INFERENCE ON THE QUANTILE REGRESSION PROCESS

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Abstract. Tests based on the quantile regression process can be formulated like the classical Kolmogorov-Smirnov and Cramér-von-Mises tests of goodness-of-fit employing the theory of Bessel processes as in Kiefer (1959). However, it is frequently desirable to formulate hypotheses involving unknown nuisance parameters, thereby jeopardizing the distribution free character of these tests. We characterize this situation as "the Durbin problem" since it was posed in Durbin (1973), for parametric empirical processes.

In this paper we consider an approach to the Durbin problem involving a martingale transformation of the parametric empirical process suggested by Khmaladze (1981) and show that it can be adapted to a wide variety of inference problems involving the quantile regression process. In particular, we suggest new tests of the location shift and location-scale shift models that underlie much of classical econometric inference.

The methods are illustrated with a reanalysis of data on unemployment durations from the Pennsylvania Reemployment Bonus Experiments. The Pennsylvania experiments, conducted in 1988-89, were designed to test the efficacy of cash bonuses paid for early reemployment in shortening the duration of insured unemployment spells.

1. Introduction

Quantile regression, as introduced by Koenker and Bassett (1978), is gradually evolving into a comprehensive approach to the statistical analysis of linear and nonlinear response models for conditional quantile functions. Just as classical linear regression methods based on minimizing sums of squared residuals enable one to estimate models for conditional mean functions, quantile regression methods based on minimizing asymmetrically weighted *absolute* residuals offer a mechanism for estimating models for the conditional median function, and the full range of other conditional quantile functions. By supplementing least squares estimation of conditional mean functions with techniques for estimating a full family of conditional quantile functions, quantile regression is capable of providing a much more complete statistical analysis of the stochastic relationships among random variables.

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There is already a well-developed theory of asymptotic inference for many important aspects of quantile regression. Rank-based inference based on the approach of Gutenbrunner, Jurečková, Koenker, and Portnoy (1993) appears particularly attractive for a wide variety of quantile regression inference problems including the construction of confidence intervals for individual quantile regression parameter estimates. There has also been considerable attention devoted to various resampling strategies. See e.g. Hahn (1995), Horowitz (1998), Bilias, Chen, and Ying (1999), He and Hu (1999). In Koenker and Machado (1999) some initial steps have been taken toward a theory of inference based on the entire quantile regression process. These steps have clarified the close tie to classical Kolmogorov-Smirnov goodness of fit results, and related literature p-sample goodness-of-fit tests based on Bessel processes initiated by Kiefer (1959).

This paper describes some further steps in this direction. These new steps depend crucially on an ingenious suggestion by Khmaladze (1981) for dealing with tests of composite null hypotheses based on empirical processes. Khmaladze's results were rather slow to percolate into statistics generally, but the approach has recently played an important role in work on specification tests of the form of the regression function by Stute, Thies, and Zhu (1998) and Koul and Stute (1999). In econometrics, Bai (1998) was apparently the first to recognize the potential importance of these methods, considering tests of model specification based on the empirical process in parametric time series models.

In contrast to the prior literature, which has focused on tests based directly on the empirical process, we consider tests based on the quantile regression process. We focus primarily on tests of the hypothesis that the regression effect of covariates exerts either a pure location shift, or a location-scale shift of the conditional distribution of the response, versus alternatives under which covariates may alter the shape of the conditional distribution as well as its location and scale. These tests may be viewed as semiparametric in the sense that we need not specify a parametric form for the error distribution.

Khmaladze's martingale transformation provides a general strategy for purging the effect of estimated nuisance parameters from the first order asymptotic representation of the empirical process and related processes and thereby restoring the feasibility of "asymptotically distribution free" tests. We will argue that the approach is especially attractive in the quantile regression setting and is capable of greatly expanding the scope of inferential methods described in earlier work. Although there are close formal connections between the theory of tests based on the quantile regression process and the existing literature there are some significant differences as well. Some of these differences are immediately apparent from the classical Bahadur-Kiefer theorems and the literature on confidence bands for QQ-plots, *e.g.* Nair (1982). But the principal differences emerge from the regression specification of the conditional quantile functions. In particular, we introduce tests of the classical regression specifications that the covariates affect only the location, or the location and scale, of the conditional distribution of the response variable without requiring a parametric specification of the innovation distribution.

An alternative general approach to tests based on empirical processes with estimated parameters entails resampling the test statistic under conditions consistent with the null hypothesis to obtain critical values. The origins of this approach may be traced to Bickel (1969). Romano (1988) describes implementations for a variety of problems. Andrews (1997) introduces a conditional Kolmogorov-Smirnov test for model specification in parametric settings and uses resampling to obtain critical values, and Abadie (2000) develops related methods for investigating treatment effects in the two sample setting. We hope to explore this approach to quantile regression inference in subsequent work.

The remainder of the paper is organized as follows. In the next section, we introduce a general paradigm for quantile regression inference focusing initially on the canonical two-sample treatment-control model. In Section 3, we briefly introduce Khmaladze's approach to handling empirical processes with estimated nuisance parameters. Section 4 extends this approach to general problems of inference based on the quantile regression process. Section 5 treats some practical problems of implementing the tests. Section 6 describes an empirical application to the analysis of unemployment durations. Section 7 contains some concluding remarks.

2. The Inference Paradigm

To motivate our approach it is helpful to begin by reconsidering the classical two-sample treatment-control problem. In the simplest possible setting we can imagine a random sample of size, n, drawn from a homogeneous population and randomized into n_1 treatment observations, and n_0 control observations. We observe a response variable, Y_i, and are interested in evaluating the effect of the treatment on this response.

In a typical clinical trial application, for example, the treatment would be some form of medical procedure, and Y_i, might be log survival time. In the initial analysis we might be satisfied to know simply the mean treatment effect, that is, the difference in means for the two groups. This we could evaluate by "running the regression" of the observed y_i's on an treatment indicator variable. Of course this regression would presume, implicitly, that the variability of the two subsamples was the exactly same; this observation opens the door to the possibility that the treatment alters other features of the response distribution as well. Although we are accustomed to thinking about regression models in which the covariates affect only the location of the conditional distribution of the response – this is the force of the iid error assumption – there is no compelling reason to believe that covariates must only operate in this restrictive fashion.

2.1. **Quantile Treatment Effects.** Lehmann (1974) introduced the following general formulation of the two sample treatment effect in which the treatment response is assumed to be $x + \Delta(x)$ when the response of the untreated subject would be x. In this model, the distribution G of the treatment responses is that of the random variable $X + \Delta(X)$ where X is distributed according to the control distribution F.

Doksum (1974) provides a detailed axiomatic analysis of this formulation, showing that if we define $\Delta(x)$ as the "horizontal distance" between F and G at x, so $F(x) = G(x + \Delta(x))$ then $\Delta(x)$ is uniquely defined and can be expressed as $\Delta(x) = G^{-1}(F(x)) - x$. Changing variables, so $\tau = F(x)$ we obtain what we will call the *quantile treatment effect*,

$$\delta(\tau) = \Delta(F^{-1}(\tau)) = G^{-1}(\tau) - F^{-1}(\tau).$$

In the two sample setting this quantity is naturally estimable by

$$\hat{\delta}(\tau) = G_{n_1}^{-1}(\tau) - F_{n_0}^{-1}(\tau)$$

where G_{n_1}, F_{n_0} denote the empirical distribution functions of the treatment and control observations respectively, and $F_n^{-1} = \inf\{x | F_n(x) \geq \tau\}$, as usual. Since we cannot observe subjects in *both* the treated and control states – this platitude may be regarded as the fundamental "uncertainty principle" underlying the "causal effects" literature – it seems reasonable to regard $\delta(\tau)$ as a complete description of the treatment effect.[1]

Of course, it is possible that the two distributions differ only by a location shift, so $\delta(\tau) = \delta_0$, or that they differ by a scale shift so $\delta(\tau) = \delta_1 F^{-1}(\tau)$ or that they differ by a location and scale shift so $\delta(\tau) = \delta_0 + \delta_1 F^{-1}(\tau)$. Indeed, most of the regression literature deals with just such models. These hypotheses are all nicely nested within Lehmann's general framework. And yet, as we shall see, testing them against the general alternatives represented by the Lehmann-Doksum quantile treatment effect poses some challenges.

2.2. **Inference on the Quantile Regression Process.** In the two-sample treatment-control model there are a multitude of tests designed to answer the question: "Is the treatment effect significant." The most familiar of these, like the two-sample Student-t and Mann-Whitney-Wilcoxon tests are designed to reveal location shift alternatives. Others are designed for scale shift alternatives. Still others, like the two sample Kolmogorov-Smirnov test, are intended to encompass omnibus non-parametric

[1] The foregoing discussion can be interpreted in terms of the "potential outcomes" of the causal effects literature. The Lehmann formulation essentially assumes that the ranks of the control observations would be preserved were they to be treated. Heckman, Smith, and Clements (1997) consider a model which allows a specified degree of "slippage" in the quantile ranks of the distributions in the context of a bounds analysis. Abadie, Angrist, and Imbens (2001) have suggested an IV estimator for quantile treatment effects when the treatment is endogonous. Oja (1981) considers orderings of distributions based on location, scale, skewness, kurtosis, etc., based on the function $\Delta(x)$.

alternatives. When the non-parametric null is posed in a form free of nuisance parameters we have an elegant distribution-free theory for a variety of tests, including the Kolmogorov-Smirnov test, that are based on the empirical distribution function.

Non-parametric testing in the presence of nuisance parameters under the null, however, poses some new problems. Suppose, for example, that we wish to test the hypothesis that the response distribution under the treatment, G, differs from the control distribution, F, by a pure location shift, that is for all $\tau \in [0,1]$,

$$G^{-1}(\tau) = F^{-1}(\tau) + \delta_0$$

for some real δ_0, or that they differ by a location-scale shift, so,

$$G^{-1}(\tau) = \delta_1 F^{-1}(\tau) + \delta_0.$$

In such cases we can easily estimate the nuisance parameters, δ_0, δ_1, but the introduction of the estimated parameters into the asymptotic theory of the empirical process destroys the distribution free character of the resulting Kolmogorov-Smirnov test. Analogous problems arise in the theory of the one sample Kolmogorov-Smirnov test when there are estimated parameters under the null, and have been considered by Durbin (1973) and others.

In the general linear quantile regression model specified as,

$$Q_{y|x}(\tau|x) = x^\top \beta(\tau)$$

such models may be represented by the linear hypothesis:

$$\beta(\tau) = \alpha + \gamma F_0^{-1}(\tau)$$

for α and γ in $\mathbb{R}^p$ and F_0^{-1} a univariate quantile function. Thus, all p coordinates of the quantile regression coefficient vector are required to be affine functions of the same univariate quantile function, F_0^{-1}. Such models may be viewed as arising from linearly heteroscedastic model

$$y_i = x_i^\top \alpha + (x_i^\top \gamma) u_i$$

with the $\{u_i\}$ iid from the df F_0. They can be estimated by solving the linear programming problem,

$$\min_{b \in \mathbb{R}^p} \sum_{i=1}^n \rho_\tau(y_i - x_i^\top b)$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$.

When the vectors α and γ are fully specified under the null hypothesis tests may be formulated as suggested in Koenker and Machado (1999). However, when they are left unspecified, and therefore must be estimated, this alters fundamentally the asymptotic behavior of the tests and leads us to Khmaladze (1981).

3. A Heuristic Introduction to Khmaladzation

Arguably the most fundamental problem of statistical inference is the classical goodness-of-fit problem: given a random sample, $\{y_1, \ldots, y_n\}$, on a real-valued random variable, Y, test the hypothesis that Y comes from distribution function, F_0. Tests based on the empirical distribution function, $F_n(y) = n^{-1} \sum I(Y_i \leq y)$, like the Kolmogorov-Smirnov statistic

$$K_n = \sup_{y \in \mathbb{R}} \sqrt{n} |F_n(y) - F_0(y)|,$$

are especially attractive because they are asymptotically distribution-free. The limiting distribution of K_n is the same for every continuous distribution function F_0. This remarkable fact follows by noting that the process, $\sqrt{n}(F_n(y) - F_0(y))$, can be transformed to a test of uniformity, via the change of variable, $y \to F_0^{-1}(t)$, based on

$$v_n(t) = \sqrt{n}(F_n(F_0^{-1}(t)) - t).$$

It is well known that $v_n(t)$ converges weakly to a Brownian bridge process, $v_0(t)$.

3.1. **The Durbin Problem.** It is rare in practice, however, that we are willing to specify F_0 completely. More commonly, our hypothesis places F in some parametric family $\mathcal{F}_\theta$ with $\theta \in \Theta \subseteq \mathbb{R}^p$. We are thus led to consider, following Durbin (1973), the parametric empirical process,

$$U_n(y) = \sqrt{n}(F_n(y) - F_{\hat{\theta}_n}(y)).$$

Again changing variables, so $y \to F_{\theta_0}^{-1}(t)$, we may equivalently consider

$$\hat{v}_n(t) = \sqrt{n}(G_n(t) - G_{\hat{\theta}_n}(t))$$

where $G_n(t) = F_n(F_{\theta_0}^{-1}(t))$ and $G_{\hat{\theta}_n}(t) = F_{\hat{\theta}_n}(F_{\theta_0}^{-1}(t))$ so $G_{\theta_0}(t) = t$. Under mild conditions on the sequence $\{\hat{\theta}_n\}$ we have the linear (Bahadur) representation,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \int_0^1 h_0(s) dv_n(s) + o_p(1).$$

So provided the mapping $\theta \to G_\theta$ has a Fréchet derivative, $g = g_{\theta_0}$, that is, $\sup_t |G_{\theta+h}(t) - G_\theta(t) - h^\top g(t)| = o(||h||)$ as $h \to 0$, see van der Vaart (1998, p.278), we may write,

$$G_{\hat{\theta}_n}(t) = t + (\hat{\theta}_n - \theta_0)^\top g(t) + o_p(||\hat{\theta}_n - \theta_0||),$$

and thus obtain, with $r_n(t) = o_p(1)$,

$$\hat{v}_n(t) = v_n(t) - g(t)^\top \int_0^1 h_0(s) dv_n(s) + r_n(t), \tag{3.1}$$

which converges weakly to the Gaussian process,

$$u_0(t) = v_0(t) - g(t)^\top \int_0^1 h_0(s) dv_0(s).$$

The necessity of estimating θ_0 introduces the drift component $g(t)^\top \int_0^1 h_0(s)dv_0(s)$. Instead of the simple Brownian bridge process, $v_0(t)$, we obtain a more complicated Gaussian process with covariance function

$$Eu_0(t)u_0(s) = s \wedge t - ts - g(t)^\top \mathcal{H}_0(s) - g(s)^\top \mathcal{H}_0(t) + g(s)^\top \mathcal{J}_0 g(t)$$

where $\mathcal{H}_0(t) = \int_0^t h_0(s)ds$ and $\mathcal{J}_0 = \int_0^1\int_0^1 h_0(t)h_0(s)^\top dtds$. When $\hat{\theta}_n$ is the mle, so $h_0(s) = -(E\nabla_\theta \psi)^{-1}\psi(F^{-1}(s))$ with $\psi = \nabla_\theta \log f$, the covariance function simplifies nicely to

$$Eu_0(t)u_0(s) = s \wedge t - ts - g(s)^\top \mathcal{I}_0 g(t)$$

where $\mathcal{I}_0$ denotes Fisher's information matrix. See Durbin (1973) and Shorack and Wellner (1986) for further details on this case.

The practical consequence of the drift term involving the function $g(t)$ is to invalidate the distribution-free character of the original test. Tests based on the parametric empirical process $\hat{v}_n(t)$ require special consideration of the process $u_0(t)$ and its dependence on F in each particular case. Koul (1992) and Shorack and Wellner (1986) discuss several leading examples. Durbin (1973) describes a general numerical approach based on Fourier inversion, but also expresses doubts about feasibility of the method when the parametric dimension of θ exceeds one. Although the problem of finding a viable, general approach to inference based on the parametric empirical process had been addressed by several previous authors, notably Darling (1955), and Kac, Kiefer, and Wolfowitz (1955), we will, in the spirit of Stigler's (1980) law of eponymy, refer to this as "the Durbin problem."

3.2. **Martingales and the Doob-Meyer Decomposition.** Khmaladze's general approach to the Durbin problem can be motivated as a natural elaboration of the Doob-Meyer decomposition for the parametric empirical process. The Doob-Meyer decomposition asserts that for any nonnegative submartingale, x, there exists an increasing right continuous predictable process, $a(t)$, such that $Ea(t) < \infty$, and a right continuous martingale m, such that

$$x(t) = a(t) + m(t) \qquad a.s.$$

See e.g. Fleming and Harrington (1991). For an extensive account of recent developments see Prakasa Rao (1999).

Let $X_1, \ldots, X_n$ be iid from F_0, so $Y_i = F_0(X_i), \quad i = 1, \ldots, n$ are iid uniform, $U[0,1]$. The empirical distribution function

$$G_n(t) = F_n(F_0^{-1}(t)) = n^{-1}\sum_{i=1}^n I(Y_i \le t).$$

viewed as a process, is a submartingale. We have an associated filtration $\mathcal{F}^{G_n} = \{\mathcal{F}_t^{G_n} : 0 \le t \le 1\}$ and the order statistics $Y_{(1)}, \ldots, Y_{(n)}$ are Markov times with respect to $\mathcal{F}^{G_n}$, that is $\{Y_{(i)} \le t\} = \{F_n(t) \ge i/n\} \in \mathcal{F}_t^{G_n}$.

The process $G_n(t)$ is Markov; Khmaladze notes that for $\Delta t \geq 0$,

$$n\Delta G_n(t) = n[G_n(t+\Delta t) - G_n(t)] \sim \text{Binomial}(n(1-G_n(t)), \Delta t/(1-t))$$

with $G_n(0) = 0$, thus

$$E(\Delta G_n(t)|\mathcal{F}_t^{G_n}) = \frac{1-G_n(t)}{1-t}\Delta t. \tag{3.2}$$

This suggests the decomposition

$$G_n(t) = \int_0^t \frac{1-G_n(s)}{1-s} ds + m_n(t).$$

That $m_n(t)$ is a martingale then follows from the fact that, $E(m_n(t)|\mathcal{F}_s^{G_n}) = m_n(s)$ using (3.2), and integrability of $m_n(t)$ follows from the inequality

$$\int_0^t \frac{1-G_n(s)}{1-s} ds \leq -\log(1-Y_{(n)}),$$

which implies a finite mean for the compensator, or predictable component. Substituting for $G_n(t)$ in (3.2) we have the classical Doob-Meyer decomposition of the empirical process v_n,

$$v_n(t) = w_n(t) - \int_0^t \frac{v_n(s)}{1-s} ds,$$

where $v_n(t) = \sqrt{n}(G_n(t) - t)$ and the normalized process $w_n(t) = \sqrt{n} m_n(t)$ converges weakly to a standard Brownian motion process, $w_0(t)$, by the argument of Khmaladze(1981, §2.6).

To extend this approach to the general parametric empirical process, we now let $g(t) = (t, \bar{g}(t)^\top)^\top = (t, g_1(t), \ldots g_m(t))^\top$ be a $(m+1)$-vector of real-valued functions on $[0,1]$. Suppose that the functions $\dot{g}(t) = dg(t)/dt$ are linearly independent in a neighborhood of 1 so

$$C(t) \equiv \int_t^1 \dot{g}(s)\dot{g}(s)^\top ds$$

is non-singular, and consider the transformation

$$w_n(t) = v_n(t) - \int_0^t \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) dv_n(r) ds. \tag{3.3}$$

Here, $w_n(t)$ clearly depends upon the choice of g, and therefore differs from $w_n(t)$ defined above. But the abuse of notation may be justified by noting that in the special case $g(t) = t$, we have $C(s) = 1-s$, and $\int_s^1 \dot{g} dv_n(r) = -v_n(s)$ yielding the Doob-Meyer decomposition (3.2) as a special case. In the general case, the transformation

$$Q_g \varphi(t) = \varphi(t) - \int_0^t \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) d\varphi(r) ds$$

may be recognized as the residual from the prediction of $\varphi(t)$ based on the recursive least squares estimate using information from the interval $(t, 1]$. For functions in the span of g, the prediction is exact, that is, $Q_g g = 0$.

Now returning to the representation of the parametric empirical process, $\hat{v}_n(t)$, given in (3.1), using Khamaladze (1981, §4.2), we have,

$$\tilde{v}_n(t) = Q_g \hat{v}_n(t) = Q_g(v_n(t) + r_n(t)) = w_0(t) + o_p(1).$$

The transformation of the parametric empirical process annihilates the g component of the representation and in so doing restores the feasibility of asymptotically distribution free tests based on the transformed process $\tilde{v}_n(t)$.

3.3. **The Parametric Empirical Quantile Process.** What can be done for tests based on the parametric empirical process can also be adapted for tests based on the parametric empirical *quantile* process. In some respects the quantile domain is actually more convenient. Suppose $\{y_1, \ldots, y_n\}$ constitute a random sample on Y with distribution function F_Y. Consider testing the hypothesis, $F_Y(y) = F_0((y - \mu_0)/\sigma_0)$, so, $\alpha(\tau) \equiv F_Y^{-1}(\tau) = \mu_0 + \sigma_0 F_0^{-1}(\tau)$. Given the empirical quantile process

$$\hat{\alpha}(\tau) = \inf\{a \in \mathbb{R} | \sum_{i=1}^{n} \rho_\tau(y_i - a) = \min!\}$$

and known parameters $\theta_0 = (\mu_0, \sigma_0)$ tests may be based on

$$v_n(\tau) = \sqrt{n}\varphi_0(\tau)(\hat{\alpha}(\tau) - \alpha(\tau))/\sigma_0 \Rightarrow v_0(\tau)$$

where $\varphi_0(\tau) = f_0(F_0^{-1}(\tau))$ and $v_0(\tau)$ is the Brownian bridge process.

To test our hypothesis when θ is unknown, set $\xi(t) = (1, F_0^{-1}(t))^\top$ and for an estimator $\hat{\theta}_n$ satisfying,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \int_0^1 h_0(s) dv_n(s) + o_p(1)$$

set $\tilde{\alpha}(t) = \hat{\mu} + \hat{\sigma}_0 F_0^{-1}(t) = \hat{\theta}_n^\top \xi(t)$. Then

$$\begin{aligned} \hat{v}_n(t) &= \sqrt{n}\varphi_0(t)(\hat{\alpha}(t) - \tilde{\alpha}(t)/\sigma_0 \\ &= v_n(t) - \sqrt{n}\varphi_0(t)(\hat{\theta} - \theta_0)^\top \xi(t)/\sigma_0 \\ &= v_n(t) - \sigma_0^{-1}\varphi_0(t)\xi(t)^\top \int_0^1 h_0(s) dv_n(s) + o_p(1) \end{aligned} \tag{3.4}$$

Thus, if we take $g(t) = (t, \xi(t)^\top \varphi_0(t))^\top$, we obtain, $\dot{g}(t) = (1, \dot{f}/f, 1 + F_0^{-1}(t)\dot{f}/f)^\top$ where $\dot{f}/f$ is evaluated at $F_0^{-1}(t)$. Given the representation (3.4) and the fact that $\xi(t)$ lies in the linear span of g, we may again apply Khmaladze's martingale transformation to obtain,

$$\tilde{v}_n(t) = Q_g \hat{v}_n(t),$$

which can then be shown to converge to the standard Brownian motion process. As we have suggested in Section 2, we would like to consider a two sample version of the foregoing problem in which we leave the precise functional form of the distribution F, and therefore the form of the function, g, unspecified under the null. This is a special case of the general quantile regression tests introduced in the next section.

4. Quantile Regression Inference

The classical linear regression model asserts that the conditional mean of the response, y_i, given covariates, x_i, may be expressed as a linear function of the covariates. That is, there exists a $\beta \in \mathbb{R}^p$ such that,

$$E(y_i|x_i) = x_i^\top \beta.$$

The linear quantile regression model asserts, analogously, that the conditional quantile functions of y_i given x_i are linear in covariates,

$$F^{-1}_{y_i|x_i}(\tau|x_i) = x_i^\top \beta(\tau) \tag{4.1}$$

for τ in some index set $\mathcal{T} \subset [0,1]$. The model (4.1) will be taken to be our basic maintained hypothesis. We will restrict attention to the case that $\mathcal{T} = [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1/2)$, and to facilitate asymptotic local power analysis we will consider sequences of models for which $\beta(\tau) = \beta_n(\tau)$ depends explicitly on the sample size, n.

A leading special case is the *location-scale shift model*,

$$F^{-1}_{y_i|x_i}(\tau|x_i) = x_i^\top \alpha + x_i^\top \gamma F_0^{-1}(\tau). \tag{4.2}$$

where $F_0^{-1}(\tau)$ denotes a univariate quantile function. Covariates affect both the location and scale of the conditional distribution of y_i given x_i in this model, but the covariates have no effect on the *shape* of the conditional distribution. Typically, the vectors $\{x_i\}$ "contain an intercept" so e.g., $x_i = (1, z_i^\top)^\top$ and (4.2) may be seen as arising from the linear model

$$y_i = x_i^\top \alpha + (x_i^\top \gamma) u_i$$

where the "errors" $\{u_i\}$ are iid with distribution function F_0. Further specializing the model, we may write,

$$x_i^\top \gamma = \gamma_0 + z_i^\top \gamma_1,$$

and the restriction, $\gamma_1 = 0$, then implies that the covariates affect only the *location* of the y_i's. We will call this model

$$F^{-1}_{y_i|x_i}(\tau|x_i) = x_i^\top \alpha + \gamma_0 F_0^{-1}(\tau) \tag{4.3}$$

the *location shift model*. Although this model underlies much of classical econometric inference, it posits a very narrowly circumscribed role for the x_i. In the remainder of this section we explore ways to test the hypotheses that the general linear quantile regression model takes either the location shift or location-scale shift form.

We will consider a linear hypothesis of the general form,

$$R\beta(\tau) - r = \Psi(\tau), \qquad \tau \in \mathcal{T}, \tag{4.4}$$

where R denotes a $q \times p$ matrix, $q \leq p, r \in \mathbb{R}^q$, and $\Psi(\tau)$ denotes a known function $\Psi : \mathcal{T} \to \mathbb{R}^q$. Consider tests based on the quantile regression process,

$$\hat{\beta}(\tau) = \operatorname{argmin}_{b \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(y_i - x_i^\top b)$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$. Under the location-scale shift form of the quantile regression model (4.2) we will have under mild regularity conditions,

$$\sqrt{n}\varphi_0(\tau)\Omega^{-1/2}(\hat{\beta}(\tau) - \beta(\tau)) \Rightarrow v_0(\tau) \tag{4.5}$$

where $v_0(\tau)$ now denotes a p-dimensional independent Brownian bridge process. Here $\beta(\tau) = \alpha + \gamma F^{-1}(\tau)$, and $\Omega = H_0^{-1} J_0 H_0^{-1}$ with $J_0 = \lim n^{-1} \sum x_i x_i^\top$, and $H_0 = \lim n^{-1} \sum x_i x_i^\top / \gamma^\top x_i$.

It then follows quite easily that under the null hypothesis (4.4),

$$v_n(\tau) = \sqrt{n}\varphi_0(\tau)(R\Omega R^\top)^{-1/2}(R\hat{\beta}(\tau) - r - \Psi(\tau)) \Rightarrow v_0(\tau),$$

so tests that are asymptotically distribution free can be readily constructed. Indeed, Koenker and Machado (1999) consider tests of this type when R constitutes an exclusion restriction so e.g., $R = [0 \vdots I_q], r = 0$, and $\Psi(\tau) = 0$. In such cases it is also shown that the nuisance parameters $\varphi_0(\tau)$ and Ω can be replaced by consistent estimates without jeopardizing the distribution free character of the tests.

To formalize the foregoing discussion we introduce the following regularity conditions, which closely resemble the conditions employed in Koenker and Machado. We will assume that the $\{y_i\}$'s are, conditional on x_i, independent with linear conditional quantile functions given by (4.1) and local, in a sense specified in A.3, to the location-scale shift model (4.2).

A. 1. *The distribution function F_0, in (4.2) has a continuous Lebesgue density, f_0, with $f_0(u) > 0$ on $\{u : 0 < F_0(u) < 1\}$.*

A. 2. *The sequence of design matrices $\{X_n\} = \{(x_i)_{i=1}^n\}$ satisfy:*

- **(i):** $x_{i1} \equiv 1 \qquad i = 1, 2, \ldots$
- **(ii):** $J_n = n^{-1} X_n^\top X_n \to J_0$, *a positive definite matrix.*
- **(iii):** $H_n = n^{-1} X_n^\top \Gamma_n^{-1} X_n \to H_0$, *a positive definite matrix where* $\Gamma_n = diag(\gamma^\top x_i)$.
- **(iv):** $\max_{i=1,\ldots,n} \| x_i \| = \mathcal{O}(n^{1/4} \log n)$

A. 3. *There exists a fixed, continuous function $\zeta(\tau) : [0,1] \to \mathbb{R}^q$ such that for samples of size n,*

$$R\beta_n(\tau) - r - \Psi(\tau) = \zeta(\tau)/\sqrt{n}.$$

As noted in Koenker and Machado (1999), conditions A.1 and A.2 are quite standard in the quantile regression literature. Somewhat weaker conditions are employed by Gutenbrunner, Jurečková, Koenker, and Portnoy (1993) in an effort to extend the theory further into the tails. But this isn't required for our present purposes, so we have reverted to conditions closer to those of Gutenbrunner and Jurečková (1992). Condition A.3 enables us to explore local asymptotic power of the proposed tests employing a rather general form for the local alternatives.

We can now state our first result. Proofs of all results appear in the appendix.

Theorem 1. *Let $\mathcal{T}$ denote the closed interval $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0, 1/2)$. Under conditions A.1-3*

$$v_n(\tau) \Rightarrow v_0(\tau) + \eta(\tau) \text{ for } \tau \in \mathcal{T}$$

where $v_0(\tau)$ denotes a q-variate standard Brownian bridge process and

$$\eta(\tau) = \varphi_0(\tau)(R\Omega R^\top)^{-1/2}\zeta(\tau).$$

Under the null hypothesis, $\zeta(\tau) = 0$, the test statistic

$$\sup_{\tau\in\mathcal{T}} \| v_n(\tau) \| \Rightarrow \sup_{\tau\in\mathcal{T}} \| v_0(\tau) \| .$$

Typically, even if the hypothesis is fully specified, it is necessary to estimate the matrix Ω and the function $\varphi_0(t) \equiv f_0(F_0^{-1}(t))$. Fortunately, these quantities can be replaced by estimates satisfying the following condition.

A. 4. *There exist estimators $\varphi_n(\tau)$ and Ω_n satisfying*

i.: $\sup_{\tau\in\mathcal{T}} |\varphi_n(\tau) - \varphi_0(\tau)| = o_p(1)$,
ii.: $||\Omega_n - \Omega|| = o_p(1)$.

Corollary 1. *The conclusions of Theorem 1 remain valid if $\varphi_0(\tau)$ and Ω are replaced by estimates satisfying condition A.4.*

Remark. An important class of applications of Theorem 1 involves partial orderings of conditional distributions using stochastic dominance. In the simplest case of the two-sample treatment control model this involves testing the hypothesis that the treatment distribution stochastically dominates the control distribution, so $\beta_1(\tau) > 0$ for $\tau \in \mathcal{T}$. This can be accomplished using the one sided KS statistic. Similarly, tests of second order stochastic dominance can be based on the indefinite integral process of $\beta_1(\tau)$. These two sample tests extend nicely to general quantile regression settings and thus complement the important work of McFadden (1989) on testing for stochastic dominance, and Abadie (2000) on related test for treatment effects.

Theorem 1 extends slightly the results of Koenker and Machado (1999), but it still can not be used to test the location shift or location-scale shift hypothesis. It fails to answer our main question: how to deal with unknown nuisance parameters in R and r? To begin to address this question, we introduce an additional condition.

A. 5. *There exist estimators R_n and r_n satisfying $\sqrt{n}(R_n - R) = \mathcal{O}_p(1)$ and $\sqrt{n}(r_n - r) = \mathcal{O}_p(1)$.*

And we now consider the parametric quantile regression process,

$$\hat{v}_n(\tau) = \sqrt{n}\varphi_0(\tau)[R_n\Omega R_n^\top]^{-1/2}(R_n\hat{\beta}(\tau) - r_n - \Psi(\tau)).$$

The next result establishes a representation for $\hat{v}_n(\tau)$ analogous that provided in (2.2) for the univariate empirical quantile process.

Theorem 2. *Under conditions A.1-5, we have*

$$\hat{v}_n(\tau) - Z_n^\top \xi(\tau) \Rightarrow v_0(\tau) + \eta(\tau)$$

where $\xi(\tau) = \varphi_0(\tau)(1, F_0^{-1}(\tau))^\top$*, and* $Z_n = \mathcal{O}_p(1)$*, with* $v_0(\tau)$ *and* $\eta(\tau)$ *as specified in Theorem 1.*

Corollary 2. *The conclusions of Theorem 2 remain valid if* $\varphi_0(\tau)$ *and* Ω *are replaced by estimates satisfying condition A.4.*

As in the univariate case we are faced with two options. We can accept the presence of the Z_n term, and abandon the asymptotically distribution free nature of tests based upon $\hat{v}_n(\tau)$. This would presumably require some resampling strategy to determine critical values. Or we can, following Khmaladze, try to find a transformation of $\hat{v}_n(\tau)$ that annihilates the Z_n contribution, and thus restores the asymptotically distribution free nature of inference. We adopt the latter approach.

Let $g(t) = (t, \xi(t)^\top)^\top$ so $\dot{g}(t) = (1, \psi(t), \psi(t)F^{-1}(t))^\top$ with $\psi(t) = (\dot{f}/f)(F^{-1}(t))$. We will assume that $g(t)$ satisfies the following condition.

A. 6. *The function* $g(t)$ *satisfies:*

i: $\int \| \dot{g}(t) \|^2 \, dt < \infty$,

ii: $\{\dot{g}_i(t) : i = 1, \ldots, m\}$ *are linearly independent in a neighborhood of 1.*

Khmaladze (1981, §3.3) shows that A.6.ii implies $C^{-1}(\tau)$ exists for all $\tau < 1$.

We consider the transformed process $\tilde{v}_n(\tau)$ defined as,

$$\tilde{v}_n(\tau)^\top \equiv Q_g \hat{v}_n(\tau)^\top = \hat{v}_n(\tau)^\top - \int_0^\tau \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) d\hat{v}_n(r)^\top ds, \tag{4.6}$$

where the recursive least squares transformation should now be interpreted as operating coordinate by coordinate on the $\hat{v}_n$ process.

Theorem 3. *Under conditions A.1 - 6, we have*

$$\tilde{v}_n(\tau) \Rightarrow w_0(\tau) + \tilde{\eta}(\tau)$$

where $w_0(\tau)$ *denotes a q-variate standard Brownian motion, and* $\tilde{\eta}(\tau)^\top = Q_g \eta(\tau)^\top$*. Under the null hypothesis,* $\zeta(\tau) = 0$*,*

$$\sup_{\tau \in \mathcal{T}} \| \tilde{v}_n(\tau) \| \Rightarrow \sup_{\tau \in \mathcal{T}} \| w_0(\tau) \| .$$

Typically, in applications, the function $g(t)$ will *not* be specified under the null hypothesis, but will also need to be estimated. Fortunately, only one rather mild further condition is needed to enable us to replace g by an estimate.

A. 7. *There exists an estimator,* $g_n(\tau)$*, satisfying* $\sup_{\tau \in \mathcal{T}} \| \dot{g}_n(\tau) - \dot{g}(\tau) \| = o_p(1)$.

By a similar argument as Bai (1998), we have the following Corollary.

Corollary 3. *The conclusions of Theorem 3 remain valid if $\varphi_0(\tau), \Omega,$ and g are replaced by estimates satisfying conditions A.4 and A.7.*

In some applications, R is known and only r contains nuisance parameters that need to be estimated. In this case, we have reduced dimensionality in $\xi(t)$, Z_n, $g(t)$, and $\dot{g}(t)$. In particular, the asymptotic result of Theorem 2 reduces to

$$\hat{v}_n(\tau) - \varphi_0(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}(r_n - r) \Rightarrow v_0(\tau) + \eta(\tau),$$

and the corresponding $\dot{g}(\cdot)$ (or $g(\cdot)$) functions in transformation (4.6) are

$$\dot{g}(t) = (1, \psi(t))^\top \text{ and } g(t) = (t, \varphi_0(t))^\top.$$

An important example of this case is the heteroskedasticity test described in Section 4.2.

The proposed tests have non-trivial power against any local deviation $\eta(\tau)$ such that $Q_g\eta(\tau) \neq 0$. Consider the following linear operator $\mathcal{K}$:

$$\mathcal{K}x(t) = \dot{g}(t)^\top C(t)^{-1} \int_t^1 \dot{g}(r)x(r)dr$$

and denote its eigen-space with eigenvalue 1, $\{x(t) : x(t) = \mathcal{K}x(t)\}$, as $\mathcal{S}$. Our tests will have power as long as $\eta \notin \mathcal{S}$. $\mathcal{K}$ is a Fredholm integral operator defined on the space $L_2[0,1]$, and we can write,

$$\mathcal{K}x(t) = \int_0^1 K(t,s)x(s)ds$$

with Volterra kernel $K(t,s) = \dot{g}(t)^\top C(t)^{-1}\dot{g}(s)\mathbf{1}_{[t,1]}(s)$. The set $\mathcal{S}$ is the set of solutions to the Volterra equation,

$$x(t) = \int_0^1 K(t,s)x(s)ds,$$

and by Khmaladze (1981, p. 252), $\mathcal{S} = \{x : x(t) = \dot{g}(t)^\top\xi\}$. Thus, the proposed tests[2] have non-trivial power against any local deviation process $\eta(\tau)$ such that $\dot{\eta}$ is not in the linear space spanned by the elements of $\dot{g}(\tau)$.

5. Implementation of the Tests

Given a general framework for inference based on the quantile regression process, we can now elaborate some missing details. We will begin by considering tests of the location scale shift hypothesis against a general quantile regression alternative.

[2]In the above analysis, for simplicity and without loss of generality, we focus on statistics of the classical Kolmogorov-Smirnov form. The results of the paper apply more generally to statistics of the form $h(\tilde{v}_n(\cdot))$ for continuous functionals h. Besides the "sup " function, one may use other measures of the discrepancy between $\tilde{v}_n(\tau)$ and 0, depending on the alternatives of interest. For instance, we can construct a Cramér-von Mises type test statistic $\int_{\mathcal{T}} \tilde{v}_n(\tau)^\top \tilde{v}_n(\tau)d\tau$, based on $\tilde{v}_n(\tau)$. Or, more generally, $\int_{\mathcal{T}} \tilde{v}_n(\tau)^\top W(\tau)\tilde{v}_n(\tau)d\tau$ for a suitably chosen weight matrix function $W(\tau)$.

5.1. **The location-scale shift hypothesis.** We would like to test

$$F^{-1}_{y_i|x_i}(\tau|x_i) = x_i^\top \alpha + x_i^\top \gamma F_0^{-1}(\tau) \text{ vs. } F^{-1}_{y_i|x_i}(\tau|x_i) = x_i^\top \beta_n(\tau).$$

In the simplest case the univariate quantile function is known and we can formulate the hypothesis in the (4.4) notation,

$$R\beta(\tau) - r = \Psi(\tau)$$

by setting $r_i = \alpha_i/\gamma_i$, $R = \mathrm{diag}(\gamma_i^{-1})$, and $\Psi(\tau) = \iota_{p-1} F_0^{-1}(\tau)$. Obviously, there is some difficulty if there are γ_i equal to zero. In such cases, we can take $\gamma_i = 1$, and set the corresponding elements $r_i = \alpha_i$ and $\Psi_i(\tau) \equiv 0$. How should we go about estimating the parameters α and γ? Under the null hypothesis, $\beta_i(\tau) = \alpha_i + \gamma_i F_0^{-1}(\tau) : i = 1, \dots, p$, so it is natural to consider linear regression. Since $\hat{\beta}_i(\tau)$ is piecewise constant with jumps at points $\{\tau_1, \dots, \tau_J)$, it suffices to consider p bivariate linear regressions of $\hat{\beta}_i(\tau_j)$ on $\{(1, F_0^{-1}(\tau_j)) : \quad j = 1, \dots, J\}$. Each of these regressions has a known (asymptotic) Gaussian covariance structure that could be used to construct a weighted least squares estimator, but pragmatism might lead us to opt for the simpler unweighted estimator. In either case we have our required $\mathcal{O}(n^{-1/2})$ estimators $\hat{\alpha}_n$ and $\hat{\gamma}_n$.

When $F_0^{-1}(\tau)$ is (hypothetically) known the Khmaladzation process is relatively painless computationally. The function $\dot{g}(t) = (1, \psi_0(t), \psi_0(t) F_0^{-1}(t))^\top$ is known and the transformation (3.3) can be carried out by recursive least squares. Again, the discretization is based on the jumps $\{\tau_1, \dots, \tau_J\}$ of the piecewise constant $\hat{\beta}(\tau)$ process. Tests statistics based on the transformed process, $\tilde{v}_n(\tau)$, can then be easily computed. The simplest of these is probably the Kolmogorov-Smirnov sup-type statistic

$$K_n = \sup_{\tau \in \mathcal{T}} \parallel \tilde{v}_n(\tau) \parallel$$

where $\mathcal{T}$ is typically of the form $[\varepsilon, 1-\varepsilon]$ with $\varepsilon \in (0, 1/2)$. The choice of the norm $||\cdot||$ is also an issue. Euclidean norm is obviously natural, but has the possibly undesirable effect of accentuating extreme behavior in a few coordinates. Instead, we will employ the ℓ_1 norm in the simulations and the empirical application below.

In practice, $F_0^{-1}(t)$ is generally unknown under the null and it is convenient to choose one coordinate, typically the intercept coefficient, to play the role of numeraire. If the covariates are centered then the intercept component of the quantile regression process can be interpreted as an estimate of the conditional quantile function of the response at the mean covariate vector. From (4.4) we can write

$$\beta_i(\tau) = \mu_i + \sigma_i \beta_1(\tau) \qquad i = 2, \dots, p \tag{5.1}$$

where $\mu_i = \alpha_i - \alpha_1 \gamma_i / \gamma_1$ and $\sigma_i = \gamma_i / \gamma_1$, or in matrix notation as $R\beta(\tau) = r$ where $\Psi(\tau) \equiv 0$, $R = [\sigma \vdots - I_{p-1}]$ and $r = -\mu$. Estimates of the vectors μ and σ are again obtainable by regression of $\hat{\beta}_i(\tau) : \quad i = 2, \dots, p$ on the intercept coordinate $\hat{\beta}_1(\tau)$.

5.2. **Estimation of Nuisance Parameters.** Our proposed tests depend crucially on estimates of the quantile density and quantile score functions. Fortunately, there is a large related literature on estimating $\varphi_0(\tau)$, including e.g. Siddiqui (1960), Hájek (1962), Sheather and Maritz (1983), and Welsh (1988). Following Siddiqui, since, $dF_0^{-1}(t)/dt = (\varphi_0(t))^{-1}$, it is natural to use the estimator,

$$\varphi_n(t) = \frac{2h_n}{F_n^{-1}(t+h_n) - F_n^{-1}(t-h_n)}, \tag{5.2}$$

where $F_n^{-1}(s)$ is an estimate of $F_0^{-1}(s)$ and h_n is a bandwidth which tends to zero as $n \to \infty$. See Koenker and Machado (1999) for further details including discussion of the estimation of the matrix Ω appearing in the standardization of the process.

Finally, we must face the problem of estimating the function $\dot{g}$. Fortunately, there is also a large literature on estimation of score functions. For our purposes it is convenient to employ the adaptive kernel method [3] described in Portnoy and Koenker (1989). Given a uniformly consistent estimator, $\dot{g}_n$, satisfying condition A.7, see Portnoy and Koenker (1989, Lemma 3.2), Corollary 3 implies that under the null hypothesis

$$\tilde{v}_n(t) \equiv Q_{g_n} \hat{v}_n(t) \Rightarrow w_0(t)$$

and therefore tests can be based on K_n as before. Note that in this case estimation of $\dot{g}$ provides as a byproduct an estimator of the function $\varphi_0(t)$ which is needed to compute the process $\hat{v}_n(t)$.

In applications it will usually be desirable to restrict attention to a closed interval $[\tau_0, \tau_1] \subset (0,1)$. This is easily accommodated, following Koul and Stute (1999), Remark 2.3, by considering the modified test statistic,

$$K_n = \sup_{\tau \in [\tau_0, \tau_1]} ||\tilde{v}(\tau) - \tilde{v}(\tau_0)||/\sqrt{\tau_1 - \tau_0},$$

which converges weakly, just as in the unrestricted case, to $\sup_{[0,1]} ||w_0(\tau)||$. This renormalization is useful in our empirical application, for example, since we are restricted at the outset to estimating the $\hat{v}$ process on the subinterval $[\tau_0, \tau_1] = [.2, .8]$. Indeed, it may be fruitful to consider other forms of standardization as well.

5.3. **The location shift hypothesis.** An important special case of the location-scale shift model is, of course, the pure location shift model,

$$F_{y_i|x_i}^{-1}(\tau|x_i) = x_i^\top \alpha + \gamma_0 F_0^{-1}(\tau)$$

This is just the classical homoscedastic linear regression model,

$$y_i = x_i^\top \alpha + \gamma_0 u_i$$

[3] An attractive alternative to this approach has been developed by Cox (1985) and Ng (1994) based on smoothing spline methods.

where the $\{u_i\}$ are iid with distribution function F_0. If it is found to be appropriate then it is obviously sensible to consider estimation by alternative methods. For F_0 Gaussian, least squares would of course be optimal.[4]

The location-shift hypothesis can be expressed in standard form by setting $R = [0 \vdots I_{p-1}]$, $r = (\alpha_2, \ldots, \alpha_p)^\top$. It asserts simply that the quantile regression slopes are constant, independent of τ. Again, the unknown parameters in $\{R, r\}$ are easily estimated so the process $\hat{v}_n(\tau)$ is easily constructed. The transformation is obviously somewhat simpler in this case since $g(t) = (t, \varphi_0(t))$ has one fewer coordinate than in the previous case.

We can continue to view tests of the location-shift hypothesis as tests against a general quantile regression alternative represented in (A.3), or we can also consider the behavior of the tests against a more specialized class of location-scale shift alternatives for which

$$\zeta(\tau) = \zeta_0 F_0^{-1}(\tau)$$

for some fixed vector $\zeta_0 \in \mathbb{R}^{p-1}$. In the latter setting we have a test for parametric heteroscedasticity and we can compare the performance of our very general class of tests against alternative tests designed to be more narrowly focused on heteroscedastic alternatives.

5.4. **Local Asymptotic Power Comparison.** In this section we compare the heteroscedasticity tests proposed above in an effort to evaluate the cost of considering a much more general class of semiparametric alternatives instead of the strictly parametric alternatives represented by the location scale shift model.

We consider the local alternative with $\zeta(\tau) = \zeta_0 F_0^{-1}(\tau)$ in the location-shift model, and denote this hypothesis as H_n as in assumption A.3. This corresponds to the linear model with asymptotically vanishing heteroskedasticity studied by Koenker and Bassett (1982). In the location-shift model, $R = [0 \vdots I_{p-1}]$, and thus $R\Omega R^\top = \Omega_x$, where Ω_x is the lower $(p-1) \times (p-1)$ corner of Ω. Under H_n,

$$\hat{v_n}(\tau) = v_0(\tau) - \varphi_0(\tau)\Omega_x^{-1/2}\sqrt{n}\,(r_n - r) + \zeta_0\Omega_x^{-1/2}\varphi_0(\tau)F_0^{-1}(\tau) + o_p(1)$$

and the transformed process is

$$\tilde{v_n}(\tau) = w_0(\tau) + \tilde{\zeta}(\tau)\zeta_0\Omega_x^{-1/2} + o_p(1)$$

where the noncentrality process $\tilde{\zeta}(\tau)$ is simply the martingale transformation of $\zeta(\tau) = \varphi_0(\tau)F_0^{-1}(\tau)$. Provided that $\dot{\zeta}(\tau)$ is not in the space spanned by the functions $\{1, \dot{\varphi}_0(\tau)\}$, it is clear that $\tilde{\zeta}(\tau) \neq 0$, so the proposed tests have non-trivial power. For

[4]For F_0 unknown one might consider the Huber M-estimator, or its L-estimator counterpart, see Koenker and Portnoy (1987). In the location shift model it is also well-known from Bickel (1982), that the slope parameters, $(\beta_2, ..., \beta_p)$, are adaptively estimable provided F_0 has finite Fisher information for the location parameter. Thus, it would be reasonable to consider M-estimators like those described in Hsieh and Manski (1987) or the adaptive L-estimators described in Portnoy and Koenker (1989).

any given ζ_0, the asymptotic local power of the proposed Kolmogorov-Smirnov test is,

$$P(c) = \Pr\left\{\sup_{\tau\in\mathcal{T}} \left\| w_0(\tau) + \tilde{\zeta}(\tau)\zeta_0\Omega_x^{-1/2}\right\| > c_\alpha\right\},$$

where c_α is the asymptotic critical value at significance level α.

5.5. **Monte-Carlo.** We have conducted several Monte Carlo experiments to examine the finite sample performance of the proposed tests. These are reported in Koenker and Xiao (2001) to conserve space. Software is also available at this website. The Monte Carlo experiments demonstrate that the proposed tests have accurate size and respectable power over sample sizes in the range 200-800. As anticipated, bandwidth choice in the standardization step has important influence on the finite sample performance of our tests. The Monte Carlo results suggest: (1) Bandwidth rules like Bofinger and Hall-Sheather that are adaptive – wider near the median and narrower in the tails – are essential to good performance to the tests; (2) The bandwidth rule proportional to $n^{-1/3}$ suggested by Hall and Sheather (1988) provides a good lower bound while, the bandwidth suggested by Bofinger (1975) proportional to $n^{-1/5}$ provides a reasonable upper bound for bandwidth selection for the location-shift test; (3) A rescaled version of the Bofinger bandwidth rule yields good performance over a wide range of experimental settings.

5.6. **Extensions.** The proposed methods can be easily modified to attack many other inference problems, including tests for parametric conditional distributions and tests for conditional symmetry. The theory we develop also carries over with minor changes to some analytic nonlinear restrictions in place of (4.4) and associated changes in the transformation (4.6). For example, we may modify our hypothesis to the form

$$R\beta(\tau) - r = \Psi(\tau, \theta). \tag{5.3}$$

If θ were known, the previous tests would apply. More generally, if θ is unknown but may be estimated by a $\sqrt{n}$ consistent estimator, then, under regularity conditions the transformed process $\tilde{v}_n(\tau)$ defined by (4.6) can be constructed and results similar to Theorem 3 can be obtained.

6. A Reappraisal of the Pennsylvania Reemployment Bonus Experiments

A common concern about unemployment insurance (UI) systems has been the suggestion that the insurance benefit acts as a disincentive for job-seekers and thus prolongs the duration of unemployment spells. During the 1980's several controlled experiments were conducted in the U.S. to test the incentive effects alternative compensation schemes for UI. In these experiments, UI claimants were offered a cash bonus if they found a job within some specified period of time and if the job was retained for a specified duration. The question addressed by the experiments was:

would the promise of such a monetary lump-sum benefit provide a significant inducement for more intensive job-seeking and thus reduce the duration of unemployment?

In the first experiments conducted in Illinois a random sample of new UI claimants were told that they would receive a bonus of \$500 if they found full-time employment within 11 weeks after filing their initial claim, and if they retained their new job for at least 4 months. These "treatment claimants" were then compared with a control group of claimants who followed the usual rules of the Illinois UI system. The Illinois experiment provided very encouraging initial indication of the incentive effects of such policies. They showed that bonus offers resulted in a significant reduction in the duration of unemployment spells and consequent reduction of the regular amounts paid by the state to UI beneficiaries. This finding led to further "bonus experiments" in the states of New Jersey, Pennsylvania and Washington with a variety of new treatment options. An excellent review of the experiments, some general conclusions about their efficacy and a critique of their policy relevance can be found in Meyer (1995), and Meyer (1996). In this section we will focus more narrowly on a reanalysis of data from the Pennsylvania Reemployment Bonus Demonstration described in detail in Corson, Decker, Dunstan, and Keransky (1992).

The Pennsylvania experiments were conducted by the U.S. Department of Labor between July 1988 and October 1989. During the enrollment period, claimants who became unemployed and registered for unemployment benefits in one of the 12 selected local offices throughout the state were *randomly assigned* either to a control group or one of six experimental treatment groups. In the control group the existing rules of the unemployment insurance system applied. Individuals in the treatment groups were offered a cash bonus if they became reemployed in a full-time job, working more than 32 hours per week, within a specified qualification period. Two bonus levels and two qualification periods were tested, but we will restrict attention to the high bonus, long qualification period treatment which offered a cash of bonus of six times the weekly benefit for claimants establishing reemployment within 12 weeks. This restricted the sample size to 6384 observations. A detailed description of the characteristics of claimants under study is presented in Koenker and Bilias (2001) which has information on age, race, gender, number of dependents, location in the state, existence of recall expectations, and type of occupation.

Since a large portion of spells end in either the first week or the twenty seventh week, it should be stressed that the definition of the first spell of UI in the Pennsylvania study includes a waiting week and that the maximum number of uninterrupted full weekly benefits is 26. This implies that many subjects did not receive any weekly benefit and that many other claimants received continuously their full, entitled unemployment benefit.

6.1. **The Model.** Our basic model for analyzing the Pennsylvania experiment presumes that the logarithm of the duration (in weeks) of subjects' spells of UI benefits have linear conditional quantile functions of the form

$$Q_{\log(T)}(\tau|x) = x^\top \beta(\tau).$$

The choice of the log transformation is dictated primary by the desire to achieve linearity of the parametric specification and by its ease of interpretation. Multiplicative covariate effects are widely employed throughout survival analysis, and they are certainly more plausible in the present application than the assumption of additive effects. It is perhaps worth emphasizing that the role of the transformation is completely transparent in the quantile regression setting, where $Q_{h(T)}(\tau|x) = x^\top \beta(\tau)$ implies $Q_T(\tau|x) = h^{-1}(x^\top \beta(\tau))$. In contrast, the role of transformations in models of the conditional mean function are rather complicated since the transformation affects not only location, but scale and shape of the conditional distribution of the response. Our model includes the following effects:

- Indicator for the treatment group.
- Indicators for female, black and hispanic respondents.
- Number of dependents, with 2 indicating two or more dependents.
- Indicators for the 5 quarters of entry to the experiment.
- Indicator for whether the claimant "expected to be recalled" to a previous job.
- Indicators for whether the respondent was "young" – less than 35, or "old" – indicating age greater than 54.
- Indicator for whether claimant was employed in the durable goods industry.
- Indicator for whether the claimant was registered in one of the low unemployment short unemployment duration districts: Coatesville, Reading, or Lancaster.

In Figure 6.1 we present a concise visual representation of the results from the estimation of this model. Each of the panels of the Figure illustrate one coordinate of the vector-valued function, $\hat{\beta}(\tau)$, viewed as a function of $\tau \in [\tau_0, \tau_1]$. Here we choose $\tau_0 = .20$ and $\tau_1 = .80$, effectively neglecting the proportion of the sample that are immediately reemployed in week one and those whose unemployment spell exceeds the insured limit of 26 weeks. The lightly shaded region in each panel of the figure represents a 90 percent confidence band. We omit the plots for the 5 quarter of entry indicators, and for the low unemployment district variable to conserve space.

Before turning to interpretation of specific coefficients, we will try to offer some brief general remarks on how to interpret these figures. The baseline case is the pure location shift model for which we would have the classical accelerated failure time (AFT) model,

$$\log T_i = x_i^\top \beta + u_i$$

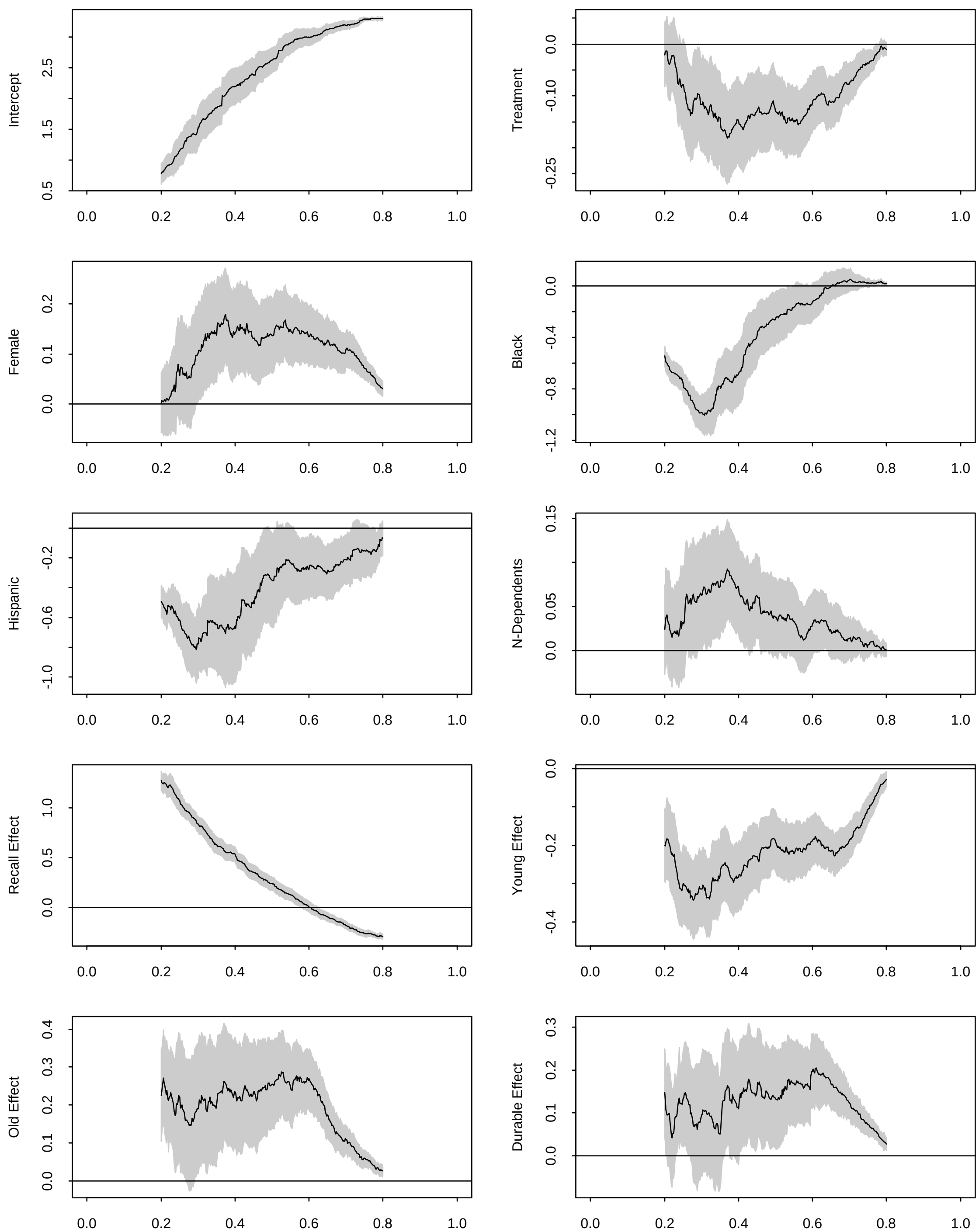
with $\{u_i\}$'s iid from some F. For F of the form $F(u) = 1 - \exp(-\exp(u))$, this is the well known Cox proportional hazard model with Weibull baseline hazard. In this case we would expect to see slope coefficients $\hat{\beta}_j(\tau)$ that fluctuate around a constant value indicating that the shift in the response due to a change in the covariate is constant over the entire estimated range of the distribution. In the linear location-scale model, the plots of the "slope" coefficients $\hat{\beta}_j(\tau)$ should mimic the "intercept" coefficient up to a location and scale shift. The intercept coefficient estimates a normalized version of the quantile function of the u_i's and all the other coefficients are simply location and scale shifts of this function.

6.2. **Interpretation of the Estimated Effects.** The treatment effect of the bonus illustrated in the upper right panel of Figure 6.1 clearly does not conform well to the location shift paradigm of the conventional models. After the log transformation of durations, a location shift would imply that the treatment exerts a constant *percentage* change in all durations. In the present instance this implication is particularly implausible since the entire point of the experiment was to alter the shape of the conditional duration distribution, concentrating mass within the qualification period, and reducing it beyond this period. In the treatment panel we see that the bonus effect gradually reduces durations from a null effect in the lower tail to a maximum reduction of 15% at the median, and then gradually again returns to a null effect in the upper tail. This finding accords perfectly with the timing imposed by the qualification period of the experiment. It might be thought that the bonus should not affect durations at all beyond the qualification period, but further consideration suggests that accelerated search in an effort to meet the qualification period deadline could easily yield "successes" that extended beyond the qualification period due to decision delay by potential employers, or other factors. No treatment effect is observed in either tail implying that the treatment had no effect in changing the probability of immediate reemployment (in week one), or in effecting the probability of durations beyond the 26 week maximum. The high bonus and long qualification period treatment, yielded roughly a 15% reduction in median duration.

The randomization of the experiment was quite effective in rendering the potentially confounding effects of other covariates orthogonal to the treatment indicator. Nevertheless, it is of some interest to explore the effect of a few other covariates in an effort to better understand determinants of the duration of unemployment.

Women are 5 to 15% slower than men to exit unemployment. Blacks and Hispanics appear much quicker than whites to become reemployed. This effect is particularly striking in the case of blacks for whom median duration is roughly half ($\approx e^{-.75}$) that of whites, and only 30% as long as controls at $\tau = .33$. The number of dependents appears to exert a rather weak positive effect on unemployment durations. The recall indicator is considerably more interesting; anticipated recall to one's prior job has a very strong and very precisely estimated detrimental effect over the entire lower tail of the distribution. However, beyond quantile $\tau = .6$, which corresponds to about 20 weeks duration for white, male controls, the anticipated recall appears to be abandoned and beyond this point expected recall becomes a significant force for early reemployment in the upper tail of the distribution. Not surprisingly the young (those

FIGURE 6.1. Quantile Regression Process for Log Duration Model



under 34) tend to find reemployment earlier than their middle aged counterparts, while the old (those over 54) do significantly worse. In both cases the effects are highly significant throughout the entire range of quantiles we have estimated.

Taken together, the results presented in Figure 6.1 do not seem to lend much support to either the location shift, or to the location-scale shift, hypotheses of the conventional regression model. In the former case we would expect to see plots that appeared essentially constant in τ while in the latter, we expect to see plots that mimic the shape of the intercept plot. Neither of these expectations are fulfilled. However, as we have emphasized earlier, it is crucial to be able evaluate these impressions by more formal statistical methods, a task that is undertaken in the next subsection.

6.3. **Inference from the Quantile Regression Process.** Our proposed inference strategy for testing the location scale shift hypothesis can be split into several intermediate steps. These steps are illustrated sequentially in Koenker and Xiao (2001). We begin by estimating the nuisance parameters of the restricted model (5.1). The model for each coordinate is estimated based the 301 estimated points of the quantile regression process. With the possible exception of the recall effect, none of these fits look very compelling, but at this stage we are already deeply mired in the Durbin problem and so it is difficult to judge the significance of departures from the fitted relationships.

Next, taking the residuals from these fits, and standardizing by the Cholesky decomposition of their (inverse) covariance matrix yields the parametric quantile regression process, $\hat{v}_n(\tau)$. Had we specified hypothetical values for the coefficients rather than estimating them, we could of course treat the resulting process as a vector of independent Brownian bridges under the null. However, the effect of the estimated nuisance parameters from the first step is to distort the variability of the process, as we have seen in Section 3. At this point we estimate the score function $\dot{g}$ and perform the martingale transformation on each slope coordinate. The transformation is applied on the restricted subinterval, $[\tau_0, \tau_1]$, as described at the end of Section 4.1, yielding the new process, $\tilde{v}_n(\tau)$, The transformed coordinates of this process are, under the null hypothesis, asymptotically, independent Brownian motions. We consider the test statistic,

$$K_n = \sup_{\tau \in \mathcal{T}} ||\tilde{v}_n(\tau) - \tilde{v}_n(\tau_0)|| / \sqrt{\tau_1 - \tau_0}$$

which takes the value 112.23. Here $\mathcal{T} = [.25, .75]$ so there is an additional .05 trimming to mollify the extreme behavior of the transformation in the tails. The critical value for this test is 16.00, employing the ℓ_1 norm, so the location-scale-shift hypothesis is decisively rejected. See Koenker and Xiao (2001) for tables and further details on asymptotic critical values for the proposed tests. These tables and their construction are closely related to those provided by Andrews (1993) for changepoint problems.

It is of some independent interest to investigate which of the coordinates contribute "most" to the joint significance of our K_n statistic. This inquiry is fraught with all the usual objections since the coordinates are not independent, but we plunge ahead, nevertheless. In place of the joint hypothesis we can consider univariate sub-hypotheses of the form,

$$\beta_i(\tau) = \mu_i + \sigma_i \beta_1(\tau)$$

for each "slope" coefficient. In effect, this approach replaces the matrix standardization used for the joint test by a scalar standardization. The martingale transformation is then applied just as in the previous case. In Table 6.1 we report the test statistics,

$$K_{ni} = \sup_{\tau \in \mathcal{T}} |\tilde{v}_{ni}(\tau) - \tilde{v}_{ni}(\tau_0)| / \sqrt{\tau_1 - \tau_0}$$

for each of the covariates. Effects for the 5 quarters of entry are not reported. The critical values for these coordinatewise tests are 1.923 at .05, and 2.420 at .01, so the treatment, race, gender and age effects are highly significant.

Variable	Location Scale Shift	Location Shift
Treatment	5.41	5.48
Female	4.47	4.42
Black	5.77	22.00
Hispanic	2.74	2.00
N-Dependents	2.47	2.83
Recall Effect	4.45	16.84
Young Effect	3.42	3.90
Old Effect	6.81	7.52
Durable Effect	3.07	2.83
Lusd Effect	3.09	3.05
Joint Effect	112.23	449.83

TABLE 6.1. Tests of the Location-Scale-Shift Hypothesis

Also reported in Table 6.1 are the corresponding test statistics for the pure location-shift hypothesis. Not surprisingly, we find that the more restrictive hypothesis of constant $\beta_i(\tau)$ effects is considerably less plausible than the location scale hypothesis. The joint test statistic is now, 449.83, with .01 critical value of 16.00, and all of the reported covariates effects are significant at level .05, with the exception of the hispanic effect.

7. CONCLUSION

The linear location shift and location-scale shift regression models are very elegant and convenient abstractions for many statistical purposes. However, they also place very stringent restrictions on the way that covariates are permitted to influence the conditional distribution of the response variable. In our unemployment duration application the location-scale shift hypothesis may be viewed as a generalized form of the familiar accelerated failure time model in which the scale of the response distribution responds linearly to the covariates. This specification is decisively rejected by the data from the Pennsylvania experiments. Not only the treatment effect of the bonus payment, but many other of the covariates appear to affect the conditional distribution of unemployment duration in ways that are poorly approximated either by pure location and/or scale shifts. One consequence of the proposed methods of inference, it may be hoped, would be a greater willingness to explore more flexible models for covariate effects in a wide variety of econometric models.

Appendix A. Proofs

Proof of Theorem 1 Notice that $R\widehat{\beta}(\tau) - r - \Psi(\tau) = R\left[\widehat{\beta}(\tau) - \beta(\tau)\right] + R\beta(\tau) - r - \Psi(\tau)$. Under Assumption A.3, $R\beta(\tau) - r - \Psi(\tau) = \zeta(\tau)/\sqrt{n}$, thus

$$R\widehat{\beta}(\tau) - r - \Psi(\tau) = R\left[\widehat{\beta}(\tau) - \beta(\tau)\right] + \zeta(\tau)/\sqrt{n}.$$

Under Assumptions A.1 and A.2, by Theorem 1 of Gutenbrunner and Jureckova (1992), we have, uniformly for $\tau \in \mathcal{T}$,

$$\sqrt{n}\left[\widehat{\beta}(\tau) - \beta(\tau)\right] \Rightarrow \frac{1}{\varphi(\tau)} H_0^{-1} J_0^{1/2} v_0(\tau)$$

where $v_0(\tau)$ is a standardized p-dimensional Brownian bridge process, $\varphi(\tau) = f(F^{-1}(\tau))$. Thus

$$\begin{aligned} v_n(\tau) &= \sqrt{n}\varphi(\tau)[R\Omega R^\top]^{-1/2}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &= \varphi(\tau)[R\Omega R^\top]^{-1/2} R\sqrt{n}\left[\widehat{\beta}(\tau) - \beta(\tau)\right] + \varphi(\tau)[R\Omega R^\top]^{-1/2}\zeta(\tau) \Rightarrow v_0(\tau) + \eta(\tau). \end{aligned}$$

∎

Proof of Corollary 1 We have

$$\begin{aligned} v_n(\tau) &= \sqrt{n}\varphi_n(\tau)[R\Omega_n R^\top]^{-1/2}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &= \sqrt{n}\varphi_0(\tau)[R\Omega R^\top]^{-1/2}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &\quad + [\varphi_n(\tau) - \varphi_0(\tau)]\,[R\Omega_n R^\top]^{-1/2}\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &\quad + \varphi_0(\tau)\left[[R\Omega_n R^\top]^{-1/2} - [R\Omega R^\top]^{-1/2}\right]\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \end{aligned}$$

Following Bai (1998), notice that

$$[R\Omega_n R^\top]^{-1/2} - [R\Omega R^\top]^{-1/2} = [R\Omega_n R^\top]^{-1/2}\left\{[R\Omega R^\top]^{1/2} - [R\Omega_n R^\top]^{1/2}\right\}[R\Omega R^\top]^{-1/2},$$

and $[R\Omega_n R^\top]^{1/2} = R\widehat{H}_0^{-1} J_0^{1/2}$, where $\Omega_n = \widehat{H}_0^{-1} J_0 \widehat{H}_0^{-1}$,

$$[R\Omega R^\top]^{1/2} - [R\Omega_n R^\top]^{1/2} = R[H_0^{-1} - \widehat{H}_0^{-1}]J^{1/2} = R\widehat{H}_0^{-1}[\widehat{H}_0 - H_0]H_0^{-1}J^{1/2}.$$

Under Assumption A.4,

$$\begin{aligned} &[\varphi_n(\tau) - \varphi_0(\tau)]\,[R\Omega_n R^\top]^{-1/2}\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] = o_p(1), \\ &\varphi_0(\tau)\left[[R\Omega_n R^\top]^{-1/2} - [R\Omega R^\top]^{-1/2}\right]\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] = o_p(1), \end{aligned}$$

thus

$$\begin{aligned} v_n(\tau) &= \sqrt{n}\varphi_n(\tau)[R\Omega_n R^\top]^{-1/2}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &= \sqrt{n}\varphi_0(\tau)[R\Omega R^\top]^{-1/2}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] + o_p(1) \Rightarrow v_0(\tau) + \eta(\tau). \end{aligned}$$

■

Proof of Theorem 2 We may write

$$\begin{aligned} \widehat{v}_n(\tau) &= \sqrt{n}\varphi_0(\tau)[R_n\Omega R_n^\top]^{-1/2}[R_n\widehat{\beta}(\tau) - r_n - \Psi(\tau)] \\ &= \varphi_0(\tau)[R_n\Omega R_n^\top]^{-1/2}\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &\quad +\varphi_0(\tau)[R_n\Omega R_n^\top]^{-1/2}\sqrt{n}[r - r_n] + \varphi(\tau)[R_n\Omega R_n^\top]^{-1/2}\sqrt{n}[R_n - R]\widehat{\beta}(\tau) \\ &= \varphi_0(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &\quad +\varphi_0(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}[r - r_n] + \varphi(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}[R_n - R]\beta(\tau) + o_p(1) \end{aligned}$$

Notice that $\beta(\tau) = \alpha + \gamma F^{-1}(\tau)$,

$$\begin{aligned} \widehat{v}_n(\tau) &= \varphi_0(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}[R\widehat{\beta}(\tau) - r - \Psi(\tau)] \\ &\quad +\varphi_0(\tau)\left\{[R\Omega R^\top]^{-1/2}\sqrt{n}[r - r_n] + [R\Omega R^\top]^{-1/2}\sqrt{n}[R_n - R]\alpha\right\} \\ &\quad +\varphi_0(\tau)F^{-1}(\tau)[R\Omega R^\top]^{-1/2}\sqrt{n}[R_n - R]\gamma + o_p(1) \\ &= v_n(\tau) + Z_n^\top\xi(\tau) + o_p(1) \end{aligned}$$

where $\xi(\tau) = (\varphi_0(\tau),\, \varphi_0(\tau)F^{-1}(\tau))^\top$, and

$$Z_n = \left[\ [R\Omega R^\top]^{-1/2}\sqrt{n}[r - r_n] + [R\Omega R^\top]^{-1/2}\sqrt{n}[R_n - R]\alpha, \quad [R\Omega R^\top]^{-1/2}\sqrt{n}[R_n - R]\gamma\ \right]^\top = O_p(1).$$

And thus by Theorem 1, $\widehat{v}_n(\tau) - Z_n^\top\xi(\tau) \Rightarrow v_0(\tau) + \eta(\tau)$. ■

Proof of Corollary 2 Similar to that of Corollary 1. ■

Proof of Theorem 3 By Theorem 2, $\widehat{v}_n(\tau) = v_0(\tau) + Z_n^\top\xi(\tau) + \eta(\tau) + o_p(1)$. Denote the transformation based on $\dot{g}$ as

$$Q_g(h(\tau)) = h(\tau) - \int_0^\tau \left[\dot{g}(s)^\top C(s)^{-1}\int_s^1 \dot{g}(r)dh(r)\right] ds,$$

Since Q_g is a linear operator, we have

$$\widetilde{v}_n(\tau)^\top = Q_g\widehat{v}_n(\tau)^\top = Q_g v_0(\tau)^\top + Q_g\xi(\tau)^\top Z_n + Q_g\eta(\tau)^\top + o_p(1).$$

By construction, $Q_g(\xi(\tau)^\top) = 0$, and by Khmaladze (1981), $Q_g v_0(\tau)^\top \Rightarrow w_0(\tau)^\top$, where w_0 is a q-variate standard Brownian motion. Thus

$$\widetilde{v}_n(\tau) \Rightarrow w_0(\tau) + \widetilde{\eta}(\tau).$$

Under the null hypothesis,

$$\sup_{\tau\in\mathcal{T}}\|\widetilde{v}_n(\tau)\| \Rightarrow \sup_{\tau\in\mathcal{T}}\|w_0(\tau)\|.$$

■

Proof of Corollary 3

The proof follows Stute et al. (1998) and Bai (1998). We denote the transformation based on $\dot{g}_n$ as $Q_{g_n}(\cdot)$, thus

$$Q_{g_n}(\widehat{v}_n(\tau)^\top) = \widehat{v}_n(\tau)^\top - \int_0^\tau \left[\dot{g}_n(s)^\top C_n(s)^{-1}\int_s^1 \dot{g}_n(r)d\widehat{v}_n(\tau)^\top\right] ds.$$

Noticing that,

$$\widehat{v}_n(\tau) = \sqrt{n}\varphi_n(\tau)[R_n\Omega_n R_n^\top]^{-1/2}[R_n\widehat{\beta}(\tau) - r_n - \Psi(\tau)] = v_n(\tau) + Z_n^\top \xi_n(\tau) + o_p(1)$$

where Z_n is an $O_p(1)$ quantity independent of τ, by construction, $Q_{g_n}(g_n) = 0$. Thus we have

$$Q_{g_n}(\widehat{v}_n(\tau)^\top) = Q_{g_n}(v_n(\tau)^\top) + o_p(1).$$

We may write,

$$\begin{aligned}
\text{(A.1)} \qquad & \int_0^\tau \left[\dot{g}_n(s)^\top C_n(s)^{-1}\int_s^1 \dot{g}_n(r)dv_n(r)\right]ds - \int_0^\tau \left[\dot{g}(s)^\top C(s)^{-1}\int_s^1 \dot{g}(r)dv_n(r)^\top\right]ds \\
&= \int_0^\tau \left[\dot{g}_n(s)^\top C_n(s)^{-1}\int_s^1 [\dot{g}_n(r) - \dot{g}(r)]\, dv_n(r)^\top\right]ds \\
&\quad + \int_0^\tau \left[\dot{g}_n(s)^\top \left[C_n(s)^{-1} - C(s)^{-1}\right]\int_s^1 \dot{g}(r)dv_n(r)^\top\right]ds \\
&\quad + \int_0^\tau \left[\left[\dot{g}_n(s)^\top - \dot{g}(s)^\top\right] C(s)^{-1}\int_s^1 \dot{g}(r)dv_n(r)^\top\right]ds
\end{aligned}$$

Since $\dot{g}_n(s)$ is a consistent estimator of $\dot{g}(s)$ uniformly on $s \in \mathcal{T}$, and $\int_s^1 [\dot{g}_n(r) - \dot{g}(r)]\, dv_n(r)^\top = o_p(1)$, we have, for all $s \in \mathcal{T}$, $\left\|C(s)^{-1}\right\| \le \left\|C(1-\varepsilon)^{-1}\right\| < \infty$, and,

$$\left\|C_n(s)^{-1}\right\| \le \left\|C_n(1-\varepsilon)^{-1}\right\| = \left\|C(1-\varepsilon)^{-1}\right\| + o_p(1) < \infty.$$

The first and third terms on the right hand side of (A.1) are $o_p(1)$, and

$$C(s) - C_n(s) = \int_s^1 \left[\dot{g}(v)\dot{g}(v)^\top - \dot{g}_n(v)\dot{g}_n(v)^\top\right] dv = o_p(1),$$

so we have,

$$\begin{aligned}
&\int_0^\tau \left[\dot{g}_n(s)^\top \left[C_n(s)^{-1} - C(s)^{-1}\right]\int_s^1 \dot{g}(r)dv_n(r)^\top\right]ds \\
&\qquad = \int_0^\tau \left[\dot{g}_n(s)^\top C_n(s)^{-1}\left[C(s) - C_n(s)\right] C(s)^{-1}\int_s^1 \dot{g}(r)dv_n(r)^\top\right]ds = o_p(1).
\end{aligned}$$

Consequently (A.1) is $o_p(1)$, $Q_{g_n}(v_n(\tau)^\top) = Q_g(v_n(\tau)^\top) + o_p(1)$, and Corollary 3 follows. ■

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