Inference on the Quantile Regression Process

Electronic Appendix

Roger Koenker and Zhijie Xiao

1 Asymptotic Critical Values

Like many other Kolmogorov-Smirnov type tests (see, e.g. Andrews (1993)), the limiting distribution $\sup_{\tau \in \mathcal{T}} \|w_0(\tau)\|$ is dependent on the norm $\|\cdot\|$, the pre-specified $\mathcal{T}$ and the dimension parameter q. Notice that the transformation is generally unstable in the extreme right tails, and the uniform convergence of existing estimators of the density and score $(f(F^{-1}(s))$ and $f'/f(F^{-1}(s)))$ usually requires that $\mathcal{T}$ be bounded away from zero and one, we consider a subset of $[0,1]$ whose closure lies in $(0,1)$.

We calculated the 1%, 5%, and 10% critical values for the test statistic

$$K_n = \sup_{\tau \in \mathcal{T}} \|\tilde{v}_n(\tau)\|$$

based on simulations where the Brownian motion was approximated by a Gaussian random walk, using a sample size $n = 2000$ and $20,000$ replications. For the norm $\|\cdot\|$, we use the ℓ_1 norm for a q-dimensional vector x, $\|x\| = \sum_{j=1}^{q} |x_j|$. Table 1 covers $\mathcal{T} = [\epsilon, 1-\epsilon]$ for $\epsilon = 0.05$, 0.1, 0.15, 0.2, 0.25, 0.3, and $q = 1, 2, ..., 20$. Although conventionally we consider symmetric intervals $\mathcal{T} = [\epsilon, 1-\epsilon]$ for some small numbers ϵ, a much wider range of intervals $\mathcal{T}$ may be considered for the proposed tests. Critical values based other choices of the interval $\mathcal{T}$ and the dimension parameter q can be similarly calculated. Gauss programs are available from the authors upon request.

2 Monte Carlo Results

In this section, we describe some Monte Carlo experiments to examine the finite sample performance of the proposed tests. In particular, we examine the effectiveness of the martingale transformation on the size and power properties of the tests. Some general rules of bandwidth selection are given based on the Monte Carlo results. We consider sample sizes $n = 100, 200, 300, 400, 500, 800,$ as representive of sample sizes in empirical analyses, and indicative of the validity of the asymptotic approximations. The numbers of iterations in all these simulations are 1000.

First of all, to investigate the effectiveness of the martingale transformation on quantile regression inference, we examine the size and power properties of the infeasible version tests where the true density and score functions are used in the standardization and the martingale transformation. We start with the heteroscedasticity test as described in Section 5.3. The data were generated from the model

$$y_i = \alpha + \beta x_i + \sigma(x_i) u_i, \tag{1}$$

where x_i and u_i are iid $\mathcal{N}(0,1)$ random variates and are mutually independent, $\alpha = 0$, and $\beta = 1$. $\sigma(x_i) = \gamma_0 + \gamma_1 x_i$ with $\gamma_0 = 1$. We examine the empirical rejection rates of the test for different choices of sample sizes and γ_1 values, at the 5% level of significance. In constructing the test, we used the OLS estimator for $\widehat{\beta}$, and the truncation parameter value $\epsilon = 0.05$ (i.e. $\mathcal{T} = [0.05, 0.95]$). Since x_i is a scalar, the limiting null distribution of the test statistic is $\sup_{0.05 \le \tau \le 0.95} |W(\tau)|$ and the 5% level critical value is 2.14. For the choices of the heteroscedasticity parameter γ_1, we consider $\gamma_1 = 0, 0.1, 0.2, 0.3, 0.5, 1, 2, 5$. When $\gamma_1 = 0$, the model is homoscedastic and the rejection rates give the empirical sizes. When $\gamma_1 \neq 0$, the model is heteroscedastic and the rejection rates deliver the empirical powers. Table 1 reports the empirical rejection rates for different values of γ_1 and n. Other values of the truncation parameter ϵ were also tried and qualitatively similar results were obtained. In addition, noticing that the true density and score function are known in this case, we also investigated the test with no truncation for similar choices of γ_1 values. The corresponding results are reported in Table 2. These Monte Carlo results indicate that, given information on the density and score, the martingale transformation yields good size and power for the proposed testing procedure in finite sample. The cases with estimated scores and density for this model are examined and reported below.

The remaining Monte Carlo experiments are based on the following two sample model

$$\begin{cases} y_{1i} = \alpha_1 + \sigma_1 u_i, \ i = 1, \ldots.., n_1, \\ y_{2i} = \alpha_2 + \sigma_2 v_i, \ i = 1, \ldots.., n_2, \end{cases} \tag{2}$$

for different values of $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$. In particular, we report results based on the following two sets of parameter values

$$\text{Location Shift} \quad : \quad \alpha_1 = 1, \alpha_2 = 0, \sigma_1 = \sigma_2 = 1, \tag{3}$$

$$\text{Location} - \text{Scale Shift} \quad : \quad \alpha_1 = 1, \alpha_2 = 0, \sigma_1 = 2, \sigma_2 = 1, \tag{4}$$

where u_t, v_t are iid $\mathcal{N}$(0,1) random variates. Alternative values of $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ have also been considered and, again, qualitatively similar results were obtained. As shown in the previous discussion, the two samples can be pooled into one regression by application of dummy variables. When the parameters take the first set of values, (2) is a location shift model. The null hypothesis of a location shift model can be tested by the procedure given in Section 5.3. When the data is generated based on the second set parameters, (2) is a location-scale model. The location-scale model hypothesis can be tested by the procedure given in Section 5.1. Table 3 reports the empirical size of these tests for different combinations of n_1 and n_2. We can see that the test has pretty good size properties in finite sample. These Monte Carlo results, together with the results on the heteroskedasticity test in Tables 1 and 2, confirm the effectiveness of the martingale transformation in quantile regression inference.

The above Monte Carlo experiments use the true density and score. Obviously it is also important to evaluate the effect of nonparametric nuisance parameter estimation on the performance of the proposed tests. In the rest of our Monte Carlo experiments, we estimate $F^{-1}(s)$ and $\varphi_0(s)$ using the approach described in the text. For the score function $\dot{g}$, we employ the adaptive kernel estimator of Portnoy and Koenker (1989).

The kernel estimation procedures for these nuisance functions are nonparametric and therefore obviously entail choices of bandwidth values. Unsuitable bandwidth selection can produce poor results. However, under broad conditions on the convergence rate of the bandwidth parameters, the nonparametric estimates are consistent and testing procedures using different bandwidth choices are (first order) asymptotically equivalent, although the finite sample performance of these tests can vary considerably with bandwidth choice. Extensive simulations have been conducted in the literature to show the importance of bandwidth choice on estimation and testing procedure that use nonparametric estimates.

It was anticipated that the estimation of $\varphi_0(t)$, used in the standardization step, would exert important influence on the finite sample performance of our tests. This is confirmed in the simulations. For this reason, we pay particular attention to the bandwidth choice in density estimation. A bandwidth rule that Hall and Sheather (1988) suggested based on Edgeworth expansion for studentized quantiles is

$$h_{HS} = n^{-1/3} z_\alpha^{2/3} [1.5 s(t)/s''(t)]^{1/3},$$

where z_α satisfies $\Phi(z_\alpha) = 1 - \alpha/2$ for the construction of $1-\alpha$ confidence intervals, and $s(t) = \varphi_0(t)^{-1}$. In the absence of other information about the form of $s(\cdot)$, we plug in the Gaussian model to select bandwidth and obtain

$$h_{HS} = n^{-1/3} z_\alpha^{2/3} [1.5 \phi^2(\Phi^{-1}(t))/(2(\Phi^{-1}(t))^2 + 1)]^{1/3}.$$

Another bandwidth selection has been proposed by Bofinger (1975). The Bofinger bandwidth h_B was derived based on minimizing the mean squared error of the density estimator and is of order $n^{-1/5}$:

$$h_B = n^{-1/5} [4.5 s^2(t)/(s''(t))^2]^{1/5}.$$

Again, we plug in the Gaussian density and obtain the following bandwidth that has been widely used in practice

$$h_B = n^{-1/5} [4.5 \phi^4(\Phi^{-1}(t))/(2(\Phi^{-1}(t))^2 + 1)^2]^{1/5}.$$

To focus attention on the effect of $\varphi_n(s)$, we first conduct Monte Carlo experiments where only the density function is estimated while the true score function is used, the Monte Carlo results of the heteroscedasticity test are reported in Table 4. From these results, we found over-rejection with the Hall-Sheather bandwidth, and under-rejection when the Bofinger bandwidth was used. Such a finding is consistent with the fact that the Bofinger bandwidth is eventually much larger than the Hall/Sheather bandwidth. (Notice that both of these bandwidth choices are varying over $t \in \mathcal{T}$. For comparison purposes, we also tried using bandwidth choices $h = n^{1/5}$, $n^{1/4}$, and $n^{1/3}$. All these bandwidth values are constant over the whole range of quantiles. The sampling performance of tests using a constant bandwidth turned out to be poor, and are inferior to bandwidth choices such as the Hall/Sheather or Bofinger bandwidth that varies over the quantiles. For this reason, in the Monte Carlo that follows, we focus on the Hall/Sheather and Bofinger bandwidth, and their variants.)

Finally we consider the estimation of both the density and the score in our tests. For the density estimation, besides the Hall/Sheather bandwidth h_{HS} and the Bofinger bandwidth h_B, we also considered several bandwidth choices which are variants of h_{HS} and h_B. Giving the above findings that the Hall-Sheather bandwidth over-rejects and the Bofinger bandwidth under-rejects, most of the bandwidths that we consider take values between h_{HS} and h_B. We denote the variants of h_B as $h_{\theta B}$, $h_{\theta B} = \theta h_B$, where θ is a scalar, we considered $\theta = 0.4$, 0.5, 0.6, 0.7, 0.8, and 1.2, in our Monte Carlo and representative results are reported. (We have also considered bandwidth of the form $h_{\delta HS} = \delta h_{HS}$ for different values of δ (mostly > 1), the results are qualitatively similar to $h_{\theta B}$ and thus we report $h_{\theta B}$ only.)

The score function was estimated by the method of Portnoy and Koenker (1989) and we choose the Silverman (1986) bandwidth in our Monte Carlo. Our simulation results show that the tests are more affected by the estimation of the $\varphi_0(t)$ than that of the score. Intuitively, the estimator of the density $\varphi_0(t)$ plays the role of a scalar and thus has the largest influence. The Monte Carlo results also indicates that the method of Portnoy and Koenker (1989) coupled with the Silverman bandwidth has reasonably good performance.

Table 5 reports the Monte Carlo results for the heteroscedasticity (location-shift) test with these bandwidth selections, and Table 6 gives the corresponding results of the location-scale test. The Monte Carlo evidence reconfirmed the fact that the bandwidth choice does have an important influence on the finite sample performance of these tests. For the location-scale test, we found that the Bofinger bandwidth h_B to be a reasonable choice. For the heteroscedasticity test, we again found over-rejection for h_{HS}, and under-rejection with h_B. From the Monte Carlo results, at least for the model and the nonparametric methods used here, it seems that the Hall/Sheather bandwidth provides a good lower bound in bandwidth selection, and the Bofinger bandwidth provides a good upper bound. Among different variants of h_B, the bandwidth $h_{0.6B}$ seems to be roughly optimal for the heteroskedasticity tests. Similarly, among variants of h_{HS}, we found that $h_{3.7HS} = 3.7h_{HS}$ to be the best.

All these Tables also show that: as the sample sizes increase, the tests do have improved size and power properties, corroborating the asymptotic theory. In summary, the Monte Carlo results indicate that, by choosing appropriate bandwidth, the proposed tests have reasonable size and power properties.

Asymptotic Critical Values

	$\varepsilon = 0.05$			$\varepsilon = 0.1$			$\varepsilon = 0.15$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
$p = 1$	2.721	2.140	1.872	2.640	2.102	1.833	2.573	2.048	1.772
$p = 2$	4.119	3.393	3.011	4.034	3.287	2.946	3.908	3.199	2.866
$p = 3$	5.350	4.523	4.091	5.267	4.384	3.984	5.074	4.269	3.871
$p = 4$	6.548	5.560	5.104	6.340	5.430	4.971	6.148	5.284	4.838
$p = 5$	7.644	6.642	6.089	7.421	6.465	5.931	7.247	6.264	5.758
$p = 6$	8.736	7.624	7.047	8.559	7.412	6.852	8.355	7.197	6.673
$p = 7$	9.876	8.578	7.950	9.573	8.368	7.770	9.335	8.125	7.536
$p = 8$	10.79	9.552	8.890	10.53	9.287	8.662	10.35	9.044	8.412
$p = 9$	11.81	10.53	9.820	11.55	10.26	9.571	11.22	9.963	9.303
$p = 10$	12.91	11.46	10.72	12.54	11.17	10.43	12.19	10.85	10.14
$p = 11$	14.03	12.41	11.59	13.58	12.10	11.29	13.27	11.77	10.98
$p = 12$	15.00	13.34	12.52	14.65	13.00	12.20	14.26	12.61	11.86
$p = 13$	15.93	14.32	13.37	15.59	13.90	13.03	15.22	13.48	12.69
$p = 14$	16.92	15.14	14.28	16.52	14.73	13.89	16.12	14.34	13.48
$p = 15$	17.93	16.11	15.19	17.53	15.67	14.76	17.01	15.24	14.36
$p = 16$	18.85	16.98	16.06	18.46	16.56	15.65	17.88	16.06	15.22
$p = 17$	19.68	17.90	16.97	19.24	17.44	16.53	18.78	16.93	16.02
$p = 18$	20.63	18.83	17.84	20.21	18.32	17.38	19.70	17.80	16.86
$p = 19$	21.59	19.72	18.73	21.06	19.24	18.24	20.53	18.68	17.70
$p = 20$	22.54	20.58	19.62	22.02	20.11	19.11	21.42	19.52	18.52

	$\varepsilon = 0.2$			$\varepsilon = 0.25$			$\varepsilon = 0.3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
$p = 1$	2.483	1.986	1.730	2.420	1.923	1.664	2.320	1.849	1.602
$p = 2$	3.742	3.100	2.781	3.633	3.000	2.693	3.529	2.904	2.602
$p = 3$	4.893	4.133	3.749	4.737	4.018	3.632	4.599	3.883	3.529
$p = 4$	6.023	5.091	4.684	5.818	4.948	4.525	5.599	4.807	4.365
$p = 5$	6.985	6.070	5.594	6.791	5.853	5.406	6.577	5.654	5.217
$p = 6$	8.147	6.985	6.464	7.922	6.760	6.241	7.579	6.539	6.024
$p = 7$	9.094	7.887	7.299	8.856	7.611	7.064	8.542	7.357	6.832
$p = 8$	10.03	8.775	8.169	9.685	8.510	7.894	9.413	8.211	7.633
$p = 9$	10.90	9.672	9.018	10.61	9.346	8.737	10.27	9.007	8.400
$p = 10$	11.89	10.52	9.843	11.48	10.17	9.517	11.15	9.832	9.192
$p = 11$	12.85	11.35	10.66	12.48	10.99	10.28	12.06	10.62	9.929
$p = 12$	13.95	12.22	11.48	13.54	11.82	11.11	12.96	11.43	10.74
$p = 13$	14.86	13.09	12.31	14.34	12.66	11.93	13.82	12.24	11.51
$p = 14$	15.69	13.92	13.11	15.26	13.46	12.67	14.64	13.03	12.28
$p = 15$	16.55	14.77	13.91	16.00	14.33	13.47	15.46	13.85	13.05
$p = 16$	17.41	15.58	14.74	16.81	15.09	14.26	16.25	14.61	13.78
$p = 17$	18.19	16.43	15.58	17.59	15.95	15.06	17.04	15.39	14.54
$p = 18$	19.05	17.30	16.37	18.49	16.78	15.83	17.85	16.14	15.30
$p = 19$	19.96	18.09	17.17	19.40	17.50	16.64	18.78	16.94	16.05
$p = 20$	20.81	18.95	17.97	20.14	18.30	17.38	19.48	17.74	16.79

TABLE 1: Size and Power of the Heteroskedasticity Test (Truncated, $\epsilon = 0.05$)

	Size	Power						
n	$\gamma_1 = 0$	$\gamma_1 = 0.1$	$\gamma_1 = 0.2$	$\gamma_1 = 0.3$	$\gamma_1 = 0.5$	$\gamma_1 = 1$	$\gamma_1 = 2$	$\gamma_1 = 5$
100	0.006	0.134	0.377	0.729	0.974	0.981	0.990	0.999
200	0.054	0.269	0.77	0.977	0.999	1.000	1.000	1.000
300	0.052	0.383	0.931	1.000	1.000	1.000	1.000	1.000
400	0.052	0.549	0.989	1.000	1.000	1.000	1.000	1.000
500	0.052	0.616	1.000	1.000	1.000	1.000	1.000	1.000
800	0.051	0.829	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 2: Size and Power of the Heteroskedasticity Test (No Truncation)

	Size	Power						
n	$\gamma_1 = 0$	$\gamma_1 = 0.1$	$\gamma_1 = 0.2$	$\gamma_1 = 0.3$	$\gamma_1 = 0.5$	$\gamma_1 = 1$	$\gamma_1 = 2$	$\gamma_1 = 5$
100	0.069	0.129	0.347	0.626	0.682	0.96	0.98	0.989
200	0.073	0.258	0.753	0.982	1.000	1.000	1.000	1.000
300	0.067	0.395	0.92	0.999	1.000	1.000	1.000	1.000
400	0.063	0.51	0.987	1.000	1.000	1.000	1.000	1.000
500	0.056	0.595	0.999	1.000	1.000	1.000	1.000	1.000
800	0.052	0.802	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 3: Application to The Two-Sample Models

Case 1: Location Shift						Case 2: Location-Scale Shift					
$\alpha_1 = 1, \alpha_2 = 0, \sigma_1 = \sigma_2 = 1$						$\alpha_1 = 1, \alpha_2 = 0, \sigma_1 = 2, \sigma_2 = 1$					
n_1	n_2	size	n_1	n_2	size	n_1	n_2	size	n_1	n_2	size
100	100	0.074	100	200	0.060	100	100	0.153	100	200	0.179
150	150	0.080	100	300	0.086	150	150	0.158	100	300	0.196
200	200	0.064	150	300	0.055	200	200	0.169	150	300	0.175
250	250	0.054	200	300	0.056	250	250	0.172	200	300	0.183
300	300	0.053	250	350	0.054	300	300	0.141	250	350	0.150
400	400	0.051	300	500	0.053	400	400	0.088	300	500	0.102

TABLE 4: The Heteroskedasticity Test (Estimated Density)

	h_{HS}				h_B			
	Size	Power			Size	Power		
n	$\gamma_1 = 0$	0.2	0.5	1	$\gamma_1 = 0$	0.2	0.5	1
100	0.29	0.71	0.989	0.998	0.026	0.130	0.392	0.762
200	0.22	0.842	1.000	1.000	0.031	0.197	0.606	0.882
300	0.18	0.92	1.000	1.000	0.031	0.316	0.997	1.000
400	0.16	0.97	1.000	1.000	0.037	0.561	0.999	1.000
500	0.13	1.000	1.000	1.000	0.039	0.688	1.000	1.000
800	0.07	1.000	1.000	1.000	0.044	0.812	1.000	1.000

TABLE 5: The Heteroskedasticity Test (Estimated Density and Scores)

	h_{HS}				h_B			
	Size	Power			Size	Power		
n	$\gamma_1 = 0$	0.2	0.5	1	$\gamma_1 = 0$	0.2	0.5	1
100	0.45	0.723	0.99	1.000	0.009	0.053	0.197	0.545
200	0.21	0.877	1.000	1.000	0.013	0.109	0.772	0.949
300	0.195	0.952	1.000	1.000	0.019	0.229	0.985	0.992
400	0.186	0.995	1.000	1.000	0.023	0.412	0.997	0.998
500	0.173	1.000	1.000	1.000	0.029	0.565	1.000	1.000
800	0.102	1.000	1.000	1.000	0.041	0.792	1.000	1.000

TABLE 5(Continued): The Heteroskedasticity Test (Estimated Density and Scores)

	$h_{0.5B}$				$h_{0.6B}$				$h_{0.7B}$			
	Size	Power			Size	Power			Size	Power		
n	$\gamma_1 = 0$	0.2	0.5	1	$\gamma_1 = 0$	0.2	0.5	1	$\gamma_1 = 0$	0.2	0.5	1
100	0.101	0.264	0.804	0.898	0.035	0.211	0.755	0.820	0.016	0.126	0.641	0.828
200	0.070	0.48	0.988	0.999	0.041	0.406	0.990	0.989	0.022	0.280	0.964	0.992
300	0.062	0.622	0.998	1.000	0.043	0.665	1.000	1.000	0.029	0.416	0.998	1.000
400	0.054	0.812	1.000	1.000	0.043	0.809	1.000	1.000	0.035	0.632	1.000	1.000
500	0.054	0.916	1.000	1.000	0.045	0.911	1.000	1.000	0.040	0.814	1.000	1.000
800	0.053	0.982	1.000	1.000	0.050	0.969	1.000	1.000	0.049	0.924	1.000	1.000

TABLE 6: Empirical Size of Location-Scale Test (Estimated Density and Scores)

n_1	n_2	h_{HS}	h_B	$h_{0.6B}$	$h_{0.7B}$	$h_{1.2B}$
50	50	0.616	0.028	0.063	0.054	0.011
75	75	0.603	0.033	0.086	0.069	0.029
100	100	0.589	0.038	0.098	0.069	0.038
150	150	0.538	0.079	0.112	0.095	0.036
200	200	0.511	0.079	0.123	0.112	0.042
250	250	0.507	0.065	0.126	0.120	0.048
300	300	0.456	0.078	0.135	0.127	0.047
400	400	0.415	0.088	0.138	0.106	0.053
500	500	0.406	0.105	0.145	0.103	0.079
600	600	0.312	0.085	0.101	0.092	0.069
700	700	0.226	0.072	0.089	0.085	0.065