# Copula-Based Quantile Autoregression

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**Abstract**

Parametric copulae are shown to be an attractive device for specifying quantile autoregressive models for nonlinear time-series. Estimation of local, quantile-specific models offers some salient advantages over classical global parametric approaches. Consistency and asymptotic normality of the proposed estimators are established, leading to a general framework for inference and model specification testing.

## 1. Introduction

Estimation of models for conditional quantiles constitutes an essential ingredient in modern risk assessment. And yet, often, such quantile estimation and prediction relies heavily on unrealistic global distributional assumptions. In this paper we consider new estimation methods for conditional quantile functions that are motivated by parametric models, but retain some semi-parametric flexibility and thus, should deliver more robust and more accurate estimates, while also being well-suited to the evaluation of misspecification.

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We employ parametric copula models to generate nonlinear-in-parameters quantile autoregression (QAR) models. Such models have several advantages over the linear QAR models previously considered in Koenker and Xiao (2006) since by construction they are globally plausible with monotone conditional quantile functions over the entire support of the conditioning variables. Rather than imposing this global structure, however, we choose instead to estimate the implied conditional quantile function independently, thereby facilitating an analysis of misspecification.

Copula models provide a rich source of potential nonlinear dynamics describing temporal dependence, and also permit us to carefully distinguish this dependence from the specification of the marginal (stationary) distribution of the response. Stationarity of the processes considered implies that only one marginal distribution is required for the specification in addition to the choice of the copula.

Choice of the parametric specification of the copula, $C$, and the marginal, distribution $F$, is a challenging problem. In this paper we restrict attention in our asymptotic analysis to settings in which the choices of $C$ and $F$ yield correctly specified conditional quantile functions. This is obviously a weaker condition than the direct assertion that we have correctly specified $C$ and $F$ themselves, since each of the conditional quantile functions we consider are permitted to have their own vector of parameters. Indeed, this distinction between global parametric models and local, quantile-specific, ones is essential throughout the quantile regression literature and facilitates inference for misspecification that arises from discrepancies in the quantile specific estimates of the model parameters.

The plan of the paper is as follows: We introduce the copula-based QAR model in Section 2. Assumptions and asymptotic properties of the proposed estimator are developed in Section 3. Section 4 briefly describes statistical inference and Section 5 concludes. For simplicity of illustration and without loss of generality, we focus our analysis on first order QAR processes in our analysis.

## 2. Copula-Based Quantile Autoregression Models

### 2.1. First-order strictly stationary Markov models

To motivate copula-based quantile autoregression models, we start with a strictly stationary Markov process of order 1, $\{Y_t\}_{t=1}^n$, whose probabilistic properties are determined by the true joint distribution of $Y_{t-1}$ and $Y_t$, say, $G^*(y_{t-1}, y_t)$. Suppose that $G^*(y_{t-1}, y_t)$ has continuous marginal distribution function $F^*(\cdot)$, then by Sklar's Theorem, there exists an unique copula function $C^*(\cdot, \cdot)$ such that

$$G^*(y_{t-1}, y_t) = C^*(F^*(y_{t-1}), F^*(y_t)).$$

Differentiating $C^*(u, v)$ with respect to $u$, and evaluate at $u = F^*(y_{t-1}), v = F^*(y_t)$, we obtain the conditional distribution of $Y_t$ given $Y_{t-1} = x$ :

$$\Pr\left[Y_t < y | Y_{t-1} = x\right] = \left.\frac{\partial C^*(u, v)}{\partial u}\right|_{u=F^*(x), v=F^*(y)} \equiv C_1^*(F^*(x), F^*(y)).$$

For any $\tau \in (0, 1)$, solving $\tau = C_1^*(F^*(x), F^*(y))$ for $y$, in terms of $\tau$, we obtain the $\tau$-th conditional quantile function of $Y_t$ given $Y_{t-1} = x$ :

$$Q_{Y_t}(\tau | x) = F^{*-1}(C_1^{*-1}(\tau; F^*(x))),$$

where $F^{*-1}(\cdot)$ signifies the inverse of $F^*(\cdot)$ and $C_1^{*-1}(\cdot; u)$ is the partial inverse of $C_1^*(u, v)$ with respect to $v$. If we denote $C_1^{*-1}(\cdot; u)$ as $h^*(\cdot)$, i.e.

$$C_1^{*-1}(\tau; u)\big|_{u=F^*(x)} = h^*(x),$$

we may write the $\tau$-th conditional quantile function of $Y_t$ as

$$Q_{Y_t}(\tau|x) = F^{*-1}(h^*(\tau; x)) = H^*(x).$$

In this paper, we will work with the class of copula-based first-order strictly stationary Markov models.

**Assumption 1:** $\{Y_t : t = 1, ..., n\}$ *is a sample of a stationary first-order Markov process generated from* $(F^*(\cdot), C^*(\cdot, \cdot))$*, where* $F^*(\cdot)$ *is the true invariant distribution which is absolutely continuous with respect to Lebesgue measure on the real line;* $C^*(\cdot, \cdot)$ *is the true copula for* $(Y_{t-1}, Y_t)$*, is absolutely continuous with respect to Lebesgue measure on* $[0, 1]^2$*, and is neither the Fréchet-Hoeffding upper nor lower bound.*

Assumption 1 is equivalent to assume that $\{Y_t : t = 1, ..., n\}$ *is a sample of a stationary first-order Markov process generated from* $(f^*(\cdot), g(\cdot|\cdot))$, where

$$g^*(y_t|y_{t-1}) \equiv f^*(y_t)c^*(F^*(y_{t-1}), F^*(y_t)), \tag{2.1}$$

where $g^*(\cdot|y_{t-1})$ is the true conditional density function of $Y_t$ given $Y_{t-1} = y_{t-1}$, $c^*(\cdot, \cdot)$ is the copula density of $C^*(\cdot, \cdot)$, and $f^*(\cdot)$ is the density of the marginal distribution $F^*(\cdot)$, which is unspecified.

### 2.1.1. Transformation model

As demonstrated in Chen and Fan (2006), all the copula-based first order Markov models can be expressed in terms of an autoregression transformation model. Let $U_t = F^*(Y_t)$, then under assumption 1, $\{U_t\}$ is strictly stationary first-order Markov with the joint distribution of $U_t$ and $U_{t-1}$ is given by the copula $C^*(\cdot, \cdot)$

(with corresponding density denoted as $c^*(\cdot,\cdot)$). Let $H_1()$ be any increasing transformation, then

$$H_1(F^*(Y_t)) = H_2(F^*(Y_{t-1})) + \sigma(F^*(Y_{t-1}))\varepsilon_t$$

or equivalently,

$$U_t = F^*(Y_t) = H_1^{-1}\left(H_2(U_{t-1}) + \sigma(U_{t-1})\varepsilon_t\right),$$

where the conditional density of $\varepsilon_t$ given $U_{t-1} = F^*(Y_{t-1}) = u_{t-1}$ is

$$\begin{aligned} f^*_{\varepsilon|F^*(Y_{t-1})=u_{t-1}}(\varepsilon) &= c^*(u_{t-1}, H_1^{-1}(H_2(u_{t-1}) + \sigma(u_{t-1})\varepsilon))/D(u_{t-1}) \\ &= c^*(F^*(Y_{t-1}), H_1^{-1}(H_2(F^*(Y_{t-1})) + \sigma(F^*(Y_{t-1}))\varepsilon))/D(F^*(Y_{t-1})) \end{aligned}$$

where

$$D(u) = \frac{dH_1(H_2(u)) + \sigma(u)\varepsilon))}{d\varepsilon},$$

and satisfies the condition that

$$H_2(u_{t-1}) = E\left[H_1(U_t)|\, U_{t-1} = u_{t-1}\right] = \int_0^1 H_1(u) \times c^*(u_{t-1}, u)du.$$

In the special case that $H_1(u) = u$, we obtain

$$U_t = H_2(U_{t-1}) + \sigma(U_{t-1})\varepsilon_t,$$

i.e.

$$F^*(Y_t) = H_2(F^*(Y_{t-1})) + \sigma(F^*(Y_{t-1}))\varepsilon_t.$$

Letting

$$\frac{\partial C^*(u_{t-1}, u)}{\partial u_{t-1}} = C_1^*(u_{t-1}, u)$$

then

$$\begin{aligned} H_2(u_{t-1}) &= E\left[H_1(U_t)\right| U_{t-1} = u_{t-1}] = \int_0^1 uc^*(u_{t-1}, u)du \\ &= \int_0^1 udC_1^*(u_{t-1}, u) = 1 - \int_0^1 C_1^*(u_{t-1}, u)du. \end{aligned}$$

### 2.2. Copula-based parametric quantile autoregression models

In practice, neither the true copula function $C^*(\cdot,\cdot)$ nor the true marginal distribution function $F^*(\cdot)$ of $Y$ is known. If we model both parametrically, say $C(\cdot,\cdot;\alpha)$ and $F(y;\beta)$, depending on unknown parameters $\alpha$, $\beta$, the $\tau$-th conditional quantile function of $Y_t$, $Q_{Y_t}(\tau|x)$, becomes a function of unknown parameters $\alpha$ and $\beta$, i.e.

$$Q_{Y_t}(\tau|x) = F^{-1}(C_1^{-1}(\tau; F(x,\beta),\alpha),\beta).$$

Denoting $\theta = (\alpha', \beta')'$, we will write,

$$Q_{Y_t}(\tau|x) = F^{-1}(C_1^{-1}(\tau; F(x,\beta),\alpha),\beta) = H(x;\theta). \tag{2.2}$$

This copula formulation of the conditional quantile functions provides a rich source of potential nonlinear dynamics. By varying the choice of the copula specification we can induce a wide variety of nonlinear QAR(1) dependence, and the choice of the marginal, $F$ enables us to consider a wide range of possible tail behavior as well.

Copula-based models have been widely used in finance, especially in estimating conditional quantiles as required for Value-at-Risk (VaR) assessment, motivated by possible nonlinearity in financial time series dynamics. However, in many financial time series applications, correlation structure may vary over the quantiles of the conditional distribution. We would like to stress that although the conditional quantile function specification in the above representation assumes the parameters to be identical across quantiles, our estimation methods do not impose this restriction. Thus, we permit the estimated parameters to vary with $\tau$ and this provides an important diagnostic feature of the methodology.

The proposed QAR model is based on (2.2) but we permit different parameter values over $\tau$, and write the vector of unknown parameters as $\theta(\tau) = (\alpha(\tau)', \beta(\tau)')'$. We obtain the following nonlinear QAR model:

$$Q_{Y_t}(\tau|Y_{t-1}) = H(Y_{t-1}, \theta(\tau)) = F^{-1}(C_1^{-1}(\tau; F(Y_{t-1}, \beta(\tau)), \alpha(\tau)), \beta(\tau)). \tag{2.3}$$

This nonlinear form of the QAR model can capture a wide range of systematic influences of conditioning variables on the conditional distribution of the response. Koenker and Xiao (2006) considered linear-in-parameter QAR processes in studying similar specifications. Maintaining a linear specification in the QAR model, requires rather strong regularity assumptions on the domain of the associated random variables imposed to ensure quantile monotonicity. Relaxing those assumptions implies that the conditional quantile functions are no longer linear. From this point of view, copula-based models provide an important way of extending constant coefficient linear QAR models to nonlinear quantile autoregression specifications that – under the parametric model – are ensured to be globally coherent.

**Remark**. The above analysis may be extended to $k$-th order nonlinear QAR models, but we will resist the temptation to tax the readers patience with the notation required for this.

### 2.3. Examples

**Example 1: Gaussian Copula**

Let $\Phi_\alpha(\cdot,\cdot)$ be the distribution function of bivariate normal distribution with mean zeros, variances 1, and correlation coefficient $\alpha$, and $\Phi$ be the CDF of a univariate standard normal, the bivariate Gaussian copula is given by

$$\begin{aligned} & C(u,v;\alpha) = \Phi_\alpha(\Phi^{-1}(u),\Phi^{-1}(v)) \\ = & \frac{1}{2\pi\sqrt{1-\alpha^2}}\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \exp\left\{-\frac{(s^2-2\alpha st+t^2)}{2(1-\alpha^2)}\right\} dsdt. \end{aligned}$$

Let $\{Y_t\}$ be a stationary Markov process of order 1 and with true marginal distribution $F^*(\cdot)$, in addition, denote that $U_t = F^*(Y_t)$, and $Z_t = \Phi^{-1}(U_t) = \Phi^{-1}(F^*(Y_t))$, if the correlation between $U_t$ and $U_{t-1}$ is characterized by a Gaussian copula, i.e. the joint CDF of $U_t$ and $U_{t-1}$ is

$$C(u_{t-1},u_t;\alpha) = \Phi_\alpha(\Phi^{-1}(u_{t-1}),\Phi^{-1}(u_t)).$$

Differentiating $C(u,v;\alpha)$ with respect to $u$, we obtain the conditional distribution of $U_t$ given $U_{t-1}$ :

$$C_1(u_{t-1},u_t;\alpha) = \Phi\left(\frac{\Phi^{-1}(u_t)-\alpha\Phi^{-1}(u_{t-1})}{\sqrt{1-\alpha^2}}\right)$$

For any $\tau\in[0,1]$, solving

$$\tau = C_1(u_{t-1},u_t;\alpha) = \Phi\left(\frac{\Phi^{-1}(u_t)-\alpha\Phi^{-1}(u_{t-1})}{\sqrt{1-\alpha^2}}\right)$$

for $u_t$, we obtain the $\tau$-th conditional quantile function of $U_t$ given $u_{t-1}$ :

$$\begin{aligned} Q_{U_t}(\tau|u_{t-1}) &= \Phi\left(\alpha\Phi^{-1}(u_{t-1})+\sqrt{1-\alpha^2}\Phi^{-1}(\tau)\right) \\ &= \Phi\left(\alpha\Phi^{-1}(F^*(y_{t-1}))+\sqrt{1-\alpha^2}\Phi^{-1}(\tau)\right) = h^*(F^*(y_{t-1}),\tau;\alpha). \end{aligned}$$

Also, $Z_t = \Phi^{-1}(U_t)$ is a Gaussian AR(1) process that can be represented by

$$Z_t = \alpha Z_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim N(0, (1-\alpha^2))$ and is independent of $Z_{t-1}$. We obtain the $\tau$-th conditional quantile function of $Z_t$ given $Z_{t-1}$ :

$$Q(\tau|Z_{t-1}) = b(\tau) + \alpha Z_{t-1}, \quad \text{with} \quad b(\tau) = \sqrt{1-\alpha^2}\Phi^{-1}(\tau),$$

a formulation that leaves us with the familiar linear AR(1) specification that induces the simplest linear QAR model.

**Example 2: Student-$t$ copula**

Let $\mathbf{t}_{\nu,\rho}(\cdot,\cdot)$ be the distribution function of bivariate student-$t$ distribution with mean zeros, variances 1, correlation coefficient $\rho$, and degrees of freedom $\nu$. And let $t_\nu(\cdot)$ be the CDF of a univariate student-$t$ distribution with mean zero, variance 1, and degrees of freedom $\nu$. The bivariate $t-$ copula is given by, with $\alpha = (\nu, \rho)$

$$\begin{aligned} & C(u, v; \alpha) = \mathbf{t}_{\nu,\rho}(t_\nu^{-1}(u), t_\nu^{-1}(v)) \\ = & \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \left\{ 1 + \frac{(s^2 - 2\rho st + t^2)}{\nu(1-\rho^2)} \right\}^{-(\nu+2)/2} dsdt. \end{aligned}$$

If $\{Y_t\}$ is a stationary Markov process of order 1 characterized by a standard bivariate $t_\nu$-copula function $C^* = C(\cdot, \cdot; \alpha)$ and marginal distribution function $F^*(\cdot)$, then let $t_\nu$ be the CDF of a $t_\nu$ random variable, let $U_t = F^*(Y_t)$, then the $\tau$-th conditional quantile function of $U_t$ is given by

$$Q_{U_t}(\tau|\mathcal{F}_{t-1}) = t_\nu \left( \rho t_\nu^{-1}(F^*(Y_{t-1})) + \sigma(F^*(Y_{t-1})) t_{\nu+1}^{-1}(\tau) \right) = h^*(F^*(Y_{t-1}), \tau; \rho, \nu),$$

where

$$\sigma(F^*(Y_{t-1})) = \sqrt{\frac{\nu + \left[t_\nu^{-1}(F^*(Y_{t-1}))\right]^2}{\nu+1}(1-\rho^2)}.$$

Moreover, the transformed variable $\{Z_t = t_\nu^{-1}(F^*(Y_t))\}$ is a student-$t$ process that can be represented by

$$Z_t = \rho Z_{t-1} + \sigma(Z_{t-1})e_t,$$

where $e_t \sim t_{\nu+1}$, and is independent of $Y_{t-1}$,

$$\sigma(Z_{t-1}) = \sqrt{\frac{\nu + Z_{t-1}^2}{\nu+1}(1-\rho^2)}$$

is a known function of $Z_{t-1} = t_\nu^{-1}(F^*(Y_{t-1}))$. (If the true marginal distribution $F^*$ is also $t_\nu$ then $t_\nu^{-1}(F^*(Y_t)) = Y_t$). The $\tau$-th conditional quantile function of $Z_t$, given $Z_{t-1}$, is then given by

$$Q_{Z_t}(\tau|\mathcal{F}_{t-1}) = \rho Z_{t-1} + \sigma(Z_{t-1})t_{\nu+1}^{-1}(\tau).$$

Let $\theta(\tau) = (\rho, \alpha(\tau), \beta(\tau))$, where

$$\alpha(\tau) = \frac{\nu(1-\rho^2)t_{\nu+1}^{-1}(\tau)^2}{1+\nu}, \quad \beta(\tau) = \frac{(1-\rho^2)t_{\nu+1}^{-1}(\tau)^2}{1+\nu}$$

we can rewrite the conditional quantile function as

$$Q_{Z_t}(\tau|\mathcal{F}_{t-1}) = \rho Z_{t-1} + \sqrt{\alpha(\tau) + \beta(\tau)Z_{t-1}^2} = h^*(Z_{t-1}; \theta(\tau)).$$

The above example applies to any elliptical copula, that is any copula generated from an elliptically symmetric bivariate distribution, where the conditional mean is linear and conditional variance is homoskedestic if and only if the copula is Normal copula; otherwise the conditional variance is heteroskedastic.

**Example 3: Joe-Clayton copula**

The Joe-Clayton copula is given by:

$$C(u, v; \alpha) = 1 - \{1 - [(1-\bar{u}^k)^{-\gamma} + (1-\overline{v}^k)^{-\gamma} - 1]^{-1/\gamma}\}^{1/k}, \tag{2.4}$$

where $\bar{u} = 1 - u$, $\alpha = (k, \gamma)'$ and $k \geq 1$, $\gamma 0$. It is known that the lower tail dependence parameter for this family is $\lambda_L = 2^{-1/\gamma}$ and the upper tail dependence parameter is $\lambda_U = 2 - 2^{1/k}$. When $k = 1$, the Joe-Clayton copula reduces to the Clayton copula:

$$C(u, v; \alpha) = [u^{-\alpha} + v^{-\alpha} - 1]^{-1/\alpha}, \quad \text{where} \quad \alpha = \gamma > 0. \tag{2.5}$$

When $\gamma \to 0$, the Joe-Clayton copula approaches the Joe copula whose concordance ordering and upper tail dependence increase as $k$ increases. For other properties of the Joe-Clayton copula, see Joe (1997). When coupled with fat-tailed marginal distributions such as the Student's t distribution, this family of copulas can generate time series with clusters of extreme values and hence provide alternative models for economic and financial time series that exhibit such clusters.

For the Joe-Clayton copula, one can easily verify that

$$\begin{aligned} C_1(u_{t-1}, u_t; \alpha) &= (1 - u_{t-1})^{k-1}(1 - \bar{u}_{t-1}^k)^{-(\gamma+1)} \\ &\quad \times [(1 - \bar{u}_{t-1}^k)^{-\gamma} + (1 - \bar{u}_t^k)^{-\gamma} - 1]^{-(\gamma^{-1}+1)} \\ &\quad \times [1 - \{(1 - \bar{u}_{t-1}^k)^{-\gamma} + (1 - \bar{u}_t^k)^{-\gamma} - 1\}^{-1/\gamma}]^{k^{-1}-1}. \end{aligned}$$

For any $\tau \in [0, 1]$, solving $\tau = C_1(u_{t-1}, u_t; \alpha)$ for $u_t$, we obtain the $\tau$-th conditional quantile function of $U_t$ given $u_{t-1}$ based on the Clayton copula:

$$Q_{U_t}(\tau | u_{t-1}) = [(\tau^{-\alpha/(1+\alpha)} - 1)u_{t-1}^{-\alpha} + 1]^{-1/\alpha}$$

Note that this expression and the similar expressions in the foregoing examples provide a convenient mechanism with which to simulate observations from the respective models.

## 3. Asymptotic Analysis

In this section, we study estimation of the copula-based QAR model (2.3). The vector of parameters $\theta(\tau)$ and thus the conditional quantile of $Y_t$ can be estimated by the following nonlinear quantile autoregression:

$$\min_{\theta \in \Re^k} \sum_t \rho_\tau(Y_t - H(Y_{t-1}, \theta)), \tag{3.1}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ is the usual check function (Koenker and Bassett (1978)). We denote the solution as $\widehat{\theta}(\tau)$. Given $\widehat{\theta}(\tau)$, the $\tau$-th conditional quantile of $Y_t$, conditional on the past information $Y_{t-1}$, can be estimated by

$$\widehat{Q}_{Y_t}(\tau | Y_{t-1} = x) = H(x, \widehat{\theta}(\tau)).$$

### 3.1. Assumptions

We derive the asymptotic properties of the QAR estimator $\widehat{\theta}(\tau)$ based on (3.1). To facilitate our asymptotic analysis, we introduce the following regularity assumptions.

A1. The parameter space $\Theta$ is compact.

A2. Let $F(y) = F(y; \beta)$ and $C(u, v) = C(u, v; \alpha)$ be the CDF and copula functions corresponding to the quantile function $H(x, \theta)$, the associated quantile function $F^{-1}(\tau) = F^{-1}(\tau; \beta)$ is twice continuously differentiable in $\beta$, and the copula function $C(u, v; \alpha)$ is second order differentiable with respect to $u$ and $v$, and has copula density $c(u, v; \alpha)$. $C_1(u, v; \alpha) = \partial C(u, v; \alpha)/\partial u$ is invertible in its second argument and the corresponding inverse function

$(C_1^{-1}(F(x), \tau; \alpha))$ is continuously differentiable in $\alpha$ and measurable in $x$ for each $\alpha$.

A3. The true $\tau$-th conditional quantile of $Y_t$ given $Y_{t-1} = x$, takes the form of a QAR model (2.3). The true unknown conditional density of $Y_t$ given $Y_{t-1} = x$, $g^*(\cdot|x)$, is continuously differentiable and bounded away from 0 and $\infty$.

A4. The smallest eigenvalue of matrix $V(\tau)$ is strictly positive, where $V(\tau)$ is defined by (3.2).

A5. (1) There exists a $a_{0t}$ such that $\sup_{\theta \in \Theta} |H(x_t, \theta)| \leq a_{0t}$ and $E(|a_{0t}|) < \infty$. (2) $\{Y_t\}$ is stationary, ergodic and satisfies assumption 1.

A6. $\Omega(\tau)$ is finite, where $\Omega(\tau)$ is defined by (3.2).

The above assumptions are similar to those usually imposed in nonlinear time-series models. These assumptions are given for the convenience of asymptotic analysis; we do not seek to achieve the weakest possible regularity conditions. The differentiability assumptions in A2 and A3 guarantee Taylor expansions of the regression function to appropriate order. Assumption A5(2) is a very mild assumption on weak dependence property of $\{Y_t\}$. Although we do not assume the correct specification of the parametric functional forms of the copula $C$ and the marginal distribution $F$, our model is nevertheless a parametric one; hence we do not need to assume beta-mixing as that imposed in Chen and Fan (2006). Beare (2008) studied dependence in copula models based on the notion of Doukhan and Louhichi (1999).

### 3.2. Large Sample Properties of the QAR Estimators

To facilitate our analysis, we introduce the following notation:

$$
\begin{aligned}
C_1(u,v;\alpha) &= \frac{\partial C(u,v;\alpha)}{\partial u};\ C_\alpha(u,v;\alpha) = \frac{\partial C(u,v;\alpha)}{\partial \alpha};\\
C_{1\alpha}(u,v;\alpha) &= \frac{\partial C(u,v;\alpha)}{\partial u \partial \alpha};\ c(u,v;\alpha) = \frac{\partial^2 C(u,v;\alpha)}{\partial u \partial v}.
\end{aligned}
$$

Let $f(\cdot)$ be the density functions corresponding to $F(\cdot)$ and $C_1^{-1}(u,\tau;\alpha)$ denote the inverse function of $C_1(u,v;\alpha)$ with respect to the argument $v$, and

$$
\begin{aligned}
h(x,\alpha) &= C_1^{-1}(\tau;u,\alpha)\big|_{u=F(x)}\\
H(x,\theta) &= F^{-1}(h(x,\alpha);\beta),\ \dot{H}_\theta(x,\theta) = \frac{\partial H(x_t;\theta)}{\partial \theta}
\end{aligned}
$$

and finally,

$$
\dot{h}_\alpha(x,\alpha) = \frac{\partial h(x,\alpha)}{\partial \alpha},\ \dot{h}_{\alpha\alpha}(x,\alpha) = \frac{\partial^2 h(x,\alpha)}{\partial \alpha \partial \alpha'},\ \frac{\partial F^{-1}(C,\beta)}{\partial \beta} = F_\beta^{-1}(C,\beta).
$$

Consistency and asymptotic normality of the copula-based QAR estimator are summarized in the following theorem.

Theorem: (1) Under Assumptions A1 - A5, $\widehat{\theta}(\tau) \to \theta(\tau)$, as $n \to \infty$. (2) Under Assumptions A1 - A6,

$$
\sqrt{n}\left(\widehat{\theta}(\tau) - \theta(\tau)\right) \Rightarrow N(0, \tau(1-\tau)V(\tau)^{-1}\Omega(\tau)V(\tau)^{-1}),
$$

where

$$
V(\tau) = \begin{bmatrix} V_{\alpha\alpha}(\tau) & V_{\alpha\beta}(\tau) \\ V_{\beta\alpha}(\tau) & V_{\beta\beta}(\tau) \end{bmatrix},\ \Omega(\tau) = \begin{bmatrix} \Omega_{\alpha\alpha}(\tau) & \Omega_{\alpha\beta}(\tau) \\ \Omega_{\beta\alpha}(\tau) & \Omega_{\beta\beta}(\tau) \end{bmatrix} \qquad (3.2)
$$

and

$$
\begin{aligned}
V_{\alpha\alpha}(\tau) &= \mathrm{E}\left[\frac{g^*(Q_{Y_t}(\tau|Y_{t-1})|Y_{t-1})}{\{f(Q_{Y_t}(\tau|Y_{t-1}))\}^2}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))\dot{h}_\alpha(Y_{t-1};\alpha(\tau))^\top\right] \\
V_{\alpha\beta}(\tau) &= \mathrm{E}\left[\frac{g^*(Q_{Y_t}(\tau|Y_{t-1})|Y_{t-1})}{f(Q_{Y_t}(\tau|Y_{t-1}))}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))^\top\right] \\
V_{\beta\beta}(\tau) &= \mathrm{E}\left[g^*(Q_{Y_t}(\tau|Y_{t-1})|Y_{t-1})F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))^\top\right] \\
V_{\beta\alpha}(\tau) &= V_{\alpha\beta}(\tau)^\top \\
\Omega_{\alpha\alpha}(\tau) &= \mathrm{E}\left[\frac{1}{\{f(Q_{Y_t}(\tau|Y_{t-1}))\}^2}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))\dot{h}_\alpha(Y_{t-1};\alpha(\tau))^\top\right] \\
\Omega_{\alpha\beta}(\tau) &= \mathrm{E}\left[\frac{1}{f(Q_{Y_t}(\tau|Y_{t-1}))}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))^\top\right] \\
\Omega_{\beta\beta}(\tau) &= \mathrm{E}\left[F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))F_\beta^{-1}(h(Y_{t-1};\alpha(\tau)),\beta(\tau))^\top\right] \\
\Omega_{\beta\alpha}(\tau) &= \Omega_{\alpha\beta}(\tau)^\top.
\end{aligned}
$$

**Remark 1.** *In the simple case where the marginal distribution function of $Y$ is known $F(y,\beta) = F$*

$$
\begin{aligned}
V(\tau) &= E\left\{\frac{g^*(Q_{Y_t}(\tau|Y_{t-1})|Y_{t-1})}{[f(Q_{Y_t}(\tau|Y_{t-1}))]^2}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))\dot{h}_\alpha(Y_{t-1};\alpha(\tau))^\top\right\}, \\
\Omega(\tau) &= E\left[\frac{1}{[f(Q_{Y_t}(\tau|Y_{t-1}))]^2}\dot{h}_\alpha(Y_{t-1};\alpha(\tau))\dot{h}_\alpha(Y_{t-1};\alpha(\tau))^\top\right].
\end{aligned}
$$

**Remark 2.** *When both the copula function* $C^*(u,v) = C(u,v;\alpha)$ *and the marginal distribution* $F^*(y) = F(y;\beta)$ *are correctly specified, the parameters* $\theta(\tau) = (\alpha(\tau)', \beta(\tau)')'$ *become constants over all* $\tau \in [0,1]$, *and the covariance matrix in the above Theorem reduces to the following simplified form,*

$$V(\tau) = \begin{bmatrix} V_{\alpha\alpha}(\tau) & V_{\alpha\beta}(\tau) \\ V_{\beta\alpha}(\tau) & V_{\beta\beta}(\tau) \end{bmatrix},$$

*with*

$$\begin{aligned}
V_{\alpha\alpha}(\tau) &= E\left[\frac{c(F(Y_{t-1}), F(Q_{Y_t}(\tau|Y_{t-1})); \alpha)}{f(Q_{Y_t}(\tau|Y_{t-1}))}\dot{h}_\alpha(Y_{t-1};\alpha)\dot{h}_\alpha(Y_{t-1};\alpha)^\top\right] \\
V_{\alpha\beta}(\tau) &= E\left[c(F(Y_{t-1}), F(Q_{Y_t}(\tau|Y_{t-1})); \alpha)\dot{h}_\alpha(Y_{t-1};\alpha)F_\beta^{-1}(h(Y_{t-1};\alpha),\beta)^\top\right] \\
V_{\beta\beta}(\tau) &= E[f(Q_{Y_t}(\tau|Y_{t-1}))c(F(Y_{t-1}), F(Q_{Y_t}(\tau|Y_{t-1})); \alpha) \\
&\quad \times F_\beta^{-1}(h(y_{t-1};\alpha),\beta)F_\beta^{-1}(h(y_{t-1};\alpha),\beta)^\top] \\
V_{\beta\alpha}(\tau) &= V_{\alpha\beta}(\tau)^\top.
\end{aligned}$$

## 4. Inference Based on Asymptotic Normality

The asymptotic normality of the QAR estimate also facilitates inference. In order to standardize the QAR estimator and remove nuisance parameters from the limiting distribution, we need to estimate the asymptotic covariance Matrix. In particular, we need to estimate $\Omega(\tau)$ and $V(\tau)$. Let

$$\widehat{Q}_{Y_t}(\tau|Y_{t-1}) \equiv H(Y_{t-1}, \widehat{\theta}(\tau)),$$

and let $\widehat{f}$, $\widehat{F}$, $\widehat{C}$, $\widehat{F}_\beta$ be the marginal density function, distribution function, copula function and etc. evaluated at the estimated parameters $\widehat{\theta}(\tau)$. Then $\Omega(\tau)$ can be estimated by

$$\widehat{\Omega}_n(\tau) = \begin{bmatrix} \widehat{\Omega}_{n,\alpha\alpha}(\tau) & \widehat{\Omega}_{n,\alpha\beta}(\tau) \\ \widehat{\Omega}_{n,\beta\alpha}(\tau) & \widehat{\Omega}_{n,\beta\beta}(\tau) \end{bmatrix},$$

with

$$\begin{aligned}
\widehat{\Omega}_{n,\alpha\alpha}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\{\widehat{f}(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))\}^2}\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))^\top \\
\widehat{\Omega}_{n,\alpha\beta}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\widehat{f}(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))}\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))^\top \\
\widehat{\Omega}_{n,\beta\beta}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))^\top \\
\widehat{\Omega}_{n,\beta\alpha}(\tau) &= \widehat{\Omega}_{n,\alpha\beta}(\tau)^\top.
\end{aligned}$$

Next, the true (unknown) conditional density of $Y_t$ given $Y_{t-1}$, $g^*(Q_{Y_t}(\tau|Y_{t-1})|Y_{t-1})$, can be estimated by the difference quotients,

$$\widehat{g}_t(\widehat{Q}_{Y_t}(\tau|Y_{t-1})) = (\tau_i - \tau_{i-1})/(\hat{Q}_{Y_t}(\tau_i|Y_{t-1}) - \hat{Q}_{Y_t}(\tau_{i-1}|Y_{t-1})),$$

for some appropriately chosen sequence of $\{\tau_i\}$'s. Then the matrix $V(\tau)$ can be estimated by

$$\widehat{V}_n(\tau) = \begin{bmatrix} \widehat{V}_{n,\alpha\alpha}(\tau) & \widehat{V}_{n,\alpha\beta}(\tau) \\ \widehat{V}_{n,\beta\alpha}(\tau) & \widehat{V}_{n,\beta\beta}(\tau) \end{bmatrix}$$

with

$$\begin{aligned}
\widehat{V}_{n,\alpha\alpha}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\frac{\widehat{g}_t(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))}{\{\widehat{f}(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))\}^2}\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))^\top \\
\widehat{V}_{n,\alpha\beta}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\frac{\widehat{g}_t(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))}{\widehat{f}(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))}\dot{h}_\alpha(Y_{t-1};\widehat{\alpha}(\tau))\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))^\top \\
\widehat{V}_{n,\beta\beta}(\tau) &= \frac{1}{n}\sum_{t=1}^{n}\widehat{g}_t(\widehat{Q}_{Y_t}(\tau|Y_{t-1}))\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))\widehat{F}_\beta^{-1}(h(Y_{t-1};\widehat{\alpha}(\tau)),\widehat{\beta}(\tau))^\top \\
\widehat{V}_{n,\beta\alpha}(\tau) &= \widehat{V}_{n,\alpha\beta}(\tau)^\top.
\end{aligned}$$

Wald type tests can then be constructed immediately based on the standardized QAR estimators using $\widehat{\Omega}_n(\tau)$ and $\widehat{V}_n(\tau)$. The copula-based QAR models and related quantile regression estimation also provide important information about specification. Specification of, say, the copula function may be investigated based on parameter constancy over quantiles, along the lines of Koenker and Xiao (2006). In addition, specification of conditional quantile models can be studied based on the quantile autoregression residuals. For example, if we want to test the hypothesis of a general form:

$$\mathrm{H}_0\colon\ R(\theta(\tau)) = 0$$

where $R(\theta)$ is an $q$-dimensional vector of smooth functions of $\theta$, with derivatives to the second order, the asymptotic normality derived from the previous section facilitates the construction of a Wald statistic. Let

$$\dot{R}(\theta(\tau)) = \left[\frac{\partial R_1(\theta)}{\partial \theta}, \cdots, \frac{\partial R_q(\theta)}{\partial \theta}\right]\bigg|_{\theta=\theta(\tau)},$$

denote a $p \times q$ matrix of derivatives of $R(\theta)$, we can construct the following regression Wald statistic

$$W_{n,\tau} \equiv nR(\widehat{\theta}(\tau))^{\top}\left[\tau(1-\tau)\dot{R}(\widehat{\theta}(\tau))^{\top}\widehat{V}_n(\tau)^{-1}\widehat{\Omega}_n(\tau)\widehat{V}_n(\tau)^{-1}\dot{R}(\widehat{\theta}(\tau))\right]^{-1}R(\widehat{\theta}(\tau)).$$

Under the hypothesis and our regularity conditions, we have

$$W_{n,\tau} \Rightarrow \chi_q^2$$

where $\chi_q^2$ has a central chi-square distribution with $q$ degrees of freedom.

## 5. Conclusion

There are many competing approaches to broadening the scope of nonlinear time-series modeling. We have argued that parametric copulas offer an attractive framework for specifying nonlinear quantile autoregression models. In contrast to fully parametric methods like maximum likelihood that impose a global parametric structure, estimation of distinct QAR models retains considerable semiparametric flexibility by permitting local, quantile-specific parameters.

There are many possible directions for future development. Inference and specification diagnostics is clearly a priority. Extensions to methods based on nonparametric estimation of the invariant marginal are possible. Finally, semiparametric modeling of the copula itself as a sieve appears to be feasible strategy for expanding the menu of existing parametric copulas currently available.

## 6. Appendix: Proof of The Theorem

Consistency. We denote $Y_{t-1}$ as $x_t$. Notice that minimization of the objective function is equivalent to minimizing

$$Q_n(\theta) = \frac{1}{n}\sum_t \rho_\tau(Y_t - H(x_t,\theta)) - \frac{1}{n}\sum_t \rho_\tau(\varepsilon_{t\tau})$$

where

$$\varepsilon_{t\tau} \equiv Y_t - Q_{Y_t}(\tau|x_t) \equiv Y_t - H(x_t, \theta(\tau)),$$

$$Q_{Y_t}(\tau|x) = H(x,\theta(\tau)), \quad H(x,\theta) \equiv F^{-1}(C_1^{-1}(\tau; F(x,\beta),\alpha),\beta),$$

and thus

$$Q_{\varepsilon_t}(\tau|x_t) = 0.$$

Denote

$$\overline{H}_t = H(x_t,\theta) - H(x_t,\theta(\tau)), \text{ and } q_\tau(Y_t,x_t,\theta) = \rho_\tau(\varepsilon_{t\tau} - \overline{H}_t) - \rho_\tau(\varepsilon_{t\tau}),$$

we may rewrite the objective function as

$$Q_n(\theta) = \frac{1}{n}\sum_t q_\tau(Y_t,x_t,\theta).$$

In order to establish consistency, we verify uniform convergence of $Q_n(\theta)$ and the identification condition of $E\left[Q_n(\theta)\right]$ (i.e. $E\left[Q_n(\theta)\right]$ is uniquely minimized at $\theta(\tau)$.)

Under Assumption A5, we have pointwise weak law of large numbers for $Q_n(\theta)$, i.e.

$$Q_n(\theta) - E\left[Q_n(\theta)\right] \xrightarrow{P} 0, \text{ for any } \theta \in \Theta.$$

In addition, $\Theta$ is compact by Assumption A1. We next verify stochastic equicontinuity. Let $B(\theta,\eta)$ be a $\eta$-neighbourhood around $\theta$, we need to show that for any given $\varepsilon > 0$,

$$\Pr\left(\sup_{\theta\in\Theta}\sup_{\theta'\in B(\theta,\eta)}\left|Q_n(\theta)-Q_n(\theta')-E\left(Q_n(\theta)-Q_n(\theta')\right)\right|>\varepsilon\right)\to 0,\text{ as }\eta\to 0.$$

Notice that under Assumption A5,

$$\begin{aligned}\sup_{\theta\in\Theta}|q_\tau(Y_t,x_t,\theta)| &= \sup_{\theta\in\Theta}\left|\rho_\tau(\varepsilon_{t\tau}-\overline{H}_t)-\rho_\tau(\varepsilon_{t\tau})\right| \\ &\le \sup_{\theta\in\Theta}|H(x_t,\theta)-H(x_t,\theta(\tau))| \le 2a_{0t}\end{aligned}$$

thus $E\left(\sup_{\theta\in\Theta}|q_\tau(Y_t,x_t,\theta)|\right)<\infty$. Let

$$\Delta_{t\eta}=\sup_{\theta\in\Theta}\sup_{\theta'\in B(\theta,\eta)}\left|q_\tau(Y_t,x_t,\theta)-q_\tau(Y_t,x_t,\theta')\right|,$$

then

$$\begin{aligned}&\Pr\left(\sup_{\theta\in\Theta}\sup_{\theta'\in B(\theta,\eta)}\left|Q_n(\theta)-Q_n(\theta')-E\left(Q_n(\theta)-Q_n(\theta')\right)\right|>\varepsilon\right)\\ \le\;&\Pr\left(\frac{1}{n}\sum_t\left[\Delta_{t\eta}+E\Delta_{t\eta}\right]>\varepsilon\right)\\ \le\;&\frac{E\left(\frac{1}{n}\sum_t\left[\Delta_{t\eta}+E\Delta_{t\eta}\right]\right)}{\varepsilon}=\frac{2}{\varepsilon}E\Delta_{t\eta}\end{aligned}$$

which converges to zero by dominated convergence Theorem, noticing that (a) $\Delta_{t\eta}\to 0$, a.s., as $\eta\to 0$ because $q_\tau(Y_t,x_t,\theta)$ is continuous in $\theta$ under assumptions A2 and A3; (b) $\Delta_{t\eta}\le 2\sup_{\theta\in\Theta}|q_\tau(Y_t,x_t,\theta)|$ and (c) $E\left(\sup_{\theta\in\Theta}|q_\tau(Y_t,x_t,\theta)|\right)<\infty$. Thus, we have

$$\lim\left\|Q_n(\theta)-E\left[Q_n(\theta)\right]\right\|=0,\text{ a.s. uniformly in }\theta\in\Theta.$$

Next we verify that $Q(\theta) = E\left[Q_n(\theta)\right]$ is uniquely minimized at $\theta(\tau)$. Recall that the true but unknown conditional density and distribution function of $Y_t$ given $x_t$ are $g^*(\cdot|x_t)$ and $G^*(\cdot|x_t)$ respectively, and use the following identity

$$\begin{aligned}
\rho_\tau(u-v) - \rho_\tau(u) &= -v\psi_\tau(u) + (u-v)\{I(0>u>v) - I(0<u<v)\} \\
&= -v\psi_\tau(u) + \int_0^v \{I(u \le s) - I(u<0)\}ds, \qquad (6.1)
\end{aligned}$$

where

$$\psi_\tau(u) \equiv \tau - I(u<0),$$

we have

$$\rho_\tau(\varepsilon_{t\tau} - \overline{H}_t) - \rho_\tau(\varepsilon_{t\tau}) = -\overline{H}_t\psi_\tau(\varepsilon_{t\tau}) + \int_0^{\overline{H}_t} \{I(\varepsilon_{t\tau} \le s) - I(\varepsilon_{t\tau} < 0)\}ds.$$

thus

$$\begin{aligned}
\overline{Q}_n(\theta) &= \frac{1}{n}\sum_t \mathrm{E}\left\{ \rho_\tau(\varepsilon_{t\tau} - \overline{H}_t) - \rho_\tau(\varepsilon_{t\tau}) \middle| \, x_t \right\} \\
&= \frac{1}{n}\sum_{t=1}^n \mathrm{E}\left\{ \int_0^{\overline{H}_t} \{I(\varepsilon_{t\tau} \le s) - I(\varepsilon_{t\tau} < 0)\}ds \middle| \, x_t \right\} \\
&= \frac{1}{n}\sum_{t=1}^n 1\left(\overline{H}_t > 0\right) \mathrm{E}\left\{ \int_0^{\overline{H}_t} I(0 \le \varepsilon_{t\tau} \le s)ds \middle| \, x_t \right\} \\
&\quad + \frac{1}{n}\sum_{t=1}^n 1\left(\overline{H}_t < 0\right) \mathrm{E}\left\{ \int_{\overline{H}_t}^0 I(s \le \varepsilon_{t\tau} \le 0)ds \middle| \, x_t \right\}
\end{aligned}$$

where the second equality is obtained by the fact that $\mathrm{E}[\psi_\tau(\varepsilon_{t\tau})|x_t] = 0$.

Under Assumptions A2 and A3,

$$\begin{aligned}
& \frac{1}{n}\sum_{t=1}^{n} 1\left(\overline{H}_t > 0\right) \mathrm{E}\left\{ \int_0^{\overline{H}_t} I(0 \le \varepsilon_{t\tau} \le s) ds \middle| x_t \right\} \\
= & \frac{1}{n}\sum_{t=1}^{n} 1\left(\overline{H}_t > 0\right) \mathrm{E}\{ \int_0^{\overline{H}_t} I(H(x_t, \theta(\tau)) \le Y_t \le s + H(x_t, \theta(\tau))) | x_t\} ds \\
= & \frac{1}{n}\sum_{t=1}^{n} 1\left(\overline{H}_t > 0\right) \int_0^{\overline{H}_t} \left[ \int_{Q_{Y_t}(\tau|x_t)}^{s+Q_{Y_t}(\tau|x_t)} g^*(y|x_t) dy \right] ds \\
= & \frac{1}{2n}\left[ \sum_{t=1}^{n} 1\left(\overline{H}_t > 0\right) g^*(Q_{Y_t}(\tau|x_t)|x_t) \overline{H}_t' \overline{H}_t \right] + o_p(\left\|\overline{H}_t\right\|^2),
\end{aligned}$$

and similar result can be obtained for the case $\overline{H}_t < 0$. Thus,

$$\begin{aligned}
\overline{Q}_n(\theta) & = \frac{1}{n}\sum_t \mathrm{E}\left\{ \rho_\tau(\varepsilon_{t\tau} - \overline{H}_t) - \rho_\tau(\varepsilon_{t\tau}) \middle| x_t \right\} \\
& = \frac{1}{2n}\left[ \sum_{t=1}^{n} g^*(Q_{Y_t}(\tau|x_t)|x_t) \overline{H}_t' \overline{H}_t \right] + o_p(\left\|\overline{H}_t\right\|^2).
\end{aligned}$$

Recall that $\theta = (\alpha, \beta)$ and

$$\overline{H}_t = H(x_t, \theta) - H(x_t, \theta(\tau)), \text{ and } H(x_t, \theta) = F^{-1}(C_1^{-1}(\tau; F(x_t, \beta), \alpha), \beta),$$

under Assumptions A2 and A3, by a Taylor expansion of $H(x_t, \theta)$ around $\theta(\tau)$, and notice that

$$\frac{\partial F^{-1}(u, \beta)}{\partial u} = \frac{1}{f\left(F^{-1}(u, \beta)\right)}, \quad \frac{\partial H(x; \theta)}{\partial \alpha} = \frac{1}{f(H(x, \theta))} \frac{\partial h(x; \alpha)}{\partial \alpha}$$

and

$$\dot{H}_\theta(x_t, \theta) = \frac{\partial H(x_t; \theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial H(x_t; \theta)}{\partial \alpha} \\ \frac{\partial H(x_t; \theta)}{\partial \beta} \end{bmatrix} = \begin{bmatrix} f(H(x, \theta))^{-1} \partial h(x; \alpha) / \partial \alpha, \\ F_\beta^{-1}(h(x, \alpha), \beta) \end{bmatrix},$$

we obtain

$$\overline{Q}_n(\theta) = \frac{1}{2}\left(\theta - \theta(\tau)\right)^\top V_n(\tau) \left(\theta - \theta(\tau)\right) + o_p(\|\theta - \theta(\tau)\|^2)$$

where

$$V_n(\tau) = \sum_{t=1}^{n} v_t(\tau),$$

with

$$\begin{aligned} v_t(\tau) &= g^*(Q_{Y_t}(\tau|x_t)|x_t)\dot{H}_\theta(x_t,\theta(\tau))\dot{H}_\theta(x_t,\theta(\tau))^\top \\ &= \begin{bmatrix} v_{t,\alpha\alpha}(\tau) & v_{t,\alpha\beta}(\tau) \\ v_{t,\beta\alpha}(\tau) & v_{t,\beta\beta}(\tau) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} v_{t,\alpha\alpha}(\tau) &= \frac{g^*(Q_{Y_t}(\tau|x_t)|x_t)}{[f(Q_{Y_t}(\tau|x_t))]^2}\dot{h}_\alpha(x_t;\alpha(\tau))\dot{h}_\alpha(x_t;\alpha(\tau))^\top \\ v_{t,\beta\beta}(\tau) &= g^*(Q_{Y_t}(\tau|x_t)|x_t)F_\beta^{-1}(h(x_t;\alpha(\tau)),\beta(\tau))F_\beta^{-1}(h(x_t;\alpha(\tau)),\beta(\tau))^\top \\ v_{t,\alpha\beta}(\tau) &= \frac{g^*(Q_{Y_t}(\tau|x_t)|x_t)}{f(Q_{Y_t}(\tau|x_t))}\dot{h}_\alpha(x_t;\alpha(\tau))F_\beta^{-1}(h(x_t;\alpha(\tau)),\beta(\tau))^\top \\ v_{t,\beta\alpha}(\tau) &= v_{t,\alpha\beta}(\tau)^\top. \end{aligned}$$

Thus,

$$\begin{aligned} E\left[\overline{Q}_n(\theta)\right] &= \frac{1}{2}\left(\theta-\theta(\tau)\right)^\top V(\tau)\left(\theta-\theta(\tau)\right) + o_p(\|\theta-\theta(\tau)\|^2) \\ &\geq \frac{1}{2}\lambda_{\min}\left\|(\theta-\theta(\tau))\right\|^2 + o_p(\|\theta-\theta(\tau)\|^2) \end{aligned}$$

where $\lambda_{\min}$ is the smallest eigenvalue of $V(\tau)$ which, under Assumption A4, is strictly positive. Thus for any $\varepsilon > 0$, $\overline{Q}_n(\theta)$ is bounded away from zero, uniformly in $\theta$ for $\|\theta-\theta(\tau)\| \geq \varepsilon$.

Limiting Distribution. Let $\sqrt{n}\,(\theta-\theta(\tau)) = v$, we may reparameterize the objective function $Q_n(\theta)$ as a function of $v$:

$$\begin{aligned} Q_n^*(v) &= \sum_t \left[\rho_\tau(Y_t - H(x_t,\theta(\tau)+n^{-1/2}v)) - \rho_\tau(\varepsilon_{t\tau})\right] \\ &= -\sum_t \overline{H}_t(v)\psi_\tau(\varepsilon_{t\tau}) + \sum_t \int_0^{\overline{H}_t(v)} \{I(\varepsilon_{t\tau}\leq s) - I(\varepsilon_{t\tau}<0)\}ds \end{aligned}$$

We first consider the term, $\sum_t \overline{H}_t(v)\psi_\tau(\varepsilon_{t\tau})$. Under Assumptions A2 and A3, $H(x_t, \theta)$ is twice continuously differentiable with respect to $\theta$ at $\theta(\tau)$. By a Taylor expansion of $H(x_t, \theta)$ around $\theta(\tau)$, and notice that $\mathrm{E}[\psi_\tau(\varepsilon_{t\tau})|x_t] = 0$, we have

$$\sum_t \overline{H}_t(v)\psi_\tau(\varepsilon_{t\tau}) = n^{-1/2}\sum_t \dot{H}_\theta(x_t, \theta(\tau))\psi_\tau(\varepsilon_{t\tau})v + o_p(1).$$

By stationary ergodic martingale difference CLT, under Assumption A6, we have:

$$\sum_t \overline{H}_t(v)\psi_\tau(\varepsilon_{t\tau}) = n^{-1/2}\sum_t \dot{H}_\theta(x_t, \theta(\tau))\psi_\tau(\varepsilon_{t\tau})v + o_p(1) \Rightarrow v \times N(0, \tau(1-\tau)\Omega(\tau)).$$

For the second term, if we define

$$\xi_t(v) = \int_0^{\overline{H}_t(v)} \{I(\varepsilon_{t\tau} \le s) - I(\varepsilon_{t\tau} < 0)\}ds, \text{ and } \overline{\xi}_t(v) = E\{\xi_t(v)|x_t\},$$

Then

$$\begin{aligned} & \sum_t \int_0^{\overline{H}_t(v)} \{I(\varepsilon_{t\tau} \le s) - I(\varepsilon_{t\tau} < 0)\}ds \\ = \; & \sum_{t=1}^n \overline{\xi}_t(v) + \sum_{t=1}^n \left[\xi_t(v) - \overline{\xi}_t(v)\right] = \sum_{t=1}^n \overline{\xi}_t(v) + o_p(1) \end{aligned}$$

where the second equality holds following Pollard (1984, p171). For the leading term, under Assumptions A2 and A3, by a Taylor expansion of $H(x_t, \theta)$ around $\theta(\tau)$ as in the previous discussion, we obtain

$$\sum_{t=1}^n \overline{\xi}_t(v) = \frac{1}{2}v^\top \left[\frac{1}{n}\sum_{t=1}^n g^*(Q_{Y_t}(\tau|x_t)|x_t)\dot{H}_\theta(x_t, \theta(\tau))\dot{H}_\theta(x_t, \theta(\tau))^\top\right] v + o_p(1)$$

Thus, we have

$$\begin{aligned} Q_n^*(v) &= \sum_t \left[\rho_\tau(Y_t - H(x_t, \theta(\tau) + n^{-1/2}v)) - \rho_\tau(\varepsilon_{t\tau})\right] \\ &= -\frac{1}{\sqrt{n}}\sum_t \psi_\tau(\varepsilon_{t\tau})\dot{H}_\theta(x_t, \theta(\tau))v + \frac{1}{2}v^\top V_n(\tau)v + o_p(1) \\ &\Rightarrow -v \times N(0, \tau(1-\tau)\Omega(\tau)) + \frac{1}{2}v^\top V(\tau)v = Q^*(v). \end{aligned}$$

Following Knight (1989) and Pollard (1991), note that $Q_n^*(v)$ and $Q^*(v)$ are minimized at $\widehat{v} = \sqrt{n}\left(\widehat{\theta}(\tau) - \theta(\tau)\right)$ and $\mathcal{V} = N(0, \tau(1-\tau)V(\tau)^{-1}\Omega(\tau)V(\tau)^{-1})$ respectively, Lemma A of Knight (1989) ensures

$$\sqrt{n}\left(\widehat{\theta}(\tau) - \theta(\tau)\right) \Rightarrow V(\tau)^{-1} \times N(0, \tau(1-\tau)\Omega(\tau)).$$

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