BIVARIATE EXCHANGEABILITY

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1. Introduction

A couple of weeks ago googling around for additional insight into exchangeability I ran across Persi Diaconis's review in the AMS Bulletin of Kallenberg's *Probabilistic symmetries and invariance principles.* The review is exemplary in many respects, but among these it provides a quite readable introduction to bivariate exchangeability and some connections to the literature on visual perception in the work of Bela Julesz at Bell Labs in the 1970s. The latter connections are spelled out in more detail in Diaconis and Freedman (1981).

The formalism of all this is confined to "coin tossing" where in the simplest setting of Bernoulli sequences we have de Finetti's theorem that tells us that any exchangeable probability measure P can be expressed as a mixture,

$$P = \int_0^1 P_\theta \mu(d\theta),$$

where P_θ is Bernoulli with parameter θ and μ is a unique probability measure on $[0,1]$. For arrays this generalizes nicely, but there are some unexpected subtleties when invariance is restricted to only rows and columns. Diaconis considers matrices with 0-1 entries, x_{ij} such that,

$$P(x_{ij} : 1 \le i,j \le n) = P(x_{\sigma(i)\tau(j)} : 1 \le i,j \le n)$$

for permutations, σ and τ and all n. In the univariate setting we can imagine constructing sequences in the following way: first, a θ is drawn from the distribution μ, then a coin is flipped which lands heads with probability θ, if heads is observed our entry is 1, otherwise 0. In the bivariate setting we consider an arbitrary Borel function $\varphi : [0,1]^2 \to [0,1]$. We generate a pair of independent uniforms (U_i, V_j) on $[0,1]$, and now flip a $\varphi(U_i, V_j)$ coin entering a 1 if it lands heads, 0 if tails. It transpires that any row/column exchangeable P is a mixture of such φ-processes,

$$P = \int P_\varphi \mu(d\varphi),$$

so these P_φ are extreme points from a Choquet perspective. Apparently, uniqueness of μ for this variant turned out to be quite difficult.

2. Perception of Randomness

Suppose we generate a 0-1 matrix and plot it, can we tell if it has some structure that distinguishes it from merely choosing each entry with a flip of a fair coin? This was the subject of one line of Julesz's research. He conjectured that if the process generating entries had the same first and second order probabilities, that is if:

(1) the chance of a 1 appearing in any given position was 1/2, and
(2) the chance of two 1's appearing in any two positions was 1/4,

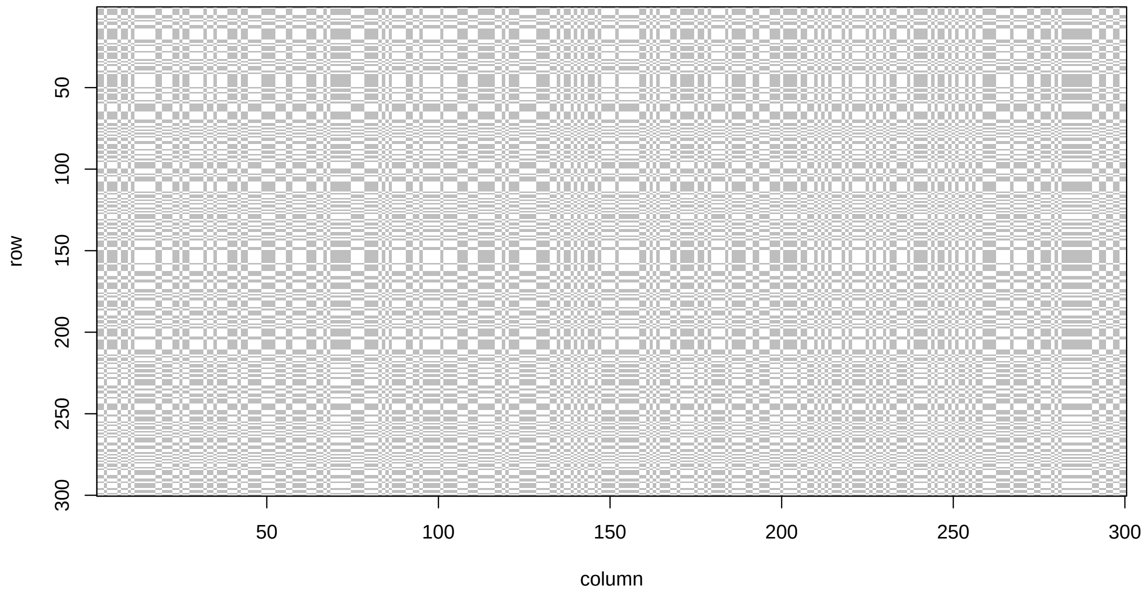

Figure 1. Diaconis Counterexample to the Julesz Conjecture

then the matrix would be indisguishable from the pure coin tossing realization. Diaconis and Freedman (1981) produced a family of counterexamples that were quite simple, yet easily distinguishable. The simplest of these examples is illustrated above and can be constructed in the following way: generate the first row as a fair Bernoulli sequence, then again flip fair coin if it falls heads repeat the first row as the second row, if tails make the second row $x_{2i} = (x_{1i} + 1) \bmod(2)$ $i = 1, 2, \ldots n$, and continue in this fashion to fill the remaining rows. As can be seen the texture of the resulting figure is quite "unrandom" looking, although it has the same first, second (and third order) probabilities!

3. Conclusion

It seems that φ-processes of the sort employed here are closely related to copula functions. There might be a temptation to think that they are also related to rank 1 matrices, since they depend upon only two Boolean n-vectors, but they are not. Extending their ideas to infinite domains offers some further challenges that bring Voronoi tesselations into the picture.

References

Diaconis, p. (2009), 'Review of *probabilistic symmetries and invariance principles*, by Olav Kallenberg', *AMS Bulletin* **46**, 691–696.

Diaconis, P. and Freedman, D. (1981), 'On the statistics of vision: The Julesz conjecture', *J. of Mathematical Psychology* **24**, 112–138.