y) P(Z, S) - C(Z - y). \tag{8.7}$$

Multiplying both sides of (8.7) by $\dfrac{P(Z, S')}{P(Z, S)}$ we get

$$\left(Z' - y\right) P\left(Z', S'\right) - \frac{P(Z, S')}{P(Z, S)} C\left(Z' - y\right) \geq (Z - y) P\left(Z, S'\right) - \frac{P(Z, S')}{P(Z, S)} C(Z - y). \tag{8.8}$$

Since, by A1, $\dfrac{P(Z,S')}{P(Z,S)} \geq 1$ and, by A2, $C(Z'-y) \geq C(Z-y)$ (8.8) implies that the following inequality hols

$$\left(Z'-y\right) P\left(Z',S'\right) - C\left(Z'-y\right) \geq \left(Z-y\right) P\left(Z,S'\right) - C\left(Z-y\right). \tag{8.9}$$

Note that (8.9) is equal to the RHS of (8.3). The proof of Lemma 3.2 follows because this means that inequality (8.6) implies inequality (8.9), which is equivalent to say that condition (8.3) is satisfied. ■

Proof of Theorem 3.1. The cross-partial derivative of the maximand in (8.2) with respect to Z and y is given by $\Delta_1(Z,y)$, which is assumed > 0 here. Hence, the maximand in (8.2) has strictly increasing differences on the lattice $\varphi_1 \triangleq \{(Z,y) : 0 \leq y \leq (n-1)k, y \leq Z \leq y+k\}$. Furthermore, the feasible correspondence $(y,S) \longrightarrow [y, y+k]$ is ascending in y. Then, by Topki's theorem (Theorem A.1), every selection from the arg max, $\widetilde{Z}$, of (8.2) is increasing in y.

Additionally, by Lemma 3.2, $\Delta_2(Z,S) \geq 0$ implies that $\widetilde{\pi}$ has the single-crossing property on the lattice $\varphi_2 \triangleq \{(Z,S) : Z \geq 0, S \geq 0\}$. Since the feasible correspondence $(y,S) \longrightarrow [y, y+k]$ does not depend on S, every selection from the arg max, $\widetilde{Z}$, is also increasing in S (Theorem A.2).

Since $\widetilde{Z}(y,S) = \widetilde{x}(y,S) + y$, the previous two paragraphs imply that, for any fixed n, every selection of B_n is increasing in (y,S). Hence, by Tarski's fixed-point theorem (Theorem A.3), B_n has a fixed point. As argued before, a fixed point in B_n is a symmetric equilibrium.

Next we show that no asymmetric equilibrium exists.

To this end, it suffices to show that the best response mapping $(y,S) \longrightarrow \widetilde{Z}$ is strictly increasing (in the sense that all its selections are strictly increasing) in y for all S. Thus, for all possible S, to each $\widetilde{Z}$ corresponds (at most) one y such that $\widetilde{Z} = \widetilde{x} + y$ with $\widetilde{Z}$ being a best-response to y and S. In other words, for the arg max $\widetilde{Z}$, each firm must be producing the same $\widetilde{x} = \widetilde{Z} - y$, with $y = (n-1)\widetilde{x}$.

We noted before that, by A4, the maximand in (8.2) has strictly increasing differences on the lattice $\varphi_1 \triangleq \{(Z,y) : 0 \leq y \leq (n-1)k, y \leq Z \leq y+k\}$. Theorem 2.8.5 in Topkis

(1998), based on Amir (1996), states that this condition is sufficient for $\widetilde{Z}$ to be strictly increasing in y for all S. Theorem 3.1 follows because, as it was argued in the previous paragraph, this condition guarantees that no asymmetric equilibria exist. ■

Proof of Theorem 3.2. Under A1-A4 and A6 there exists only one arg max, $\widetilde{Z}(y,S)$, of (8.2) that satisfies $\frac{n}{n-1}y = \widetilde{Z}(y,S)$ [Theorem 2.3, Amir and Lambson (2000)]; in Section 2, this arg max was denoted $Z^*(S,n)$. If Z^* is an interior arg max of (8.2), it should satisfy the First Order Condition (FOC), then

$$P(Z^*,S) + (Z^*/n)\, P_1(Z^*,S) - C'(Z^*/n) = 0. \tag{8.10}$$

Multiplying both sides of (8.10) by n we get

$$nP(Z^*,S) + Z^* P_1(Z^*,S) - nC'(Z^*/n) = 0. \tag{8.11}$$

Since Z^* is a function of S and n, we can use the Implicit Function Theorem to obtain

$$\frac{\partial Z^*(S,n)}{\partial S} = \frac{-\left[nP_2(Z^*,S) + Z^* P_{12}(Z^*,S)\right]}{(n+1)\, P_1(Z^*,S) + Z^* P_{11}(Z^*,S) - C''(Z^*/n)}. \tag{8.12}$$

As we already assumed interiority, we can substitute in (8.12) Z^* by its FOC and simplify the expression to get

$$\frac{\partial Z^*(S,n)}{\partial S} = \frac{-n\left\{P_1(Z^*,S)P_2(Z^*,S) + \left[C'(Z^*/n) - P(Z^*,S)\right] P_{12}(Z^*,S)\right\}}{(n+1)\, P_1^2(Z^*,S) + n\left[C'(Z^*/n) - P(Z^*,S)\right] P_{11}(Z^*,S) - P_1(Z^*,S)C''(Z^*/n)}. \tag{8.13}$$

At (fulfilled expectations) equilibrium, $Z^* = S$. Additionally, by A5, $\partial Z^*(S,n)/\partial S \geq 0$. Since $g(z,n)$ is equal to the RHS of (8.13) evaluated at $z = Z^*(S,n) = S$, Theorem 3.2 just says that if $\partial Z^*(S,n)/\partial S < 1$ at the 45^0 diagonal, *i.e.* at $z = Z^* = S$, the equilibrium is stable; and the opposite is true if $\partial Z^*(S,n)/\partial S > 1$ at the diagonal. ■

Proof of Theorem 3.3. We will show only part (a), part (b) follows direcly form these results. Assuming constant marginal costs, equation (8.12) can be expressed as

$$\frac{\partial Z^*(S,n)}{\partial S} = \frac{-\left[nP_2(Z^*,S) + Z^* P_{12}(Z^*,S)\right]}{(n+1)\, P_1(Z^*,S) + Z^* P_{11}(Z^*,S)}. \tag{8.14}$$

When n increases, the value of (8.14) changes via two channels: Directly trough n and because n affects the value of $Z^*(S,n)$. Since Theorem 3.3 analyzes the change of the slope of $Z^*(S,n)$ at the no-trade equilibrium, *i.e.* at $0 = Z^* = S$, and this equilibrium is not affected by n, we only need to consider the *direct effect* of one extra firm in the market[9]. Since $P(Z,S)$ is twice continuously differentiable, *i.e.* $P_{11}(Z,S), P_{12}(Z,S) < \infty$, the following steps show the sufficient conditions for this effect to be positive

$$\partial Z^*(0,n+1)/\partial S \geq \partial Z^*(0,n)/\partial S \tag{8.15}$$

$$\frac{-(n+1)P_2(0,0)}{(n+2)P_1(0,0)} \geq \frac{-nP_2(0,0)}{(n+1)P_1(0,0)}. \tag{8.16}$$

Rearranging terms,

$$P_1(0,0)P_2(0,0) \leq 0. \tag{8.17}$$

By A1, condition (8.17) is always true. Theorem 3.3 follows because this implies that $\partial Z^*(0,n+1)/\partial S \geq \partial Z^*(0,n)/\partial S$. ■

Proof of Theorem 3.4. The proof of this theorem is based on the proof of Theorem 3.2. In fact, to guarantee uniqueness all we need is that the condition for stability holds along the equilibrium path. In other terms, we need $g(z,n)$ to be lower than 1 $\forall z$, with $g(z,n)$ defined as in Theorem 3.2. Condition (3.1) in Theorem 3.4 reflects this restriction. ■

Proof of Theorem 4.1. The maximal and minimal selections of B_n denoted, respectively, $\overline{B}_n$ and $\underline{B}_n$, exist by Topki's theorem (Theorem A.1). Furthermore, the largest equilibrium output of $(n-1)$ firms and the largest equilibrium aggregate output, $(\overline{y}_n, \overline{Z}_n)$, constitute the largest fixed point of B_n, denoted $\overline{B}_n$. Since $\dfrac{(n-1)}{n}$ is increasing in n, B_n is increasing in n for all (y,S). Hence, the largest fixed point of B_n, $(\overline{y}_n, \overline{Z}_n)$, is also increasing in n (Theorem A.4). A similar argument, using the selection $\underline{B}_n$, establishes that $\left(\underline{y}_n, \underline{Z}_n\right)$ is increasing in n. ■

Proof of Theorem 4.2. Lets consider the following mapping

$$M_n : [0, nK] \to [0, K]$$

[9] Note that, under the assumptions of Theorem 3.3, the no-trade equilibrium exists by Lemma 3.1.

$$z \rightarrow \widetilde{x} = \{x : P(z,z) + xP_1(z,z) - C'(x) = 0\}.$$

M_n maps aggregate output into the best response per-firm output, when the profit function of the firm is evaluated at the equilibrium path, *i.e.* at $z = Z = S$. If we totally differentiate M_n with respect to n, we obtain

$$\left\{P_1(z,z) + P_2(z,z) + \widetilde{x}\left[P_{11}(z,z) + P_{12}(z,z)\right]\right\} \frac{dz}{dn}. \tag{8.18}$$

WLG lets substitute in (8.18) x by its FOC and rearrange terms, to get

$$-\frac{1}{P_1(z,z)} \left\{\left[P(z,z) - C'(z/n)\right]\left[P_{11}(z,z) + P_{12}(z,z)\right] - P_1(z,z)\left[P_1(z,z) + P_2(z,z)\right]\right\} \frac{dz}{dn}. \tag{8.19}$$

Lets define a new function

$$f(z) = \left[P(z,z) - C'(z/n)\right]\left[P_{11}(z,z) + P_{12}(z,z)\right] - P_1(z,z)\left[P_1(z,z) + P_2(z,z)\right].$$

Substituting $f(z)$ in (8.19), we get

$$-\frac{1}{P_1(z,z)} f(z) \frac{dz}{dn}. \tag{8.20}$$

By A1 $-\dfrac{1}{P_1(z,z)} > 0$. Additionally, by Theorem 4.1, at the extremal equilibria $\dfrac{dz}{dn} \geq 0$. Then, if $f(z) \geq 0$ over $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ and $\left[\underline{Z}_n, \underline{Z}_{n+1}\right]$ the mapping M_n is increasing in n at the extremal equilibria, and the opposite holds whenever $f(z) \leq 0$ over $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ and $\left[\underline{Z}_n, \underline{Z}_{n+1}\right]$. Theorem 4.2 follows, because if M_n increases [decreases] in n at the extremal equilibria, then $\overline{x}_n$ and $\underline{x}_n$ also increase [decrease] with this parameter (Theorem A.4). ■

Proof of Theorem 4.3.

(a) By Theorem 4.2, under A1-A5, $f(z) \geq 0$ over $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$ and $\left[\underline{Z}_n, \underline{Z}_{n+1}\right]$ is sufficient for the extremal per-firm equilibrium outputs to be increasing in n, *i.e.* $\overline{x}_{n+1} \geq \overline{x}_n$ and $\underline{x}_{n+1} \geq \underline{x}_n$. Using this result, the following steps prove the first part of Theorem 4.3

$$\begin{aligned}
&\overline{\pi}_{n+1} = \\
&= \overline{x}_{n+1} P\left(\overline{x}_{n+1} + \overline{y}_{n+1}, \overline{Z}_{n+1}\right) - C\left(\overline{x}_{n+1}\right) \\
&\geq \overline{x}_n \, P\left(\overline{x}_n + \overline{y}_{n+1}, \overline{Z}_{n+1}\right) - C\left(\overline{x}_n\right), \text{ by the Cournot property}
\end{aligned}$$

$$\geq \overline{x}_n \ P\left(\overline{x}_{n+1}+\overline{y}_{n+1},\overline{Z}_{n+1}\right)-C\left(\overline{x}_n\ \right), \text{ as } \overline{x}_{n+1}\geq \overline{x}_n \text{ and } P_1\left(Z,S\right)<0 \text{ by (A1)}$$
$$\geq \overline{x}_n \ P\left(\overline{x}_n\ +\overline{y}_n,\overline{Z}_n\right)-C\left(\overline{x}_n\ \right)=\overline{\pi}_n, \text{ since } P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)\geq P\left(\overline{Z}_n,\overline{Z}_n\right).$$

Therefore, under the aforementioned assumptions, $\overline{\pi}_{n+1} \geq \overline{\pi}_n$, *i.e.* the highest per-firm equilibrium profits increase when an extra firm enters the market. By a similar argument it can be shown that this is also true for the lowest equilibrium per-firm profits.

(b) We omit this proof because is similar to the previous one. ■

Proof of Theorem 6.1. By A4, $-P_1\left(Z,S\right)+C''\left(Z-y\right)>0$ on $\varphi_1 \triangleq \left\{(Z,y): y\geq 0, Z\geq y\right\}$. Thus, $U_n\left(Z,S\right) \triangleq \int_0^Z P\left(t,S\right)dt - nC\left(Z/n\right)$ is strictly concave in Z as $U_{n_{11}}\left(Z,S\right) = P_1\left(Z,S\right)-\frac{1}{n}C''\left(Z/n\right)<0$. Now consider,

$$\begin{aligned}
&W_n^{'}-W_n = \\
&= \int_0^{Z_n^{'}} P\left(t,Z_n^{'}\right)dt - nC(Z_n^{'}/n) - \left[\int_0^{Z_n} P\left(t,Z_n\right)dt - nC\left(Z_n/n\right)\right] \\
&\geq \int_0^{Z_n^{'}} P\left(t,Z_n^{'}\right)dt - nC(Z_n^{'}/n) - \left[\int_0^{Z_n} P\left(t,Z_n^{'}\right)dt - nC\left(Z_n/n\right)\right], \text{ by A1} \\
&= U_n\left(Z_n^{'},Z_n^{'}\right)-U_n\left(Z_n,Z_n^{'}\right) \\
&> U_{n_1}\left(Z_n^{'},Z_n^{'}\right)\left(Z_n^{'}-Z_n\right), \text{ since } U_n\left(Z,S\right) \text{ is strictly concave in } Z \text{ and } Z_n^{'}>Z_n \\
&= \left[P\left(Z_n^{'},Z_n^{'}\right)-C'(Z_n^{'}/n)\right]\left(Z_n^{'}-Z_n\right) \\
&\geq 0, \text{ since } Z_n^{'}>Z_n \text{ and } P\left(Z_n^{'},Z_n^{'}\right)\geq C'(Z_n^{'}/n).
\end{aligned}$$

The second part of the proof follows directly from this result because $\overline{Z}_n$ is, by definition, the highest aggregate equilibrium output. ■

Proof of Theorem 6.2. We omit this proof because is similar to the proof of Theorem 5.2. ■

Proof of Theorem 6.3.

(a) Assume that (x_n, y_n, Z_n) and (x_n', y_n', Z_n') are two equilibria that satisfy $(x_n', y_n', Z_n') > (x_n, y_n, Z_n)$, and consider the following steps:

$$\pi_n' =$$

$$= x_n' P\left(x_n' + y_n', Z_n'\right) - C\left(x_n'\right),$$
$$\geq x_n P\left(x_n + y_n', Z_n'\right) - C\left(x_n\right), \text{ by the Cournot property}$$
$$\geq x_n P\left(x_n + y_n, Z_n\right) - C\left(x_n\right) \text{ if } P\left(x_n + y_n', Z_n'\right) \geq P\left(x_n + y_n, Z_n\right)$$
$$= \pi_n$$

These inequalities imply that, if $P\left(x_n + y_n', Z_n'\right) \geq P\left(x_n + y_n, Z_n\right)$ then $n\pi_n' \geq n\pi_n$. The proof of Theorem 6.3 follows because $P\left(x_n + y_n', Z_n'\right) = \left(\frac{Z_n - Z_n'}{n} + Z_n', Z_n'\right)$ and $P\left(x_n + y_n, Z_n\right) = P\left(Z_n, Z_n\right)$.

(b) We omit this proof because it is similar to the previous one. ■

APPENDIX

The content of this appendix is based on the appendix of Amir and Lambson (2000) and the survey of Amir (2005).

In an attempt to make this paper self-contained, we provide a summary of all lattice-theoretic notions and results used here. Since this paper deals with real decision and parameter spaces, every theorem that follows is a special case of the original one.

A function F: $R_+^2 \rightarrow R$ is supermodular [submodular] if, for $x_1 \geq x_2$, $y_1 \geq y_2$

$$F\left(x_1, y_1\right) - F\left(x_2, y_1\right) \geq [\leq]\, F\left(x_1, y_2\right) - F\left(x_2, y_2\right). \tag{1.1}$$

If is twice continuously differentiable, Topki's (1978) Characterization Theorem says that supermodularity [submodularity] is equivalent to $\dfrac{\partial^2 F}{\partial x \partial y} \geq [\leq]\, 0$, for all x, y. Furthermore, $\dfrac{\partial^2 F}{\partial x \partial y} > [<]\, 0$ implies that F is strictly supermodular [submodular], the latter notion being defined by a strictly inequality in (1.1). Supermodularity is usually interpreted as a complementariety property: Having more of one variable increases the marginal returns to having more of the other variable.

F has the single-crossing property or SCP (dual SCP) in (x, y) if

$$F\left(x_1, y_1\right) - F\left(x_2, y_2\right) \geq [\leq]\, 0 \Longrightarrow F\left(x_1, y_2\right) - F\left(x_2, y_1\right) \geq [\leq]\, 0 \tag{1.2}$$

Note that [1.1] implies [1.2], while the converse is generally not true. Additionally, (1.1) is a cardinal notion while (1.2) is ordinal. Thus, the SCP is sometimes also referred to as ordinal supermodularity.

For $x \in R_+$, let $A(x) = [a_1(x), a_2(x)] \subset R_+$, with $a_1(.)$ and $a_2(.)$ being real-valued functions. $A(.)$ is ascending [descending] (in x) if a_1 and a_2 are increasing [decreasing] in x. The following results on monotone maximizers are central to our approach.

Theorem A.1. [Topkis (1978)]. Assume that (i) F is upper-semi continuous (or u.s.c.) and supermodular [submodular] in (x, y) and (ii) $A(.)$ is ascending [descending]. Then, the maximal and minimal selections of $y^*(x) \triangleq \arg\max_{y \in A(x)} F(x, y)$ are increasing [decreasing] functions. Furthermore, if F is strongly supermodular [submodular], then every selection of $y^*(.)$ is increasing [decreasing].

Theorem A.2. [Milgron and Shanon (1994)]. Assume that (i) F is u.s.c. and has the SCP [DSCP] in (x, y) and (ii) $A(.)$ is ascending [descending]. Then, the conclusion of Theorem A.1. holds.

The theorem that follows is well known as Tarski's Fixed Point Theorem.

Theorem A.3. Let $C \subset R_+$ be a compact interval, and $B : C \to C$ be an increasing function. Then B has a fixed point.

Our equilibrium comparisons are based on the following result due to Milgrom and Roberts (1990, 1994) and Sobel (1988).

Theorem A.4. Let $C \subset R_+$ be a compact interval, and $B_t : C \to C$ be an increasing function $(\forall t \geq 0)$, such that $B_t(x)$ is also increasing in t, $\forall x$. Then the minimal and maximal fixed-points of B_t are increasing in t.

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