Core Concepts in Economies where Information is Almost Complete

Stefan Krasa and Wayne Shafer*

August 17, 2000

Abstract

The paper analyzes the properties of cores with differential information, as economies converge to complete information. Two core concepts are investigated: the private core, in which agents' net trades are measurable with respect to agents' private information, and the incentive compatible core, in which coalitions of agents are restricted to incentive compatible allocations.

*We wish to thank an anonymous referee for very helpful comments.

Address of the authors: Department of Economics, University of Illinois, 1206 South Sixth Street, Champaign IL 61820.

1 Introduction

The main focus of this paper can be seen in two ways. First, is the complete information core a good predictor in environments with "almost complete" information? Second, are existing notions of a core with differential information close to the complete information core when informational asymmetries are small?

We consider two alternative core concepts with differential information that represent the main approaches in the literature. The first concept is that of the private core of Yannelis (1991). In his concept coalitions of agents are restricted to allocations that are measurable with respect to each agent's private information. The second concept that we consider is the incentive compatible core of Allen (1994), Ichiishi and Idzik (1996), and Vohra (1999). In this concept coalitions of agents are restricted to those allocations that are Bayesian incentive compatible. These two approaches differ substantially. However, because there is no clear benchmark against which the predictions of these two very different concepts can be compared when there are significant asymmetries of information, one of the main motivations of this paper is to compare them when informational asymmetries are small and when the standard (complete information) core can be used as such a benchmark.

Consider a pure exchange economy with differential information in which each agent receives a possibly noisy signal about the true state. Thus, agents' information can be specified by means of a prior over the signals and the true states. There is complete information if the prior assigns probability 1 to each agent receiving the correct signal. In order to describe what it means to be "close to complete information," we use the priors to parameterize economies. Behavior close to complete information is analyzed by considering sequences of priors that converge to the complete information prior.

Our first Theorem provides a generic result on the convergence behavior of the private core. We show that the private core does not converge to the standard complete information core for all sequences of priors, for which information is asymmetric before the limit. More precisely, we prove that generically the set of limit points of private core allocations has empty intersection with the standard (complete information) core, as the noise in the agents' signals converges to zero. Thus, the complete information core cannot be seen as an approximation of private cores of economies with almost complete information. The intuition for this result is that the private core models the difficulty of information sharing by assuming that agents base trades only on their private information. Therefore even "small" informational asymmetries lead to very different outcomes when compared to the core with complete information.

Our second and third Theorem analyze the incentive compatible core. In contrast to the private core, the incentive compatible core need not exist in general

(Allen (1994), Vohra (1999)). However, in Theorem 2 we show that it does exist close to complete information. Moreover, Theorem 2 also shows that almost every standard core allocation is the limit point of incentive compatible core allocations. Does this imply that the incentive compatible core behaves more like the standard core close to complete information? It turns out that this is not the case. Theorem 3 shows that there is a robust class of economies for which the set of limit points of incentive compatible cores is strictly larger than the standard core.

As mentioned above, the two core notions analyzed in this paper are representative of two tracks of research. Specifically, in the literature on core concepts with differential information authors either impose restrictions on how information is shared by coalitions of agents (see Wilson (1978), Yannelis (1991), Allen (1992), Berliant (1992), Koutsougeras and Yannelis (1993), and Koutsougeras (1998)), or they impose incentive compatibility restrictions on the allocations a coalition of agents can obtain (Boyd, Prescott and Smith (1988), Allen (1994), Ichiishi and Idzik (1996), Vohra (1999), Ichiishi and Sertel (1998)).

In addition to the private core, the first group of papers also investigates other core concepts, most notably the coarse core and the fine core. In the coarse core, coalitions of agent are restricted to trades that are measurable with respect to common knowledge information. In contrast, in the fine core a coalition can use the pooled information of its members (c.f., Yannelis (1991) or Koutsougeras and Yannelis (1993)). In this paper, we investigate the private core because it has been shown to have desirable properties (c.f., Koutsougeras and Yannelis (1993)). That is, the private core exists in general, it takes informational asymmetries into account, and it is incentive compatible.[1] Moreover, our main result for the private core implies that the same result holds for the coarse core.

In the second group of papers, where authors impose incentive compatibility restrictions, the concepts differ with respect to the participation constrained used, i.e., whether the blocking notion is ex-ante or interim. In this paper we use the ex-ante notion, because it avoids information leakage problems that arise when coalitions can block in the interim period (c.f., Krasa (1999)).

2 The Model

Consider an exchange economy with n agents, indexed by $i \in I = \{1, \ldots, n\}$. There is uncertainty over the state of nature $\omega \in \Omega$, where Ω is finite. Each agent i receives a possibly noisy signal $\phi_i \in \Phi_i$ about ω. For simplicity assume that $\Phi_i = \Omega$ for all

[1] Krasa and Yannelis (1994) show that if the grand coalition cannot block, then coalitional incentive compatible notions are fulfilled. These incentive notions are stronger than Bayesian incentive compatibility, and therefore imply Bayesian incentive compatibility.

agents i. Let $\Phi = \prod_{i=1}^{n} \Phi_i$. Any $\phi = (\phi_1, \dots, \phi_n) \in \Phi$ will be also denoted by (ϕ_{-i}, ϕ_i). Let π be a probability on $\Omega \times \Phi$ which is the common prior of all agents over states and signals. Let μ be the marginal probability on Ω.

Assume there are ℓ goods, and let $X_i = \mathbb{R}_+^\ell$ be the consumption space of agent i. Each agent i's preference ordering is given by a state dependent von Neumann-Morgenstern utility function $u_i\colon \Omega \times X_i \to \mathbb{R}$. Note that an agent's utility depends directly only on consumption and the true state ω. Consumption itself, however, will depend on the signals. A consumption bundle for agent i is therefore given by $x_i\colon \Omega \times \Phi \to \mathbb{R}_+^\ell$. An allocation x is a collection of consumption bundles x_i, $i \in I$ for all agents. Agent i's ex-ante expected utility is then given by

$$V_i(x_i) = \int_{\Omega \times \Phi} u_i\big(\omega, x_i(\omega, \phi)\big)\, d\pi(\omega, \phi).$$

Agent i's endowment is given by $e_i\colon \Omega \times \Phi \to \mathbb{R}_+^\ell$. We assume that the endowment e_i only depends on the true state ω. Thus, with a slight abuse of notation, we will often write $e_i(\omega)$ to denote agent i's endowment in state ω.

In a complete information economy, each agent i observes the true state ω. Thus, the signal ϕ is given by $\phi = \delta(\omega) = (\omega, \dots, \omega)$. Let $\Delta = \{(\omega, \delta(\omega)) | \omega \in \Omega\}$. Then $\pi(\Delta) = 1$ in a complete information economy, and only the consumption in $(\omega, \delta(\omega))$ matters. As a consequence, for complete information economies we will often denote agent i's consumption in $(\omega, \delta(\omega))$ by $x_i(\omega)$.

Finally, we describe our notion of convergence of allocations of the incomplete information economies to an allocation in a complete information economy.

For each $k \in \mathbb{N}$, let x_i^k, $i \in I$ be an allocation of the incomplete information economy with prior π_k such that $\lim_{k\to\infty} \pi_k(\Delta) = 1$. Then x_i^k, $i \in I$ converges to x_i, $i \in I$ if $\lim_{k\to\infty} x_i^k(\omega, \delta(\omega)) = x_i(\omega, \delta(\omega))$ for all $\omega \in \Omega$ and all agents i.

3 The Core Concepts

3.1 Complete Information Economies

Consider an economy, in which the signals perfectly reveal the true state. Thus, there is uncertainty at $t = 0$ about the state ω, but ω becomes known to all agents at $t = 1$. As mentioned above, a consumption bundle can then be written as a function of ω alone, i.e., $x_i\colon \Omega \to \mathbb{R}_+^\ell$. Agent i's ex-ante expected utility is then given by $V_i(x_i) = \int_\Omega u_i(\omega, x_i(\omega))\, d\mu(\omega)$. Thus, we can use the standard definition of the core of an exchange economy.

Definition 1 *An allocation x is in the **core** of the complete information economy if and only if*

(i) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, *μ-a.e. (feasibility);*

(ii) *The following does not hold:*

There exists a coalition $S \subset I$ *and* $y_i\colon \Omega \to \mathbb{R}^{\ell}_{+}$, $i \in S$ *with*

(ii.i) $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, *μ-a.e.;*

(ii.ii) $V_i(y_i) \geq V_i(x_i)$ *for all* $i \in S$*, where at least one inequality is strict.*

An allocation x *is a* ***strict core*** *allocation if* x *is a core allocation and if the same utilities cannot be obtained by any strict subcoalition, i.e., there do not exist* $S \subsetneqq I$ *and* $y_i\colon \Omega \to \mathbb{R}^{\ell}_{+}$ *with* $\sum_{i \in S} y_i = \sum_{i \in S} e_i$*, μ-a.e., and* $V_i(y_i) = V_i(x_i)$ *for all* $i \in S$*.*

For example, it is easy to show that if there exists a competitive equilibrium (x, p) with the property that $\sum_{i \in S} x_i \neq \sum_{i \in S} e_i$ for all coalitions $S \subsetneqq I$ then x is also a strict core allocation. Note that under the above assumption, p is no longer a competitive equilibrium price vector if the economy is decomposed into two parts. Thus, the existence of a strict core allocation (which we require in Theorem 2 below) can be viewed as an indecomposibility assumption on the economy.

3.2 Economies with Differential Information

If the signals are noisy, then agents are differentially informed. We provide two different core notions for differential information economies.

3.2.1 Definition of the Private Core

In the private core of Yannelis (1991), each agent i is restricted to consumption bundles that are measurable with respect to his private information F_i. We first provide the definition of the private core, and then describe in (1) how F_i is derived from the signal ϕ_i and the observed endowment realization $e_i(\omega)$. Also note that consumption bundles and endowments are now written as functions of ω and ϕ.

Definition 2 *An allocation* x *is in the* ***private core*** *of the differential information economy if and only if*

(i) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, *π-a.e. (feasibility);*

(ii) x_i *is* F_i*-measurable for all agents* i*.*

(iii) *The following does not hold:*

There exist a coalition $S \subset I$ *and* $y_i\colon \Omega \times \Phi \to \mathbb{R}^{\ell}_{+}$, $i \in S$ *with*

(iii.i) y^i *is* F_i*-measurable for all* $i \in S$*;*

(iii.ii) $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, π-*a.e.;*

(iii.iii) $V_i(y_i) \geq V_i(x_i)$ *for all* $i \in S$*, where at least one inequality is strict.*

Note that the main difference between the private core and the complete information core is the measurability restriction imposed on the core allocation and on any blocking allocation. Yannelis (1991) provides a very general existence result for the private core.

It now remains to describe how each agent i's private information F_i is derived. Let $\sigma(e_i)$ be the information generated by e_i, which can be interpreted as a partition of $\Omega \times \Phi$.[2] In addition, agent i knows the signal ϕ_i. Thus,

$$F_i = \sigma(e_i) \vee \left\{ \Omega \times \Phi_{-i} \times \{\phi_i\} \;\middle|\; \phi_i \in \Phi_i \right\}. \tag{1}$$

For example, consider the case where all agents' signals are accurate. We now show that any complete information core allocation x corresponds to a private core allocation $\hat{x}$. Define $\hat{x}_i(\omega, \phi_{-i}, \phi_i) = x_i(\phi_i)$, where (with a slight abuse of notation) $x_i(\phi_i)$ corresponds to agent i's consumption in the complete information core allocation if the state is $\omega = \phi_i$. Then each $\hat{x}_i$ is F_i measurable. Because the signal is accurate, $\sum_{i\in I} \hat{x}_i = \sum_{i\in I} e_i$, π-a.e., i.e., the allocation is feasible. Finally, $\hat{x}$ cannot be dominated by another F_i-measurable allocation for any coalition S. Thus, $\hat{x}$ is in the private core.

3.2.2 Definition of the Incentive Compatible Core

We first provide the standard definition of incentive compatibility.

Definition 3 *A consumption bundle* x_i *is incentive compatible for agent* i *if and only if*

$$\int_{\Omega\times\Phi_{-i}} u_i(\omega, x_i(\omega,\phi))\, d\pi\big(\omega, \phi_{-i} \;\big|\; \phi_i, e_i\big) \geq \int_{\Omega\times\Phi_{-i}} u_i(\omega, x_i(\omega, \phi_{-i}, \phi_i'))\, d\pi\big(\omega, \phi_{-i} \;\big|\; \phi_i, e_i\big),$$

for all $\phi_i, \phi_i' \in \Phi_i$.

We now provide the definition of the incentive compatible core. The main difference between this core and the core of a complete information economy is that the core allocation itself and any allocation used by a blocking coalition are required to be incentive compatible. The trades of members of coalition S must be at least measurable with respect to the pooled information of all of its members. Otherwise, the coalition could not execute the trades. Again, F_i is given by (1).

[2] Thus, $\sigma(e_i)$ is the information generated by the sets of the form $\{\omega | e_i(\omega) = \bar{e}_i\} \times \Phi$, where $\bar{e}_i \in \mathbb{R}^{\ell}_{+}$.

Definition 4 *An allocation x is in the* ***incentive compatible core*** *of the differential information economy if and only if*

(i) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, *$\pi$-a.e. (feasibility);*

(ii) *x_i is incentive compatible and $\bigvee_{i \in I} F_i$-measurable for all agents $i \in I$.*

(iii) *The following does not hold:*

There exist a coalition $S \subset I$ and $y_i \colon \Omega \times \Phi \to \mathbb{R}^{\ell}_{+}$, $i \in S$ with

(iii.i) *y^i is incentive compatible and measurable with respect to $\bigvee_{i \in S} F_i$ for all $i \in S$;*

(iii.ii) $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, *$\pi$-a.e.;*

(iii.iii) *$V_i(y_i) \geq V_i(x_i)$ for all $i \in S$, where at least one inequality is strict.*

As indicated in Allen (1994) and Vohra (1999), the incentive compatible core does not exist in general. However, in Theorem 2 we show that it exists for economies that are sufficiently close to a complete information economy.

4 The Convergence Results

In this section we analyze whether or not the private core and the incentive compatible core are close to the complete information core if the economy is close to a complete information economy. Our economies can be parameterized by endowments, preferences, and priors. When we characterize properties of the core with "almost" complete information, we fix endowments and preferences. An economy is then close to complete information if $\pi(\Delta)$ is close to 1, i.e., if the probability that all agents receive the correct signal is close to 1. In all of our Theorems we therefore consider sequences of priors $\lim_{k\to\infty} \pi_k = \pi$, where $\pi(\Delta) = 1$, and investigate whether or not the limit points of sequences of incomplete information core allocations coincide with the complete information core. The Theorems investigate for what type of sequences π_k, $k \in \mathbb{N}$ convergence can be obtained.

4.1 Convergence of the Private Core

We now show that private core allocations of economies with almost complete information will in general differ substantially from complete information core allocations. In particular, Theorem 1 below shows that generically the set of limit points of private core allocations has an empty intersection with the complete information core. We first illustrate the main intuition of Theorem 1 by means of an example.

Example 1 Assume there are two agents $i = 1,2$ and three states ω_i, $i = 1,2,3$, each of which occurs with positive probability. There is one good in each state. The agents' preferences are given by

$$u_1(\omega, x) = \begin{cases} \sqrt{x} & \text{if } \omega = \omega_1; \\ x & \text{otherwise;} \end{cases} \quad u_2(\omega, x) = x, \text{ for all } \omega.$$

Each agent has a state independent endowment of 1 unit of the consumption good.

Assume that agent 1 can perfectly observe ω_1, but that his signals about states ω_2 and ω_3 are incorrect with probability $\varepsilon > 0$. In contrast, agent 2 correctly observes state 3, but his signals about states ω_1 and ω_2 are also incorrect with probability ε. Because the endowments are state independent, agent i's consumption in the private core can only depend on ϕ_i, and can therefore be denoted by $x_i(\phi_i)$. Feasibility requires that $x_1(\phi_1) + x_2(\phi_2) = e_1 + e_2$ for π_k a.e. $\phi = (\phi_1, \phi_2)$. Given the noise in the signals described above, all $\phi \in \Phi$ occur with positive probability. Thus, feasibility implies that x_1 and x_2 are independent of the signals. Hence, the private core consists only of the agents' endowments.

In contrast, it easy to see that in a complete information core allocation, trade will always occur. Agent 1 will give up a strictly positive quantity of the good in state ω_1 in exchange for an increased consumption in states ω_2, ω_3.

The result illustrated in Example 1 will hold for a generic economy, with genericity over agents' preferences. It is easy to see that we can only get a generic result. For example, consider an economy, in which the endowment is Pareto efficient in the complete information economy. Then no trade is also the only private core allocation, and both core notions will therefore coincide.

In order to provide a generic result, we parameterize each agent i's utility function in state $\omega \in \Omega$ by $\theta_i(\omega) \in \Theta_{i,\omega}$ where $\Theta_{i,\omega}$ is an open subset of $\mathbb{R}^\ell$. Agent i's utility is therefore given by $u^i(\omega, x, \theta_i(\omega))$. As we allow agent specific perturbations of utility functions in different states ω, the entire parameter space, Θ, has dimension $\ell|\Omega|n$. We say that a result holds for a generic set of economies, if there exists a set $\tilde{\Theta}$ which is closed in Θ and has Lebesgue measure 0, such that the result holds for all economies except possibly those in $\tilde{\Theta}$.

In the following, let $\theta \in \Theta_{i,\omega}$, and $x \in \mathbb{R}^\ell_{++}$. For the genericity argument, the following standard assumptions must be fulfilled.

Assumption A1

(1) Each $u_i(\omega, x, \theta)$ is smooth, has strictly positive first derivatives with respect to x, and has a negative definite matrix of second derivatives $D^2_{xx} u_i(\omega, x, \theta)$.

(2) $D^2_{x\theta}u_i(\omega, x, \theta)$ is non-singular.

(3) For all $\omega \in \Omega$, and for all sequence x^k, $k \in \mathbb{N}$, with $x^k \in \mathbb{R}^{\ell}_{++}$ and $\lim_{k\to\infty} x^k_l = 0$ for some good l, it follows that $\lim_{k\to\infty} \|D_x u_i(\omega, x^k, \theta)\| = \infty$.

(4) Each agent's endowment is strictly positive.

The main result of this section, Theorem 1, shows that the private core does not converge to the complete information core as long as for each agent *i*, one of the signals is noisy in a state that agent *i* does not learn about from the endowment realization.

Theorem 1 *Assume that the economy fulfills Assumption A1 and that there are at least two goods in each state. Then for a generic set of economies the following holds:*

Let $\pi_k \to \pi$ be an arbitrary sequence of priors that fulfills

1. $\pi(\Delta) = 1$;
2. *for every $k \in \mathbb{N}$ and for each agent i there exist states ω_i, ω'_i with $e^i(\omega_i) = e^i(\omega'_i)$ and $\pi_k(\phi_i = \omega'_i \mid \omega_i) > 0$.*

Let E_k be the economy with prior π_k. For each $k \in \mathbb{N}$, let x^k be a private core allocation of E_k. Then none of the limit points of x^k, $k \in \mathbb{N}$ is a complete information core allocation.

Before proving the Theorem, we need Lemma 1 below. Lemma 1 shows that generically Pareto efficient allocations of the complete information economy will provide agents a different level of consumption in different states. Because there are *n* agents, we add more than *n* independent restrictions on Pareto efficient allocations if we require each agent *i*'s consumption to be the same in two different states for all goods. Lemma 1 therefore follows from the fact that the Pareto set itself has only dimension $n-1$. The proof of Lemma 1 is in the Appendix.

Lemma 1 *For all agents i, let $\omega_i \neq \omega'_i$. Let P_θ be the set of all Pareto efficient, (ex-ante) individually rational allocations with $x_i(\omega_i) = x_i(\omega'_i)$, for all agents i. Then $P_\theta = \emptyset$ for generic θ.*

We now prove Theorem 1.

Proof of Theorem 1. Let $\omega_i \neq \omega'_i$, $i \in I$ be arbitrary. Then Lemma 1 implies that there exists a generic set of economies such that no Pareto efficient allocation fulfills $x_i(\omega_i) = x_i(\omega'_i)$. We next show that the limit of private core allocations must always fulfill such restrictions.

Let x^k be a private core allocation for the economy with prior π_k. Then because each agent i has a noisy signal, there exist $\phi_i' = \omega_i' \neq \omega_i$, such that $\pi_k(\phi_i'|\omega_i) > 0$. Then

$$x_i^k(\omega_i, \delta_{-i}(\omega_i), \phi_i') = x_i^k(\omega_i', \delta_{-i}(\omega_i'), \phi_i'), \tag{2}$$

because agent i's consumption must be measurable with respect to his information F_i (he can neither distinguish the states from observing his signal, nor from learning the endowment realization). Now note that in state ω_i, all agents other than agent i cannot determine whether agent i received signal ϕ_i or signal ϕ_i', because the signal is private information to agent i. Thus $x_j^k(\omega_i, \delta(\omega_i)) = x_j^k(\omega_i, \delta_{-i}(\omega_i), \phi_i')$ in all private core allocations for all agents $j \neq i$. Because the aggregate endowment only depends on ω and not on the signals, and because both ϕ_i and ϕ_i' can occur with positive probability in state ω_i, feasibility implies

$$x_i^k(\omega_i, \delta(\omega_i)) = x_i^k(\omega_i, \delta_{-i}(\omega_i), \phi_i'). \tag{3}$$

Thus, (2) and (3), and the fact that $\omega_i' = \phi_i'$ imply

$$x_i^k(\omega_i, \delta(\omega_i)) = x_i^k(\omega_i', \delta(\omega_i')). \tag{4}$$

Now consider a limit point $x(\omega)$ of the sequence $x^k(\omega, \delta(\omega))$, $k \in \mathbb{N}$ of private core allocations. We can assume without loss of generality that (4) holds for the same states ω_i, ω_i', for all elements in the subsequence of x^k, $k \in \mathbb{N}$ that converges to x. Therefore $x_i(\omega_i, \delta(\omega_i)) = x_i(\omega_i', \delta(\omega_i'))$. Thus, for a generic economy the limit points of private core allocations are not Pareto efficient in the complete information economy, and therefore not in the complete information core. This proves the Theorem. ■

4.2 Convergence of the Incentive Compatible Core

We now investigate the convergence of the incentive compatible core. First, we require that there are at least three agents. If there are only 2 agents, we cannot expect to get convergence as Example 2 indicates.

Example 2 Assume there are two agents $i = 1,2$ and two states ω_1, ω_2. Each state occurs with probability $1/2$. There is one good in each state. Agents' preferences are given by

$$u_1(\omega, x) = \begin{cases} \sqrt{x} & \text{if } \omega = \omega_1; \\ x & \text{if } \omega = \omega_2; \end{cases} \qquad u_2(\omega, x) = \begin{cases} x & \text{if } \omega = \omega_1; \\ \sqrt{x} & \text{if } \omega = \omega_2. \end{cases}$$

Each agent has a state independent endowment of one unit of the consumption good.

First, consider the complete information economy, where each of the agents learns the state ω when it is realized. It is easy to see that any ex-ante individually rational and feasible allocation x must fulfill $x_1(\omega_1) \leq 1$; $x_1(\omega_2) \geq 1$; $x_2(\omega_1) \geq 1$; and $x_2(\omega_2) \leq 1$. Moreover, at least one of the inequalities must be strict for complete information core allocations. We now show that no such allocation is the limit of a sequence of incentive compatible core allocations.

Assume that each agent receives a noisy signal about ω. The signal is correct with probability $1-\varepsilon$, and incorrect with probability ε. Let x^ε be an incentive compatible core allocations for the economy with noise ε. Without loss of generality assume that $x^\varepsilon(\omega, \delta(\omega))$ converges. We denote the limit by $x(\omega)$.

Because agents' endowments are state independent, we can write their consumption as a function of the reported signals only. Thus, $x_i^\varepsilon(\omega, \omega')$ denotes agent i's consumption if agent 1 reports signal $\phi_1 = \omega$ and agent 2 reports signal $\phi_2 = \omega'$. Incentive compatibility of x^ε implies

$$E_{\Omega\times\Phi_2}\Big(u_1(\cdot, x_1^\varepsilon(\omega_1, \cdot)) \;\Big|\; \phi_1 = \omega_1\Big) \geq E_{\Omega\times\Phi_2}\Big(u_1(\cdot, x_1^\varepsilon(\omega_2, \cdot)) \;\Big|\; \phi_1 = \omega_1\Big) \qquad (5)$$

$$E_{\Omega\times\Phi_1}\Big(u_2(\cdot, x_2^\varepsilon(\cdot, \omega_2)) \;\Big|\; \phi_2 = \omega_2\Big) \geq E_{\Omega\times\Phi_1}\Big(u_2(\cdot, x_2^\varepsilon(\cdot, \omega_1)) \;\Big|\; \phi_2 = \omega_2\Big) \qquad (6)$$

where (5) is the incentive constraint for agent 1 if he observes $\phi_1 = \omega_1$, and (6) is the incentive constraint for agent 2 if he observes $\phi_2 = \omega_2$.[3]

If we take the limit on both sides of (5) and (6) for $\varepsilon \to 0$ we get $\sqrt{x_1(\omega_1)} \geq \limsup_{\varepsilon\to 0} \sqrt{x_1^\varepsilon(\omega_2, \omega_1)}$ and $\sqrt{x_2(\omega_2)} \geq \limsup_{\varepsilon\to 0} \sqrt{x_2^\varepsilon(\omega_2, \omega_1)}$, which implies

$$x_1(\omega_1) + x_2(\omega_2) \geq \limsup_{\varepsilon\to 0} x_1^\varepsilon(\omega_2, \omega_1) + x_2^\varepsilon(\omega_2, \omega_1). \qquad (7)$$

Now assume by way of contradiction that x is a complete information core allocation. Then as noted above $x_1(\omega_1) \leq 1$ and $x_2(\omega_2) \leq 1$, where at least one inequality is strict. Thus, (7) implies $x_1^\varepsilon(\omega_2, \omega_1) + x_2^\varepsilon(\omega_2, \omega_1) < 2$ for sufficiently small ε, a contradiction to feasibility.[4] Therefore any limit of incentive compatible core allocations is not in the complete information core.

Example 2 demonstrates that if there are only two agents, incentive compatible core allocations do not necessarily converge to complete information core allocations. The reason for this result is that incentive compatibility would require that

[3] Note that the expectation operator itself depends on the prior over $\Omega \times \Phi$, and therefore depends on ε.

[4] Note that throughout this paper we assume that there is not free disposal.

both agents are penalized by a low level of consumption when reports $\phi_1 = \omega_2$ and $\phi_2 = \omega_1$ are made. If there are more than two agents, penalties can be executed by transferring the consumption good to other agents. In fact, Theorem 2 below shows that with three or more agents convergence can be obtained. That is, we show that almost every complete information core allocation is the limit of incentive compatible core allocations. In particular, this result also implies existence of incentive compatible core allocations close to complete information (see the first statement in Theorem 2). This is a useful result, because as mentioned earlier, the incentive compatible core may be empty. In an interesting recent paper, McLean and Postlewaite (2000) provide alternative conditions under which such core allocations exist.

In Theorem 2 we use the following regularity assumptions.

Assumption A2

(1) Each $u_i(\omega, x)$ is smooth, has strictly positive first derivatives with respect to $x \in \mathbb{R}^{\ell}_{++}$, and has a negative definite matrix of second derivatives $D^2_{xx} u_i(\omega, x)$.

(2) For all $\omega \in \Omega$, and for all sequence x^k, $k \in \mathbb{N}$, with $x^k \in \mathbb{R}^{\ell}_{++}$ and $\lim_{k \to \infty} x^k_l = 0$ for some good l, it follows that $\lim_{k \to \infty} \| D_x u_i(\omega, x^k) \| = \infty$.

(3) Each agent's endowment is strictly positive.

Theorem 2 *Consider an economy where*

(i) $|I| \geq 3$, *(i.e., at least three agents);*

(ii) π *is a prior over* $\Omega \times S$ *with* $\pi(\Delta) = 1$ *(i.e., signals are not noisy under* π*);*

(iii) *assumption A2 holds;*

(iv) *there exists a strict core allocation* x *in the complete information economy.*

Let C *be the set of core allocations of a complete information economy. Then there exists a closed set* N *of lower dimension than* C*, such that for all sequences of priors* π_k, $k \in \mathbb{N}$ *that converge to a complete information prior* π:

1. *Incentive compatible core allocations exist in the economy with prior* π_k *for all sufficiently large* k.

2. *Every core allocation* $x \in C \setminus N$ *is the limit of a sequence of incentive compatible core allocations of the incomplete information economies with priors* π_k *(starting at a sufficiently large* k*).*

The proof of Theorem 2 is in the Appendix. We now explain the intuition.

Lemma 3 below demonstrates that any allocation of the complete information economy is the limit of incentive compatible allocations. Thus, in order to prove Theorem 2, one must show that the approximating sequence can be chosen to be in the incentive compatible core.

The existence of a strict core allocation ensures that the core has full dimension (i.e., dimension $n-1$, where n is the number of agents). Using Lemma 2 below, one can then show that all complete information core allocations except those in a set N of lower dimension than $n-1$ can be approximated by strict core allocations. Thus, it is sufficient to prove the result for all strict core allocations x.

The proof proceeds by way of contradiction. Assume we have a sequence of incentive compatible allocations x^k, $k \in \mathbb{N}$ that converges to a core allocation x of the complete information economy, but that x^k is not in the incentive compatible core for all k. One can show that each x^k can be selected such that the grand coalition cannot improve upon x^k by choosing another incentive compatible allocation. Thus, if x^k is not in the incentive compatible core, there must exist a coalition $S \subsetneqq I$, which can block it. Taking the limit as $k \to \infty$ implies that there exists a coalition $S \subsetneqq I$ which can obtain for its members the same utilities as in the strict core allocation x, a contradiction that proves the Theorem.

Finally, we state Lemma 2 and Lemma 3. The proofs are in the Appendix.

In the following let $U(S)$ be the set of attainable utilities of coalition S. Thus, $U(S) = \left\{ w \in \mathbb{R}^n \;\middle|\; \text{there exists } x \text{ with } \sum_{i \in S} x_i = \sum_{i \in S} e_i \text{ such that } w_i \leq V_i(x_i), \text{ for all } i \in S \right\}$. Let $\operatorname{bd} U(S)$ be the boundary of this set.

Lemma 2 *Assume that A2 holds. Then* $\operatorname{bd} U(S) \cap \operatorname{bd} U(T)$ *has dimension* $n-2$ *for all coalitions* $S \neq T$ *with* $\emptyset \subsetneqq S, T \subsetneqq I$.

Lemma 3 *Assume that:*

1. *There are at least three agents;*
2. x *is an allocation with* $\sum_{i \in I} x_i(\omega, \delta(\omega)) = \sum_{i \in I} e_i(\omega)$ *(feasibility if information is complete);*
3. $x_i(\omega, \delta(\omega)) \in \mathbb{R}^{\ell}_{++}$*, for all* $i \in I$*,* $\omega \in \Omega$*;*
4. π_k*,* $k \in \mathbb{N}$ *is an arbitrary sequence of priors with* $\pi_k \to \pi$*, and* $\pi(\Delta) = 1$.

Then there exists a sequence x^k*,* $k \in \mathbb{N}$*, with* $\lim_{k \to \infty} \int u_i(\cdot, x_i^k(\cdot))\, d\pi_k = V_i(x_i)$ *for all* $i \in I$*, where each* x^k *is a Bayesian incentive compatible allocation for the economy with prior* π_k.

Let $\tilde{C}$ be the set of limit points of all sequence of incentive compatible core allocations, for a given sequence of priors π_k, $k \in \mathbb{N}$. Let C denote the set of core allocations of the complete information economy. Theorem 2 shows that $\tilde{C}$ contains C, except possibly for a negligible set. Are there cases where $\tilde{C}$ is strictly larger than C? Theorem 3 below shows that this is the case. The intuition for this result is as follows.

In the incentive compatible core, blocking can be difficult for two agent coalitions. We have already pointed this difficulty in Example 2. Thus, in order to find economies where $\tilde{C}$ is strictly larger than C, it is sufficient to construct economies in which blocking by two agent coalitions matters. Apart from constructing an economy that has these required properties, Theorem 3 uses an argument similar to that of Theorem 2 to show that allocations that can only be blocked by a particular two agent coalition are limit points of incentive compatible core allocations.

Finally, we state Theorem 3. The proof is in the Appendix.

Theorem 3 *Let $|\Omega| \geq 4$ and $|I| \geq 3$. There exist economies that fulfill all conditions of Theorem 2, but for which the set of limit points of incentive compatible core allocations $\tilde{C}$ is strictly larger than the complete information core C. The set $\tilde{C} \setminus C$ is not negligible, and the economies are robust with respect to perturbations of endowments and preferences.*

5 Appendix

Proof of Lemma 1. In this proof, let $\Omega = \{\omega_1, \dots, \omega_{|\Omega|}\}$. For every agent i let $\omega_{k_i}, \omega_{k_i'}$ be the two states in which consumption should be the same. We now define agent i's expected utility by $V_i(x_i, \theta_i) = \sum_\omega \mu(\omega) u_i(\omega, x_i(\omega), \theta_i(\omega))$. Then let $\hat{P}_\theta$ be the set of all $(x_1, \dots, x_n, p, \lambda_2, \dots, \lambda_n)$ which solve

(E1) $D_{x_1} V_1(x_1, \theta_1) - p = 0$;

(E2) $D_{x_i} V_i(x_i, \theta_i) - \lambda_i p = 0$, $i = 2, \dots, n$;

(E3) $e - \sum_{i=1}^n x_i = 0$;

(E4) $x_{i1}(\omega_{k_i}) - x_{i1}(\omega_{k_i'}) = 0$, for $i < n$; and $x_{n2}(\omega_{k_n}) - x_{n2}(\omega_{k_n'}) = 0$.

Clearly, $\hat{P}_\theta$ is homeomorphic to the set of all Pareto efficient allocations for which (E4) holds, a set that contains P_θ. Therefore, it is sufficient to prove that $\hat{P}_\theta = \emptyset$ for generic θ.

The matrix of derivatives of this system of equations is given by

$$E = \begin{pmatrix} \tilde{C} & \tilde{B} \\ \tilde{A} & 0 \end{pmatrix},$$

where $\tilde{C}$, the matrix of derivatives of (E1)–(E3) with respect to x_1, …, x_n, p, λ_2, …, λ_n, is given by

$$\begin{pmatrix} D^2_{x_1x_1} V_1(x_1, \theta_1) & \cdots & 0 & \cdots & 0 & -I & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & D^2_{x_ix_i} V_i(x_i, \theta_i) & \cdots & 0 & -\lambda_i I & \cdots & -p & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & D^2_{x_nx_n} V_n(x_n, \theta_n) & -\lambda_n I & \cdots & 0 & \cdots & -p \\ -I & \cdots & -I & \cdots & -I & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

and the matrix of derivatives of (E1)–(E3) with respect to θ_1, …, θ_n is

$$\tilde{B} = \begin{pmatrix} D^2_{x_1\theta_1} V_1(x_1, \theta_1) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & D^2_{x_i\theta_i} V_i(x_i, \theta_i) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & D^2_{x_n,\theta_n} V_n(x_n \theta_n) \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}.$$

Finally, $\tilde{A} = (A_1, \cdots, A_n)$, where A_i is the derivative of (E4) with respect to x_i, λ, and p. The only non-zero entries in matrix A_i correspond to the derivatives with respect to $x_{i1}(\omega_{k_i})$ and $x_{i1}(\omega_{k_i'})$ if $i < n$, and $x_{n2}(\omega_{k_n})$ and $x_{n2}(\omega_{k_n'})$, otherwise.

We now show that matrix E has full rank. Thus, consider a linear combination of the rows of E which is equal to 0. The vector of scalars for the rows are denoted by w_1 for (E1), $w_2, \ldots, w_n$ for (E2); a for (E3); and b for (E4). Matrix $\tilde{B}$ implies that $w_i D^2_{x_i\theta} V_i(x_i) = 0$, $i = 1, \ldots, n$. Because $D^2_{x_i\theta} V_i(x_i)$ has full rank, it follows that $w_i = 0$ for $i = 1, \ldots, n$. Let $a_{1,\omega,k}$ be the scalar multiplier corresponding to the part of (E3) that ensures feasibility for good 1 in state ω_k. Because $w_i = 0$, the linear combination of the column elements of E corresponding to the derivative with respect to $x_{n1}(\omega_{k_i})$ yields $a_{1,\omega,k_i} = 0$. Similarly, it follows that $a_{1,\omega,k_i'} = 0$. Now let $i < n$ and consider the linear combination of column elements of E corresponding to the derivatives with respect to $x_{i1}(\omega_{k_i})$. Then since $w_i = 0$ and $a_{1,\omega,k_i} = 0$ we get $b_i = 0$. Similarly, we can show that $b_n = 0$. This immediately implies that $a = 0$. Hence all scalars are equal to 0 and E has therefore full rank. Because there are more equations than unknowns, the transversality theorem therefore implies that $\hat{P}_\theta = \emptyset$ except for a set $\tilde{\Theta} \subset \mathbb{R}^\ell$ that has measure 0.

We now show that $\tilde{\Theta}$ is closed. Let θ_k, $k \in \mathbb{N}$ be a sequence in $\tilde{\Theta}$ with $\lim_{k\to\infty} \theta_k = \theta$. Let (x^k, p^k, λ^k) be a solution of (E1)–(E4) given θ^k. Then the associated matrix E will not have full rank, i.e. some of the rows of E will be collinear. Without loss of generality we can assume that the same rows are collinear for all $k \in \mathbb{N}$. Because feasible allocations are bounded we can assume without loss of generality that x^k converges to x as $k \to \infty$. Since all x^k are individually rational and because of assumption A1 it follows that each x_i is not on the boundary of agent i's consumption set. (E1) therefore implies that p^k converges to p, where $p > 0$. Thus, (E2) implies that λ^k also converges. Therefore, the rows of matrix E are collinear for (x, p, λ) given θ. Thus, $\theta \in \tilde{\Theta}$. ■

Proof of Lemma 2. Let e_S and e_T be the aggregate endowments of coalitions S and T, respectively. First, note that $u \in \mathrm{bd}\, U(S) \cap \mathrm{bd}\, U(T)$ if and only if there exist allocations x, y with the following properties:

x and y are feasible for coalitions S and T, respectively; x cannot be improved upon by another allocation x' with $\sum_{i\in S} x_i' = e_S$, and similar for y; $u_i = V_i(x_i)$ for $i \in S$ and $u_i = V_i(y_i)$, for $i \in T$.

Without loss of generality we renumber the agents such that $S = \{1, \ldots, k+j\}$ and $T = \{k, \ldots, k+j, \ldots, m\}$, where $m \leq n$. Then x and y must fulfill the following equations:

(E1) $D_{x_i} V_i(x_i) - \lambda_i p = 0$, $i \in S$;

(E2) $D_{y_i}V_i(y_i) - \mu_i q = 0$, $i \in T$;

(E3) $V_i(x_i) - V_i(y_i) = 0$, $i \in S \cap T$;

(E4) $\sum_{i \in S}(x_i - e_i) = 0$;

(E5) $\sum_{i \in T}(y_i - e_i) = 0$;

where $\lambda_i, \mu_i > 0$, $\lambda_1 = \mu_k = 1$, and $p, q > 0$.

We now show that the matrix of derivatives of (E1)–(E5) with respect to x, y, p, λ, q, has full rank. The matrix A of derivatives with respect to x is given by

$$
\begin{array}{c}
\\
(E1)\\
\\
\\
\\
(E2)\\
\\
\\
(E3)\\
\\
(E4)\\
(E5)
\end{array}
\begin{array}{c}
\begin{matrix} x_1 & \dots & x_k & \dots & x_{k+j} \end{matrix} \\
\begin{pmatrix}
D^2_{x_1x_1}V_1(x_1) & \dots & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \dots & D^2_{x_kx_k}V_k(x_k) & \dots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & \dots & D^2_{x_{k+j}x_{k+j}}V_{k+j}(x_{k+j}) \\
0 & \dots & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & \dots & 0 \\
0 & \dots & D_{x_k}V_k(x_k) & \dots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & \dots & D_{x_{k+j}}V_{k+j}(x_{k+j}) \\
I & \dots & I & \dots & I \\
0 & \dots & 0 & \dots & 0
\end{pmatrix}
\end{array}
$$

The matrix B of derivatives with respect to y is

$$
\begin{array}{c|ccccc|}
 & y_k & \dots & y_{k+j} & \dots & y_m \\
\hline
 & 0 & \dots & 0 & \dots & 0 \\
(E1) & \vdots & \ddots & \vdots & \ddots & \vdots \\
 & 0 & \dots & 0 & \dots & 0 \\
 & D^2_{y_k y_k} V_k(y_k) & \dots & 0 & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots \\
(E2) & 0 & \dots & D^2_{y_{k+j} y_{k+j}} V_{k+j}(y_{k+j}) & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots \\
 & 0 & \dots & 0 & \dots & D^2_{y_m y_m} V_m(y_m) \\
 & -D_{y_k} V_k(y_k) & \dots & 0 & \dots & 0 \\
(E3) & \vdots & \ddots & \vdots & \ddots & \vdots \\
 & 0 & \dots & -D_{y_{k+j}} V_{k+j}(y_{k+j}) & \dots & 0 \\
(E4) & 0 & \dots & 0 & \dots & 0 \\
(E5) & I & \dots & I & \dots & I
\end{array}
$$

Finally, the matrix C of derivatives with respect to the remaining variables is

$$
\begin{array}{c|cccccccc|}
 & p & \dots & \lambda_k & \dots & \lambda_{k+j} & q & \dots & \mu_m \\
\hline
 & -I & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(E1) & -\lambda_k I & \dots & -p & \dots & 0 & 0 & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & -\lambda_{k+j} I & \dots & 0 & \dots & -p & 0 & \dots & 0 \\
 & 0 & \dots & 0 & \dots & 0 & -I & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(E2) & 0 & \dots & 0 & \dots & 0 & -\mu_{k+j} I & \dots & 0 \\
 & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & 0 & \dots & 0 & \dots & 0 & -\mu_m & \dots & -q \\
 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
(E3) & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
(E4) & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
(E5) & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0
\end{array}
$$

We now show that the rows of the matrix (ABC) are linearly independent. That is, consider an arbitrary linear combination of the rows which is equal to 0. Denote the vectors of scalar multipliers corresponding to (E1)–(E5) by w_i, z_i, α_i, a, and b, respectively. We must show that all multipliers are zero.

From the columns corresponding to the derivatives with respect to x_i we get

$$w_i D^2_{x_i x_i} V_i(x_i) + aI = 0, \text{ for } i < k; \tag{8}$$

$$w_i D^2_{x_i x_i} V_i(x_i) + \alpha_i \lambda_i p + aI = 0, \text{ for } i \geq k. \tag{9}$$

As $\lambda_1 = 1$, the column corresponding to the derivative with respect to p yield

$$\sum_{i=1}^{k+j} \lambda_i w_i = 0. \tag{10}$$

Finally, from the derivatives with respect to λ_i we get $pw_i = 0$ for $i \geq 2$. This, and (10) implies

$$pw_i = 0, \text{ for all } i \in I. \tag{11}$$

Now multiply from the right both sides of (8) and (9) by w_i, and use (11). This yields $w_i D^2_{x_i x_i} V_i(x_i) w_i + a w_i = 0$. Then

$$\sum_{i=1}^{k+j} \Big[\lambda_i w_i D^2_{x_i x_i} V_i(x_i) w_i + \lambda_i a w_i \Big] = 0. \tag{12}$$

Now (10) and (12) imply that $\sum_{i=1}^{k+j} \lambda_i w_i D^2_{x_i x_i} V_i(x_i) w_i = 0$. However, since $D^2_{x_i x_i} V_i(x_i)$ is negative definite it follows that $w_i D^2_{x_i x_i} V_i(x_i) w_i < 0$ if $w_i \neq 0$. Thus, $w_i = 0$ for $i = 1, \ldots, k + j$. Now equation (8) immediately implies that $a = 0$. Equation (9) therefore implies

$$w_i = -\alpha_i \lambda_i p \big(D^2_{x_i x_i} V_i(x_i)\big)^{-1}. \tag{13}$$

Thus, we get $-\alpha_i \lambda_i p \big(D^2_{x_i x_i} V_i(x_i)\big)^{-1} p = 0$, when we multiply both sides of (13) from the right with p, and use (11). Note that $p\big(D^2_{x_i x_i} V_i(x_i)\big)^{-1} p < 0$ because $p \neq 0$. Thus, $\alpha_i \lambda_i = 0$. Because $\lambda_i \neq 0$, we therefore get $\alpha_i = 0$.

Similarly, we can prove that z_i and b are zero. Thus, the matrix of derivatives of (E1)–(E5) has full rank. Because there are $m - 2$ more equations than unknowns, the set of solutions is therefore a $m - 2$ dimensional manifold. Thus $\text{bd}\, U(S) \cap \text{bd}\, U(T) \cap \mathbb{R}^m$ has dimension $m - 2$. Consequently, $\text{bd}\, U(S) \cap \text{bd}\, U(T)$ has dimension $n - 2$. ■

Proof of Lemma 3. First, note that we can assume without loss of generality that the information an agent receives from observing the endowment realization is also contained in the signal. Formally, let ω_i, $\omega_i' \in \Omega$ with $e_i(\omega) \neq e_i(\omega')$. Then $\pi(\phi_i = \omega|\omega') = 0$. We can therefore assume that allocations in the incomplete information economy depend only on all signals $\phi = (\phi_1, \dots, \phi_n)$ but not on ω.

Now let x be a feasible allocation of the complete information economy. We define an allocation $\hat{x}$ of the incomplete information economy as follows.

If $\phi = (\omega, \dots, \omega)$ then $\hat{x}_i(\phi) = x_i(\omega)$. If $\phi = (\delta_{-i}(\omega), \omega')$, for $\omega' \neq \omega$ then $\hat{x}_i(\phi) = 0$ and $\hat{x}_j(\phi) = x_j(\omega) + (1/(I-1))x_i(\omega)$. Finally, for all other signal profiles let $\hat{x}_i(\phi)$ be the agent's endowment $e_i(\omega)$.

We now show that $\hat{x}(\phi)$ is incentive compatible given prior π_k for all sufficiently large k. That is, we must show that

$$\int_{\Omega\times\Phi_{-i}} u_i\big(\omega, \hat{x}_i(\phi_{-i}, \phi_i)\big)\, d\pi_k(\omega, \phi_{-i}|\phi_i) \geq \int_{\Omega\times\Phi_{-i}} u_i\big(\omega, \hat{x}_i(\phi_{-i}, \phi_i')\big)\, d\pi_k(\omega, \phi_{-i}|\phi_i). \tag{14}$$

If $k \to \infty$ then $\pi_k\big(\omega, \phi_{-i} \bigm| \phi_i\big)$ converges to 1 if $\phi_{-i} = \delta_{-i}(\phi_i)$, $\phi_i = \omega$, and to 0 otherwise. Let $\omega_i = \phi_i$. Then the lefthand side of (14) converges to $u_i(\omega_i, \hat{x}(\delta(\phi_i))) = u_i(\omega_i, x_i(\omega_i))$. The righthand side of (14) converges to $u_i(\omega_i, \hat{x}(\delta_{-i}(\phi_i), \phi_i')) = u_i(\omega_i, 0)$. Because $x_i(\omega_i) > 0$ and u_i is strictly monotone, it follows that

$$\lim_{k\to\infty} \int_{\Omega\times\Phi_{-i}} u_i\big(\omega, \hat{x}_i(\phi_{-i}, \phi_i)\big)\, d\pi_k(\omega, \phi_{-i}|\phi_i) = u_i(\omega_i, x_i(\omega_i)) > u_i(\omega_i, 0) = \lim_{k\to\infty} \int_{\Omega\times\Phi_{-i}} u_i\big(\omega, \hat{x}_i(\phi_{-i}, \phi_i')\big)\, d\pi_k(\omega, \phi_{-i}|\phi_i).$$

Thus, for each $i \in I$ and $\phi_i \in I$ there exists $K_{i\phi_i} > 0$ such that (14) holds for all $k \geq K_{i\phi_i}$. Because the number of states and agents is finite, we can find $K > 0$ such that (14), and hence incentive compatibility holds for all $k \geq K$ and all ϕ.

Because $\lim_{k\to\infty} \pi_k = \pi$ and $\hat{x}(\delta(\omega)) = x(\omega)$ we get

$$\lim_{k\to\infty} \int u_i(\omega, \hat{x}_i(\omega, \phi))\, d\pi_k(\omega, \phi) = \int u_i(\omega, x_i(\omega, \phi))\, d\pi(\omega, \phi) = V_i(x_i).$$

Thus, we can define the sequence of allocations as follows: Let $x^k = \hat{x}$ for all $k \geq K$; and let x^k be an arbitrary incentive compatible allocation for $k < K$. This proves the Lemma. ■

Proof of Theorem 2. We first show that the core has dimension $n-1$. More precisely, the core contains a set which is homeomorphic to an open subset of $\mathbb{R}^{n-1}$.

Because utility is strictly concave, the function $\psi(x) = (V_1(x_1), \ldots, V_n(x_n))$ is a homeomorphism between Pareto efficient allocations and $U(I)$. By assumption, there exists a strict core allocation x. Let $\bar{u} = \psi(x)$ be the corresponding vector of utilities. Then by the definition of a strict core allocation $\bar{u} \notin U(S)$ for all $S \subsetneq I$.

Now recall that $U(I)$ has dimension $n-1$, i.e., it is homeomorphic to a set containing an nonempty open subset of $\mathbb{R}^{n-1}$. Since $\bar{u} \notin U(S)$ for all $S \neq I$ and since the sets $U(S)$ are closed it follows that there exists a neighborhood $W(\bar{u})$ of $\bar{u}$ in $U(I)$ with $W(\bar{u}) \cap U(S) = \emptyset$ for all $S \neq I$. Thus, $W(\bar{u})$ is homeomorphic to a subset of the set of core allocations. Because $W(\bar{u})$ has dimension $n-1$, the core has dimension $n-1$.

Now define $\hat{N} = \bigcup_{S \neq T; S,T \neq I} \operatorname{bd} U(S) \cap \operatorname{bd} U(T)$. By Lemma 2, $\hat{N}$ is a closed set of dimension at most $n-2$. Let $N = \psi^{-1}(\hat{N})$. Then the intersection of N with the set of core allocations has at most dimension $n-2$.

Let $\tilde{C}$ be the set of all core allocations y with $\psi(y) \notin U(S)$ for all $S \neq I$. Let x be a core allocation with $x \notin N$. We now show that x is in the closure of $\tilde{C}$.

Assume that $x \notin \tilde{C}$. Then $\bar{u} = \psi(x) \in U(S)$ for a coalition $S \neq I$. We now construct a sequence of allocations x^k, $k \in \mathbb{N}$ in $\tilde{C}$ that converges to x.

Let x^k, $k \in \mathbb{N}$ be a sequence of Pareto efficient allocations that fulfill $\lim_{k\to\infty} x^k = x$ and $V_i(x_i^k) > V_i(x_i)$ for all $i \in S$. Then $\bar{u}_k = \psi(x^k) \notin U(S)$. In order to show that $x^k \in \tilde{C}$ for sufficiently large k, it is remains to prove that $\bar{u}_k \notin U(T)$ for all $T \neq I$.

Assume by way of contradiction that $\bar{u}_k \in U(T)$ for a coalition $T \neq I$ for all large k. By construction $T \neq S$. Because $U(T)$ is closed, $\bar{u} \in U(T)$. Because x is a core allocation, it follows that $\bar{u} \in \operatorname{bd} U(T)$. Moreover, by assumption $\bar{u} \in U(S)$ and hence $\bar{u} \in \operatorname{bd} U(S)$. This, however, is a contradiction to $x \notin N$.

It now remains to prove that every core allocation $x \notin N$ is the limit of incentive compatible core allocations of incomplete information economies. Because the set of limit points of sequences is closed, it is sufficient to provide a proof for all $x \in \tilde{C}$.

For any consumption bundle y, let $V_i^k(y_i) = \int u_i(\omega, y_i(\omega, \phi))\, d\pi_k$. Let $x \in \tilde{C}$. By Lemma 3 there exists a sequence of Bayesian incentive compatible allocations x^k, $k \in \mathbb{N}$ for the incomplete information economies π_k, $k \in \mathbb{N}$ with $\lim_{k\to\infty} V_i^k(x_i^k) = V_i(x_i)$ for all agents $i \in I$. We show that one can assume x^k to be constrained Pareto efficient[5] for all k.

If x^k is not constrained Pareto efficient, choose a constrained Pareto efficient allocation $\tilde{x}^k$ with $V_i^k(\tilde{x}_i^k) \geq V_i^k(x_i^k)$ for all $i \in I$. Then because of compactness, $\tilde{x}_i^k$, $k \in \mathbb{N}$ has subsequences that converges. By slight abuse of notation we denote this subsequence again by $\tilde{x}_i^k$, $k \in \mathbb{N}$. Let $\tilde{x}$ be the limit. Clearly, $V_i(\tilde{x}_i) = \lim_{k\to\infty} V_i^k(\tilde{x}_i^k) \geq V_i(x_i)$. However, since x is Pareto efficient and because utility is

[5] That is, there does not exist another incentive compatible allocation that makes all agents weakly and at least some agents strictly better off.

strictly concave it therefore follows that $\tilde{x} = x$. Hence, $\tilde{x}^k$ converges to x.

It now remains to prove that x^k is in the core for sufficiently large k. We proceed by way of contradiction. Without loss of generality we can assume that there exists a coalition S that blocks x^k for all k. Thus, for every k there exist y^k with $V_i^k(y_i^k) \geq V_i^k(x_i^k)$ and $\sum_{i \in S} y_i^k = \sum_{i \in S} e_i$. By compactness we can assume without loss of generality that y^k converges to an allocation y.[6] Then $\sum_{i \in S} y_i = \sum_{i \in S} e_i$. Moreover, $V_i(y_i) = \lim_{k \to \infty} V_i^k(y_i^k) \geq \lim_{k \to \infty} V_i^k(x_i^k) = V_i(x_i)$. Thus, $\psi(x) \in U(S)$, a contradiction to the assumption that $x \in \tilde{C}$. ■

Proof of Theorem 3. The proof proceeds as follows. First, we construct an economy in which blocking by the two agent coalition $\{2,3\}$ matters. We denote by E the set of allocations that are blocked only by $\{2,3\}$ but not by any other coalition. In the economy that we construct, E has the same dimension as the core. The economy has also strict core allocations. Then we show that these properties are robust if we perturb agents' utility functions. The perturbed economies have utility functions that fulfill assumption A2. Finally, we use an argument similar to that of Theorem 2 to show that the set of limits of incentive compatible core allocations contains E, and is therefore larger than the core.

To simplify notation in the proof, we will consider the core in the set of attainable utilities rather than in the set of allocations. In particular, if $U(S)$ denotes the set of attainable utilities of coalition S, then v is in the core if $v \in U(I)$ but not in the interior of any $U(S)$.

We start by constructing the example economy.

There are $n \geq 3$ agents. Assume there are four states, ω_i, $i = 1, \ldots, 4$. The argument immediately generalizes to any number of states greater than four. There is one consumption good in each state. Agents' utility functions are given by

$$u_1(\omega, x) = \begin{cases} \sqrt{x} & \text{if } \omega = \omega_1; \\ x & \text{otherwise;} \end{cases} \quad u_2(\omega, x) = \begin{cases} \sqrt{x} & \text{if } \omega = \omega_2, \omega_3; \\ x & \text{otherwise;} \end{cases}$$

$$u_i(\omega, x) = \begin{cases} \sqrt{x} & \text{if } \omega = \omega_2; \\ x & \text{otherwise.} \end{cases} \quad \text{for } i \geq 3$$

Each agent's endowment in the four states is (a, a, a, b), where $b \geq (n+1)a$. Agents therefore know at $t = 1$ whether state 4 has occurred. However, their information about states ω_1, ω_2, and ω_3 is noisy. Note that state ω_4 is included to make utility

[6] More precisely, define $y_i^k = y_i$ for all $i \notin S$. The resulting sequence is bounded by the feasibility restriction for coalition S.

functions quasilinear and to avoid boundary problems. Thus, for the case of complete information, the economy can be transformed into a game with transferable utility. In particular, the set of allocations for a coalition S where marginal rates of substitution are equated have the property that agent i's consumption is $1/4$ in state ω if $u_i(\omega,x) = \sqrt{x}$ and there exists $j \in S$ with $u_j(\omega,x) = x$.

Now normalize each agent's utility function such that $E[u_i(e_i)] = 0$. Let $z = a - \sqrt{a} + 1/4$. Let m be the number of members of a coalition S. The payoff of any coalition S with at least two members is then given by

(i) $V(S) = (m+1)z$ if $1,2 \subset S$;

(ii) $V(S) = mz$ if $1 \in S$ but $2 \notin S$;

(iii) $V(S) = z$ if $2 \in S$ but $1 \notin S$.

The payoff of any single agent coalition is 0. Let $v = (v_1,\dots,v_n)$ be an allocation of agents' expected utilities. Let D be the set of all v that fulfill the following conditions.

$v_1 > (n-1)z$, $v_2 > z$, $v_j > 0$, for all j, and $\sum_{i=1}^n v_i = (n+1)z$. Then D is a subset of the core. In fact, D contains only strict core allocations, because $\sum_{i \in S} v_i > v(S)$ for all $v \in D$, and for all $S \subsetneqq I$. Moreover, because D has dimension $n-1$, the core has full dimension $n-1$.

Let E be the set of all v that fulfill the following conditions.

$v_1 = 4z - t_1$, $v_2 = t_2$, $v_3 = t_3$, $v_j = z - t_j$, for $j \geq 4$, $t_i > 0$ for all i, $t_1 < z$, $\sum_{j \neq 2,3} t_j < z$, and $t_2 + t_3 = \sum_{j \neq 2,3} t_j$.

Note that E has dimension $n-1$. Moreover, none of the allocations in E is in the core, because they can be blocked by coalition $S = \{2,3\}$. In particular, $v(\{2,3\}) = z$. However, since $\sum_{j \neq 2,3}^n t_j < z$ it follows that $v_2 + v_3 < z$. Thus, $\{2,3\}$ can block. Also note that $\{2,3\}$ is the only coalition that can block. In fact, $\sum_{i \in S} v_i > v(S)$, for all coalitions $S \neq I, \{2,3\}$. That is, all allocations of utilities in E are strict in the sense that the same utilities cannot be obtained for its member by a coalition $S \neq I, \{2,3\}$.

Because agents' consumption is strictly greater than 0 in all states, we can modify agents utility function such that all agents' marginal utility at 0 is infinite in all states. We now perturb the utility functions.

Let $\varepsilon > 0$ be arbitrary. Consider the set U_ε of all utility functions for the n agents $\tilde{u}_i$, $i = 1,\dots,n$ with $\left|u_i(\omega,x) - \tilde{u}_i(\omega,x)\right| < \varepsilon$ for all $\omega \in \Omega$, $i \in I$ and for all $0 \leq x \leq \sum_{i \in I} e_i$. Clearly, U_ε contains preferences which are strictly concave. Thus, the conditions of Theorem 2 are fulfilled for such preferences. Let $U(S)$ and $\tilde{U}(S)$ be the set of attainable utilities generated by u_i and by $\tilde{u}_i$, respectively. Then $U(S)$ and $\tilde{U}(S)$ will differ by less than ε. That is, let $v \in U(S)$ be arbitrary. Then there exists $\tilde{v} \in \tilde{U}(S)$ with $\|v - \tilde{v}\| < \varepsilon$. Similarly, for all $\tilde{v} \in \tilde{U}(S)$ there exists $v \in U(S)$ with $\|v - \tilde{v}\| < \varepsilon$.

Let $v \in D$. Then as shown above $v \notin U(S)$ for all $S \subsetneq I$. Now choose $\varepsilon < (1/2)\,\mathrm{dist}(x, U(S))$. Let $W_\varepsilon(v)$ be an ε-neighborhood of v. Then $v' \notin \tilde{U}(S)$ for all $v' \in W_\varepsilon(v)$, where $\tilde{U}(S)$ is generated by utility functions $\tilde{u} \in U_\varepsilon$. The core of the economy with utility functions $\tilde{u}_i$ therefore contains all $\tilde{v} \in \mathrm{bd}\tilde{U}(I)$ with $\tilde{v} \geq v'$ for some $v' \in W_\varepsilon(v)$. Thus, $\tilde{v} \notin \tilde{U}(S)$ for all $S \subsetneq I$. Thus, there exist strict core allocations in the perturbed economy. As a consequence, the core has full dimension $n-1$.

Similarly, we can pick $v \in E$ and prove that there exists a neighborhood $W_\varepsilon(v)$ such that $\tilde{v} \in \mathrm{bd}\tilde{U}(I)$ and $\tilde{v} \notin U(S)$ for all $S \neq I, \{2,3\}$.

Now pick utility functions in U_ε that fulfill the assumptions of Theorem 2. Let $v \in E$, and $\tilde{v}$ be a vector of utilities generated by a Pareto efficient allocation x with $\tilde{v} \geq v'$ for some $v' \in W_\varepsilon(v)$. Note that the set of all such allocations x has dimension $n-1$. Moreover, x is not in the core as it can be blocked by coalition $S = \{2,3\}$. It thus remains to show that x is nevertheless the limit of utilities of incentive compatible core allocations.

Let $\pi_k \to \pi$ such that agents 2 and 3 are not completely informed about states ω_1, ω_2 and ω_3, i.e., $\pi_k\big(\phi_k = \omega_i \bigm| \omega_j\big) > 0$, for $i, j = 1,2,3$ and $k = 2,3$. We now proceed as in the last part of the proof of Theorem 2.

Lemma 3 implies that there exists a sequence of Bayesian incentive compatible allocations x^k, $k \in \mathbb{N}$ for the economies with priors π_k, $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \int u_i(\omega, x_i^k(\omega,\phi))\, d\pi_k = V_i(x_i)$. Again, one can assume that x^k is constrained Pareto efficient.

Now suppose there exists a coalition S that can block x^k. Than as in Theorem 2 one can conclude that S can block x. Thus, $S = \{2,3\}$. In order to prove that x^k is an incentive compatible core allocation, it therefore remains to prove that $\{2,3\}$ cannot block x^k for all sufficiently large k.

In order for agent 2 and 3 to improve through trade and receive an allocation close to x, agent 2 must make a transfer to agent 3 in state ω_3 which is strictly larger than the transfer in the other states. Thus, similar to Example 2, the trades are not incentive compatible. Agent 2 is better off reporting $s = \omega_1$ if $s = \omega_2$ or ω_3 has occurred. Similarly, agent 3 is better off reporting ω_3 when ω_2 has occurred. Thus, the resulting allocation is not incentive compatible and agents 2 and 3 can therefore not block. Hence x is a limit of incentive compatible core allocations. ■

REFERENCES

B. ALLEN (1992), "Market Games with Asymmetric Information: The Private Information Core," Working Paper, University of Pennsylvania.

B. ALLEN (1994), "Incentives in Market Games with Asymmetric Information: Approximate NTU Cores in Large Economies," in W. Barnett, H. Moulin, M. Salles and N. Schofield eds., Social Choice, Welfare and Ethics, Cambridge University Press.

M. BERLIANT (1992), "On Income Taxation and the Core," *Journal of Economic Theory* **56**, 121–141.

J. BOYD, E. PRESCOTT, AND B. SMITH (1988), "Organizations in Economic Analysis," *Canadian Journal of Economics* **21**, 477-491.

T. ICHIISHI AND A. IDZIK (1996), "Bayesian Cooperative Choice of Strategies," *International Journal of Game Theory* **25**, 455–473.

T. ICHIISHI AND M. SERTEL (1998), "Cooperative Interim Contract and Re-Contract: Chandler's M-Form Firm," *Economic Theory* **11**, 523–543.

L. KOUTSOUGERAS (1998), "A Two Stage Core with Applications to Asset Market and Differential Information Economies," *Economic Theory* **11**, 563–584.

L. KOUTSOUGERAS AND N.C. YANNELIS (1993), "Incentive Compatibility and Information Superiority of the Core of an Economy with Differential Information," *Economic Theory* **3**, 195–216.

S. KRASA (1999), "Unimprovable Allocations in Economies with Incomplete Information," *Journal of Economic Theory* **87**, 144–168.

S. KRASA AND N. YANNELIS (1994), "The Value Allocation of an Economy with Differential Information," *Econometrica* **62**, 881-900.

R. MCLEAN AND A. POSTLEWAITE (2000), "Informational Size, Incentive Compatibility and the Core of a Game with Incomplete Information," Working Paper.

R. VOHRA (1999), "Incomplete Information, Incentive Compatibility and the Core," *Journal of Economic Theory* **86**, 123–147.

R. WILSON (1978), "Information, Efficiency, and the Core of an Economy," *Econometrica* **46**, 807–816.

N. YANNELIS (1991), "The Core of an Economy with Differential Information," *Economic Theory* **1**, 183–198.