The homework is due on Wednesday, September 20. Each question is worth 0.8 points. No partial credits.

For the graphic arguments, use the graphing paper that is attached. For the computer exercises, you need to attach a printout of each Excel worksheet.

**Question 1** Solve the following optimization problem graphically.

\[
\text{max } x_1 + x_2 \text{ subject to } \\
\begin{align*}
(i) & \quad 2x_1 + x_2 \leq 150 \\
(ii) & \quad x_1 + 2x_2 \leq 165 \\
(iii) & \quad 5x_1 + 4x_2 \leq 405 \\
(iv) & \quad x_1 \geq 0 \\
(v) & \quad x_2 \geq 0.
\end{align*}
\]

**Question 2** Suppose an oil company has supplies of four crude products. In the refinery the crude products can be used to make two refined products \(x_1\) and \(x_2\), which the company can sell at prices 3 and 6, respectively. In order to produce \(x_1\) units of the first refined product one needs 1 unit of crude product 1, 2 of product 2, 2 of product 3, and 1 of product 4. In order to produce \(x_2\) units of the second refined product one needs 2 units of crude product 1, 1 unit of product 2, 2 of product 3, and 5 of product 4. The company has a fixed supply of the crude products. In particular supplies of the crude products are 160, 160, 190, and 375 units, respectively. The company wants to maximize the total revenue from selling the product. As a consequence, the company solves the following optimization problem.

\[
\text{max } 3x_1 + 6x_2 \text{ subject to } \\
\begin{align*}
(i) & \quad x_1 + 2x_2 \leq 160 \\
(ii) & \quad 2x_1 + x_2 \leq 160 \\
(iii) & \quad 2x_1 + 2x_2 \leq 190 \\
(iv) & \quad x_1 + 5x_2 \leq 375 \\
(v) & \quad x_1 \geq 0 \\
(vi) & \quad x_2 \geq 0.
\end{align*}
\]

Determine the optimum graphically. Note: When you graph the lines representing (i)–(iv) then one of the lines will be strictly to the right of the feasible set, i.e., the boundaries of the feasible set are determined by only three of the lines in addition to the conditions that \(x_1, x_2 \geq 0\).
**Question 3** A firm wishes to produce 10 units of a product at the lowest possible cost. Two inputs are needed. The costs of the inputs are 4 and 6 Dollars, respectively. In order to produce 10 units of output, the inputs must fulfill $4x_1 + 5x_2 \geq 300$. Thus, the firm solves:

$$\min_{x_1,x_2} 6x_1 + 4x_2 \text{ subject to}$$

(i) $4x_1 + 5x_2 \geq 300$
(ii) $x_1 \geq 0$
(iii) $x_2 \geq 0$.

Determine the optimal choice of $x_1$ and $x_2$ graphically. *Note:* This is a minimization problem, i.e., moving down and to the left and your graph decreases costs.

**Question 4** Now assume that there are six different crude oil products and 4 different refined products. The optimization problem is given by

$$\max_{x_1,x_2,x_3,x_4} 3x_1 + 6x_2 + 2x_3 + 5x_4 \text{ subject to}$$

(i) $x_1 + 2x_2 + 2x_3 + x_4 \leq 400$
(ii) $3x_1 + 4x_2 + 0.35x_3 + 3x_4 \leq 600$
(iii) $2x_1 + 2x_2 + 3x_3 + x_4 \leq 700$
(iv) $x_1 + 2x_2 + 3x_3 + x_4 \leq 420$
(v) $3x_1 + x_2 + 3.5x_3 + 2x_4 \leq 630$
(vi) $2x_1 + 2x_2 + 2x_3 + 3x_4 \leq 500$
(vii) $10x_1 + 4x_2 \leq 300$
(viii) $x_1 + x_2 + x_3 + x_4 \leq 250$
(ix) $x_1 \geq 0$
(x) $x_2 \geq 0$.
(ix) $x_3 \geq 0$.
(x) $x_4 \geq 0$.

Using Excel, determine the optimal amount of the refined product that the company should produce. Also determine which constraints bind and which constraints are slack.
Question 5 This question considers a transportation problem that is similar to that in problem 2 of our lab class. All we changed are some of the numbers and we added a fifth retail store. The transportation problem is graphed above. Again, suppose that the transportation costs are 10 cents per kilometer and let \( x_{i,j} \) denote the quantity of the product shipped from store \( i \) to store \( j \). Thus, the company solves

\[
\min 18x_{1,1} + 14x_{1,2} + 10x_{1,3} + 16x_{1,4} + 22x_{1,5} + 9x_{2,1} + 20x_{2,2} + 11x_{2,3} + 2x_{2,4} + 8x_{2,5}
\]

subject to constraints. Determine the constraints and solve for the optimal values of \( x_{i,j} \) by using Excel. Also determine which constraints bind and which are slack.

Question 6 A consumer has a utility function \( u(x_1, x_2) = x_1^3 + x_2^2 \). The prices are \( p_1 = 2 \), \( p_2 = 4 \) and the person’s wealth is 90. Thus, the consumer solves

\[
\max_{x_1, x_2} x_1^3 + x_2^2 \text{ subject to}
\]

(i) \( 2x_1 + 4x_2 \leq 90 \)

(ii) \( x_1 \geq 0 \)

(iii) \( x_2 \geq 0 \).

(a) Determine the optimal consumption, i.e., the optimal values of \( x_1 \) and \( x_2 \) by using Excel. \textit{Note:} This is a nonlinear optimization problem!

(b) Determine the percentage of wealth that the person spends on each good. In
particular, to determine this percentage you compute
\[
\frac{100 \cdot 2x_1}{90}, \text{ and } \frac{100 \cdot 4x_2}{90}.
\]

(c) Now suppose that the wealth increases to 140. This changes constraint (i) to
\[2x_1 + 4x_2 \leq 140.\] Determine again the optimal values of \(x_1\), and \(x_2\) and the percentages of income the person spends on each good. Because wealth is now 140, these percentages are given by
\[
\frac{100 \cdot 2x_1}{140}, \text{ and } \frac{100 \cdot 4x_2}{140}.
\]

(d) Now suppose that the wealth is again 140, but that the price of good 1 increases to 4. This changes constraint (i) to \(4x_1 + 4x_2 \leq 140\). Determine again the optimal values of \(x_1\), and \(x_2\) and the percentages of income the person spends on each good. Now the percentages are given by
\[
\frac{100 \cdot 4x_1}{140}, \text{ and } \frac{100 \cdot 4x_2}{140}.
\]

(e) Any guess what the percentage will be if you choose for example the budget line \(8x_1 + 2x_2 = 200\)?

**Question 7** A consumer has a utility function \(u(x_1, x_2) = 20x_1^{1/3} + x_2\). The prices are \(p_1 = 2\), \(p_2 = 4\). If the person’s wealth is \(m\) then the person solves
\[
\max_{x_1, x_2} 20x_1^{1/3} + x_2 \text{ subject to }
\]
\[(i) \ 2x_1 + 4x_2 \leq m
\]
\[(ii) \ x_1 \geq 0
\]
\[(iii) \ x_2 \geq 0.
\]

(a) Determine the optimal consumption, i.e., the optimal values of \(x_1\) and \(x_2\) by using Excel if \(m = 10\). Does the person consume both goods?

(b) Now suppose wealth is \(m = 1,000\). Does the person consume both goods now?

(c) By inserting different values for \(m\) and optimizing, try to find a value of \(m\) at which the person will just start to consume both goods.