The homework is due on Wednesday, September 7. Each questions is worth 0.8 points. No partial credits.

For the graphic arguments, use the graphing paper that is attached. *Clearly indicate the feasible set by shading it. Always graph at least three lines representing the objective, and draw an arrow in the direction in which the objective increases.*

For the computer exercises, you need to attach a printout of each Excel worksheet. In addition, you need to write down all the formulae that you use.

Question 1 Suppose an oil company has supplies of five crude products. In the refinery the crude products can be used to make two refined products x_1 and x_2, which the company can sell at a price of 10 Dollars each.

The company has a fixed supply of the crude products. In particular supplies of the crude products are 34, 26, 22, 48, and 60, respectively. For technological reasons, the company must produces at least 1 unit of good 2 for every 3.75 units of good 1. The company wants to maximize the total revenue from selling the product. As a consequence, the company solves the following optimization problem.

$$\max_{x_1,x_2} 10x_1 + 10x_2 \text{ subject to}$$
$$\text{(i) } 2x_1 + x_2 \leq 34$$
$$\text{(ii) } x_1 + 2x_2 \leq 26$$
$$\text{(iii) } x_1 + x_2 \leq 22$$
$$\text{(iv) } x_1 + 4x_2 \leq 48$$
$$\text{(v) } 3x_1 + x_2 \leq 60$$
$$\text{(vi) } 4x_1 - 15x_2 \leq 0$$
$$\text{(vii) } x_1 \geq 0$$
$$\text{(viii) } x_2 \geq 0.$$

1. Determine the optimum graphically.
2. Do there exist prices for the output such that the firm will only produce good 2?
3. Do there exist prices for the output such that the firm will only produce good 1?

Question 2 A firm wishes to produce 60 units of a product at the lowest possible cost. Two inputs are needed. The costs of the inputs are 6 and 4 Dollars, respectively. In order to produce 60 units of output, the inputs must fulfill $5x_1 + 3x_2 \geq 60$. Thus, the firm solves:

$$\begin{aligned} &\min_{x_1,x_2} 6x_1 + 4x_2 \text{ subject to} \\ &\text{(i) } 5x_1 + 3x_2 \geq 60 \\ &\text{(ii) } x_1 \geq 0 \\ &\text{(iii) } x_2 \geq 0. \end{aligned}$$

1. Determine the optimal choice of x_1 and x_2 graphically. *Note:* This is a minimization problem, i.e., moving down and to the left and your graph decreases costs.
2. Determine prices for the inputs such that you do not just get a solution that is on the corner of the feasible set. Are the prices unique?

Question 3 A utility function is given by $u(x_1, x_2) = 2x_1 + x_2$. Suppose prices are $p_1 = 3$ and $p_2 = 2$. Income is $I = 60$. The utility maximization problem is therefore given by

$$\begin{aligned} &\max_{x_1,x_2} 2x_1 + x_2 \text{ subject to} \\ &\text{(i) } 3x_1 + 2x_2 \leq 60 \\ &\text{(ii) } x_1 \geq 0 \\ &\text{(iii) } x_2 \geq 0. \end{aligned}$$

Solve the optimization problem graphically.

Question 4 A person with utility function $u(x_1, x_2) = 2x_1 + 3x_2$ wants to find the least costly consumption that provides a utility of 60 when prices are $p_1 = 4$, $p_2 = 2$. To do this, you need to solve the following minimization problem.

$$\begin{aligned} &\min_{x_1,x_2} 4x_1 + 2x_2 \text{ subject to} \\ &\text{(i) } 2x_1 + 3x_2 \geq 60 \\ &\text{(ii) } x_1 \geq 0 \\ &\text{(iii) } x_2 \geq 0. \end{aligned}$$

Solve the optimization problem graphically. How much money does the person need to afford this consumption bundle.

Question 5 A firm produces three outputs using 5 inputs which are in fixed supply. The prices of the outputs are 3, 1, and 5, respectively. The input amounts are 250, 265,

180, 200, and 300. Thus, the firms solves

$$\max_{x_1,x_2} 3x_1 + x_2 + 5x_3 \text{ subject to}$$

$$\begin{aligned}
&\text{(i)} && 2x_1 + x_2 + 4x_3 \leq 250\\
&\text{(ii)} && x_1 + 2x_2 + 2x_3 \leq 265\\
&\text{(iii)} && 5x_1 + 4x_2 + x_3 \leq 180\\
&\text{(iv)} && x_1 + 3x_2 + x_3 \leq 200\\
&\text{(v)} && 5x_1 + x_2 + 3x_3 \leq 300\\
&\text{(vi)} && x_1 \geq 0\\
&\text{(vii)} && x_2 \geq 0\\
&\text{(viii)} && x_3 \geq 0
\end{aligned}$$

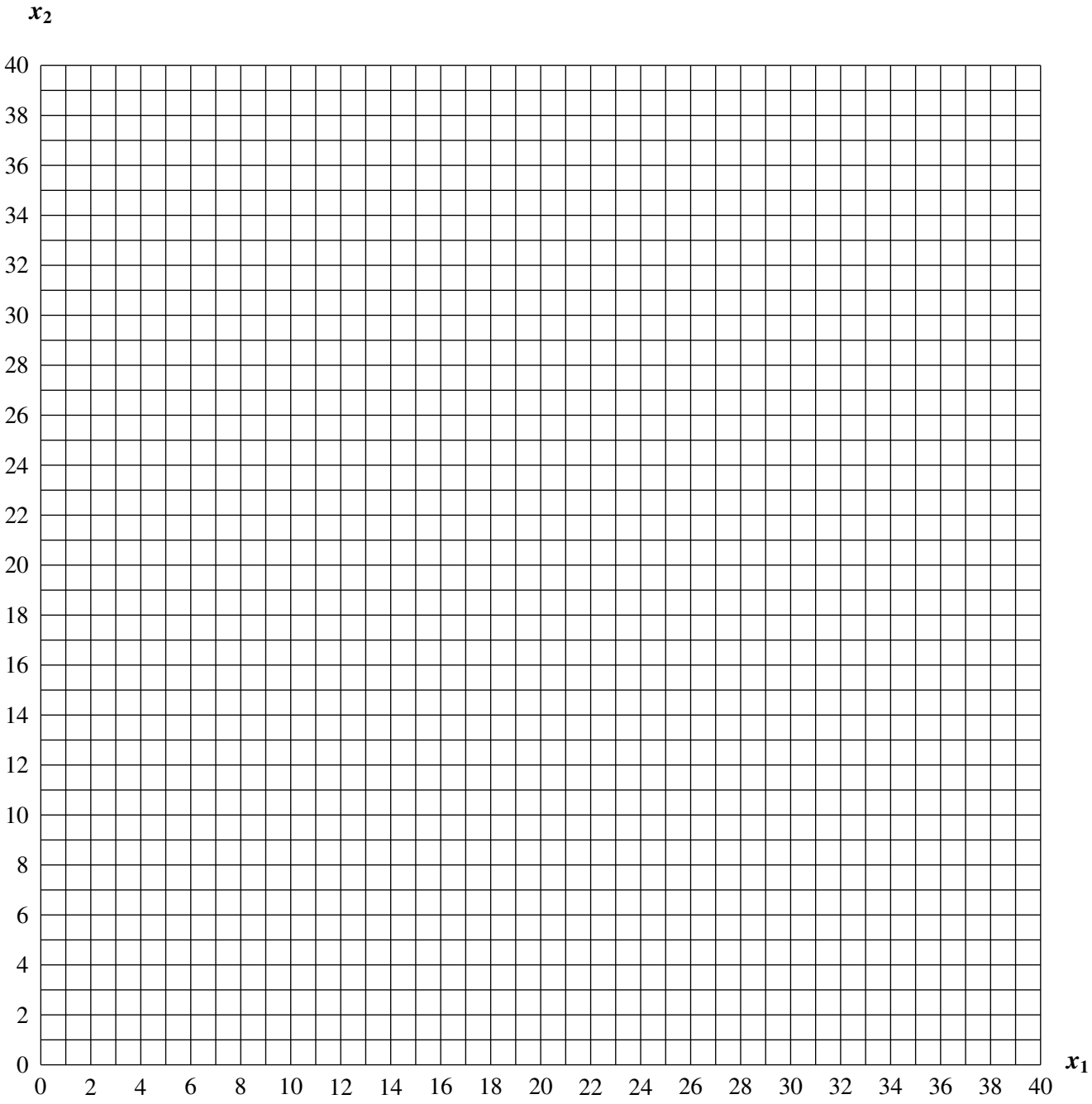
Solve the optimization problem using Excel. Determine which constraints bind and which are slack.

Question 6 A automobile parts supplier uses 10 different inputs to produce 5 outputs. The firm has already signed contracts to deliver 20 units of the first output, 30 of the second, 15 of the third, and 60 of the fourth and 20 of the fifth good. The production of outputs is linear. The prices of the inputs 12,4,3,6,11,23,8, 9, 2, and 15, respectively. Thus, the firm faces a linear programming problem, which we now specify.

$$\min_{x_1,\ldots,x_8} 12x_1 + 4x_2 + 3x_3 + 6x_4 + 11x_5 + 23x_6 + 8x_7 + 9x_8 + 2x_9 + 15x_{10} \text{ subject to}$$

$$\begin{aligned}
&\text{(i)} && 2x_1 + x_2 + 4x_3 + 5x_4 + 6x_5 + x_6 + 1.4x_7 + 2x_8 + 3x_9 + 2x_{10} \geq 20\\
&\text{(ii)} && 5x_1 + 5x_2 + 6x_3 + 4x_4 + 3x_5 + 3x_6 + 2x_7 + x_8 + x_9 + 20x_{10} \geq 30\\
&\text{(iii)} && 0.5x_1 + 2x_2 + 4x_3 + 3x_4 + 4x_5 + 7x_6 + 3x_7 + 3x_8 + 2x_9 + x_{10} \geq 15\\
&\text{(iv)} && 2.5x_1 + 2.2x_2 + 5x_3 + 6x_4 + x_5 + 3x_6 + 2x_7 + x_8 + x_9 + x_{10} \geq 60\\
&\text{(v)} && 1.5x_1 + x_2 + 3x_3 + x_4 + 0.8x_5 + 2x_6 + 6x_7 + x_8 + 3x_9 + 4x_{10} \geq 20\\
&\text{(vi)} && x_1 \geq 0, \ldots x_1 0 \geq 0.
\end{aligned}$$

Solve for the optimal values of x_i, $i = 1, \ldots, x_{10}$ by using Excel. Also determine which constraints bind and which are slack.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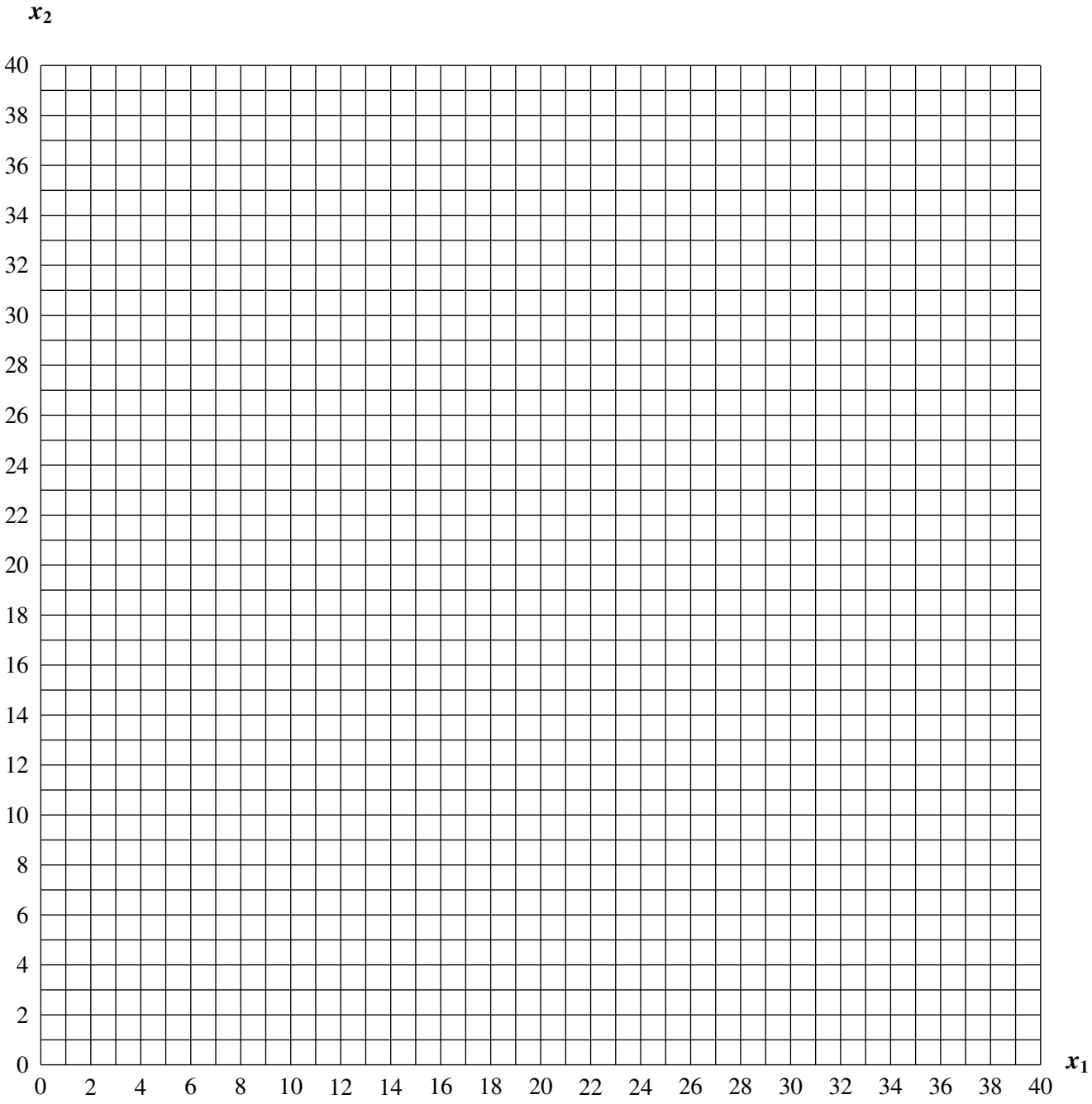
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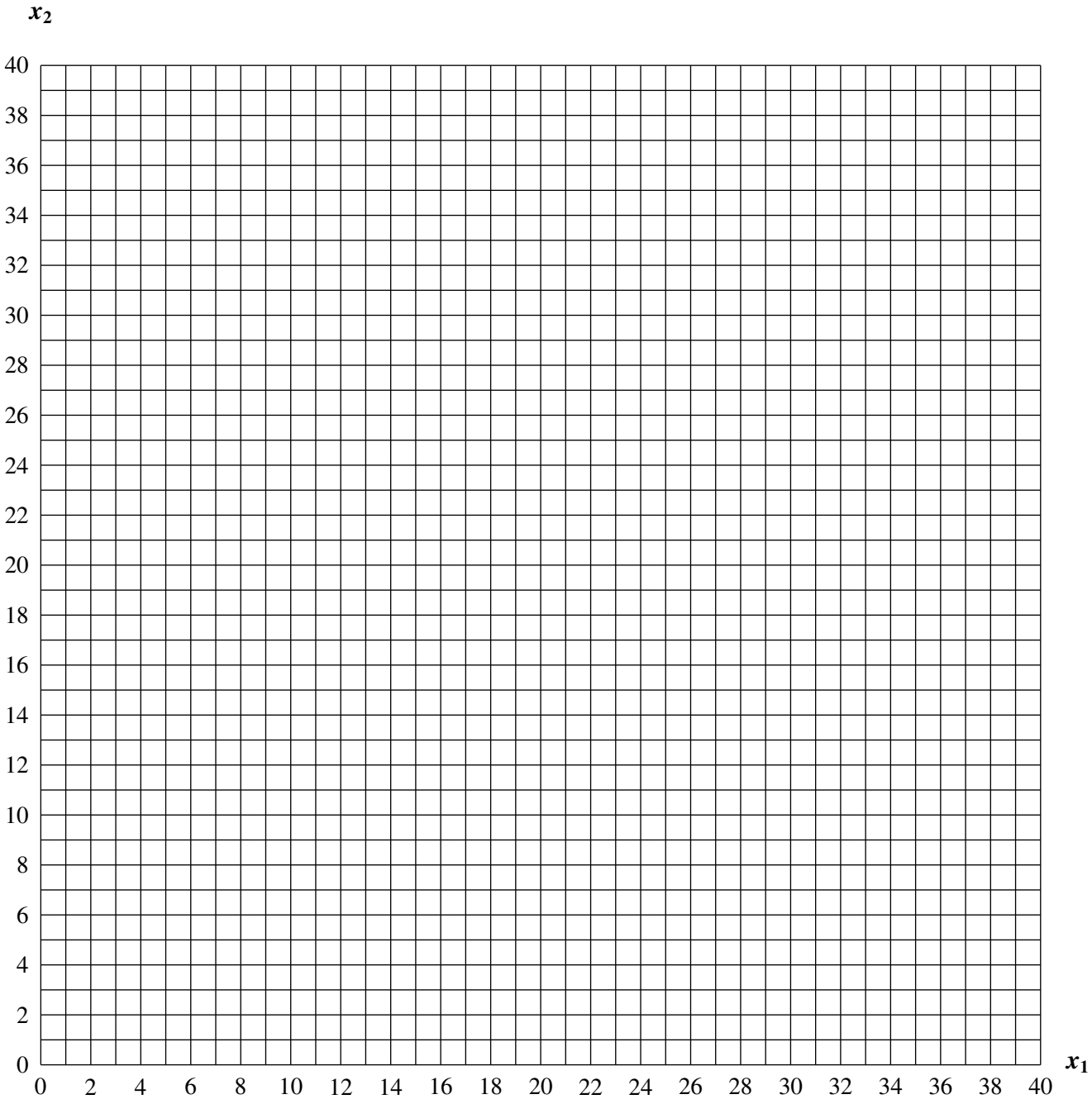
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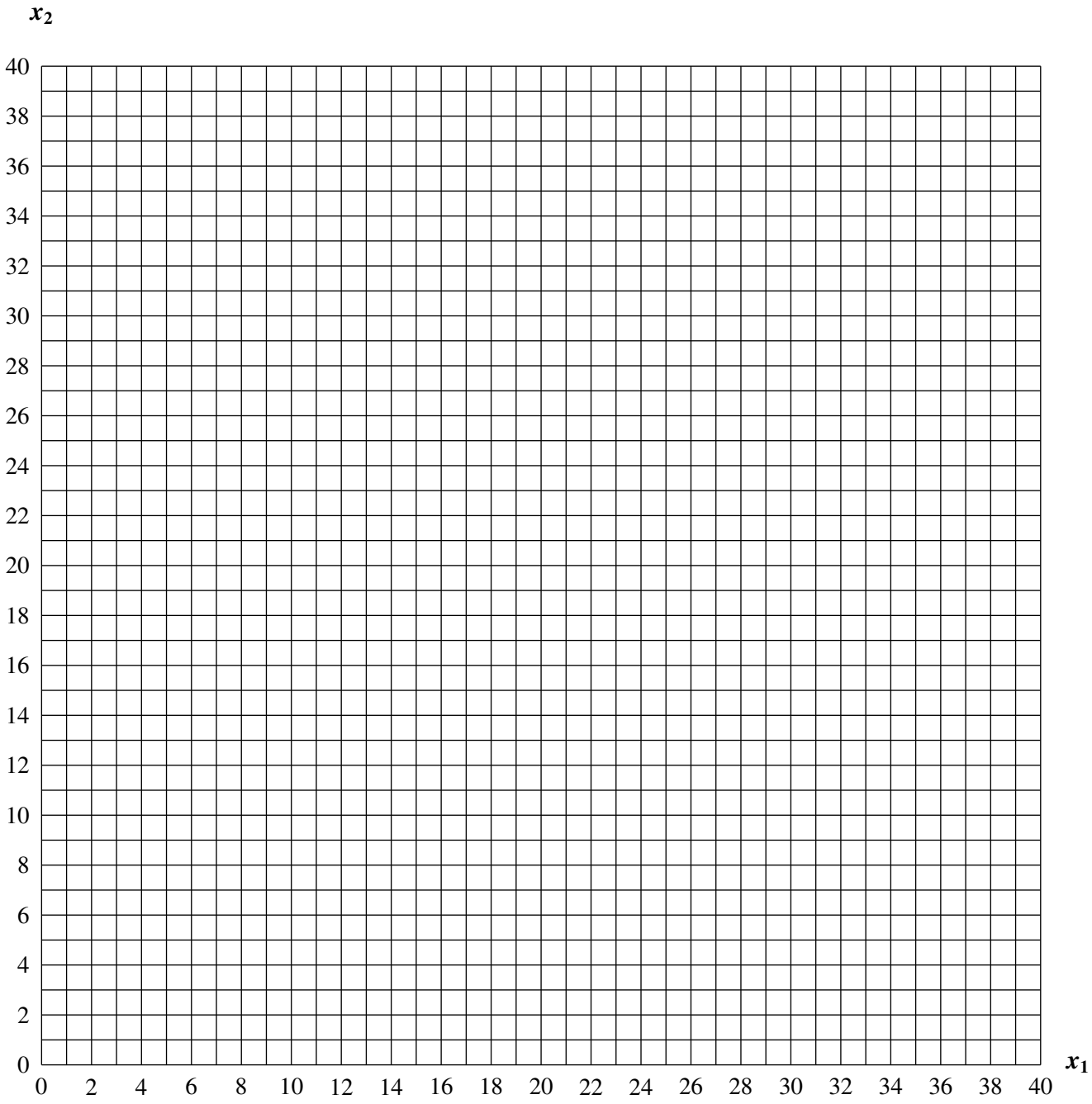
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