PROMISCUOUS BAYESIAN (UP)DATING

ROGER KOENKER

Sexual intercourse began
In nineteen sixty-three
(which was rather late for me) -
Between the end of the Chatterley ban
And the Beatles' first LP.
Phillip Larkin, *Annus Mirabilis*

The gap of more than 30 years between Metropolis et al. (1953) and Gelfand and Smith (1990) can still be partially attributed to the lack of appropriate computing power, ...
C. Robert and G. Casella, (2008) A History of MCMC.

1. Introduction

When priors are not conjugal we can't expect fertile offspring, but some forms of updating may nevertheless still be possible. Indeed, quite a lot of this seems to have been going on since about 1953, but it is only within the last two decades that it has "taken off." The simplest form of this updating is Gibbs sampling, attributed to the American physicist, J. Willard Gibbs (1839-1903), even though he – in all likelihood – had no clue about it.

I will not attempt to recapitulate the rationale for Gibbs sampling, a nice elementary exposition is available by Casella and George (1992). Instead, I will describe an application to estimating a monotone regression function as proposed by Neelon and Dunson (2004).

2. Gibbs Sampling

Gibbs sampling relies on the observation that (under conditions we will not go into) successive sampling from a complete set of conditional distributions generates a Markov chain whose stationary distribution will be a valid joint posterior distribution. To illustrate this approach we consider a simple Gaussian nonparametric regression model,

$$y_i = g(x_i) + u_i$$

Version: September 29, 2010. These notes were prepared for the 2010 Friday Reading Group on nonparametric Bayesian methods. R code for the two illustrative examples will be posted on the reading group webpage. This so-called research was partially supported by NSF grant SES-08-50060.

where the u_i are iid $\mathcal{N}(0, \sigma^2)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous monotone increasing function. Following Neelon and Dunson (2004) we will parameterize $\hat{g}$ by a linear spline with fixed knots, using the basis expansion,

$$g(x) = \beta_0 + \sum_{j=1}^{p} \varphi_j(x)\beta_j,$$

where $\varphi_j(x) = \max\{\min\{x, \xi_j\} - \xi_{j-1}, 0\}$. Given this parameterization, g is monotone iff all the $\beta_j \geq 0$, and we have the likelihood,

$$\mathcal{L}(y|\beta, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\{-\tfrac{1}{2\sigma^2}\|y - X\beta\|_2^2\}.$$

The prior on β is complicated by the constraint that the last p coordinates should be non-negative. To keep the situation semi-conjugal, the prior is formulated in terms of a latent vector, β^* taken to be Gaussian with $\beta_j = I(\beta_j^* > \delta)\beta_j^*$, for $j = 2, \cdots, p$. To impose some smoothness on $hatg$, the prior asserts that β_j^* has mean β_{j-1}^* and common variance $\nu^2 = \lambda^{-2}$ for $j = 2, \cdots, p$, i.e.

$$\pi(\beta^*) = \phi(\beta_0^*, \mu_0, \sigma_0)\phi(\beta_1^*, \mu_1, \sigma_1) \prod_{j=2}^{p} \phi(\beta_j^*, \beta_{j-1}^*, \nu).$$

Some hyperparameters are fixed, so $\mu_0 = \mu_1 = 0$, $\sigma_0 = \sigma_1 = \sqrt{10}$ and $\delta = 0.05$. The parameters σ^2 and ν^2 are taken to be inverse gamma, with rather uninformative parameters.

The conditional posterior distributions described in the Appendix of Neelon and Dunson (2004) are all straightforward conjugal computations except for the transition from their (A.1) to the next expression, say (A.2). The proportionality claimed, on closer examination, follows from the fact that the ratios of the two respective terms have the same omitted factor. Given (A.2) it is easy to compute the integrating constant for the conditional posteriors of the β_j^*'s and show that each of them takes the form of a mixture of two truncated normals. Further details are relegated to the code appearing on the group webpages.

In Figure 1 we illustrate a simple example of the use of this technique. The target function is Beta distribution function depicted in black; there are 100 observations generated with iid Gaussian error. The unconstrained least squares linear spline estimate appears in red, and the mean of the posterior based on 10,000 MCMC iterations appears in blue.

3. Metropolis-Hastings

Gibbs sampling may be regarded as a special case of the Metropolis-Hastings MCMC approach that employs a form of rejection sampling.[1] To illustrate this more general technique we will briefly describe an application to estimation and inference about univariate quantiles due to Dunson and Taylor (2005).

Bayesian inference about quantiles is problematic because there is no obvious likelihood for quantiles unless we specify a full global parametric model for the observed data and this is contrary to the think-local mindset of the quantilogue. One of the high priests of

[1] Paraphrasing Segal (1970), Gibbs means never having to reject.

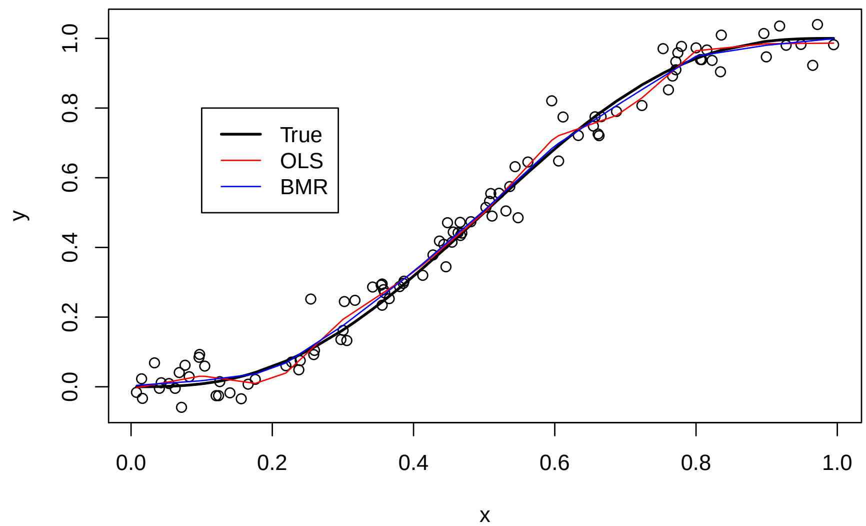

FIGURE 1. Three Normal Mixture Densities.

Bayesianism offers a way out of this impass; Jeffreys (1998) §4.4 considers inference about the median and proposes the "substitute likelihood"

$$s(\theta) = \binom{n}{k(\theta)}(\tfrac{1}{2})^2$$

where $k(\theta) = \sum_{i=1}^{n} I_{(-\infty,\theta]}(Z_i)$. This is easily generalized as in Lavine (1995) to a vector of quantiles, $\theta(\tau_i)$ for $0 < \tau_1 < \cdots < \tau_m < 1$ as

$$s(\theta) = \binom{n}{k_1(\theta) \cdots k_{m+1}(\theta)} \prod \Delta\tau_i^{k_i(\theta)}$$

with $k_j(\theta) = \sum_{i=1}^{n} I_{(\theta_{j-1},\theta_j]}(Z_i)$. Given this "likelihood" we need only formulate a prior and construct a proposal distribution for the MCMC iterations.

Dunson and Taylor (2005) adopt a simple Gaussian prior: $\pi(\theta) \sim \mathcal{N}(\theta_0, \Omega_0)$, but truncated so that $\theta_1 < \theta_2 < \cdots < \theta_m$. The proposal distribution is constructed by simply updating in the usual conjugal fashion this prior with an estimate of the usual large sample approximation of the distribution of the sample quantiles. That is, the proposal distribution is $\theta \sim \mathcal{N}(\hat{\theta}, \hat{\Omega})$ truncated to assure ordered θ_i's with

$$\hat{\theta} = \hat{\Omega}(\Omega_0^{-1}\theta_0 + \hat{\Omega}_1^{-1}\hat{\theta}_1)$$

$$\hat{\Omega} = (\Omega_0^{-1} + \hat{\Omega}_1^{-1})^{-1}$$

$\hat{\theta}_1$ is a vector of sample quantiles, and $\hat{\Omega}_1$ is an estimate of its asymptotic covariance matrix.

The Metropolis-Hastings iteration proceeds as follows:

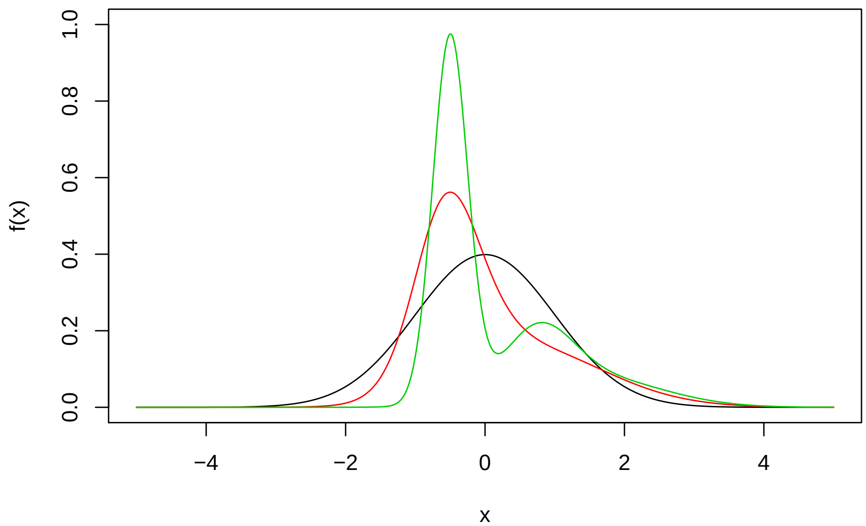

FIGURE 2. Three Normal Mixture Densities.

0.: Set $\theta^{(0)} = \hat{\theta}$.
1.: Draw a candidate θ from $\mathcal{N}(\hat{\theta}, \hat{\Omega})$
2.: Reject if θ isn't monotone,
3.: Else set $\theta^{(t)} = \theta$ with probability

$$p = \min\{1, \frac{s(\theta)\pi(\theta)}{s(\theta^{(t-1)})\pi(\theta^{(t-1)})}\}$$

or $\theta^{(t)} = \theta^{(t-1)}$ with probability $1 - p$.

A small simulation experiment designed to replicate the simulations of Dunson and Taylor (2005) was carried out. There are three target distributions, all mixtures of normals illustrated in Figure 1. Four sample sizes, $\{25, 50, 100, 200\}$ were studied, and 200 replications per setting were done. Tables 1-3 reports results for the root mean squared error comparision of the ordinary sample quantiles and the sample mean of the MCMC posterior iterations. In each simulation setting 5000 MCMC iterations were done with the last 4000 used to compute the posterior means.

There are several remarkable features of these tables. Focusing attention first on Table 1 corresponding to the standard Gaussian setting we see that the Bayes estimates do remarkably well, consistently outperforming the sample quantiles. The gap in performance declines with sample size as we would expect from the Bernstein-von-Mises theorem, but is still apparent at $n = 200$. A more puzzling feature of the table is that the sample quantiles themselves seem to be doing quite well, a fact somewhat obscured by the even better performance of the Bayes estimates. Take for example estimation of the median with $n = 100$ in

Distribution 1	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.353	0.277	0.241	0.265	0.333
$n = 50$	0.246	0.217	0.164	0.194	0.241
$n = 100$	0.177	0.127	0.109	0.124	0.160
$n = 200$	0.125	0.102	0.085	0.100	0.134
BayesQ					
$n = 25$	0.240	0.192	0.183	0.197	0.220
$n = 50$	0.180	0.169	0.157	0.153	0.177
$n = 100$	0.153	0.118	0.110	0.109	0.137
$n = 200$	0.114	0.097	0.086	0.096	0.122

Table 1. Root Mean Square Errors: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 2	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.188	0.173	0.201	0.377	0.472
$n = 50$	0.144	0.117	0.144	0.272	0.385
$n = 100$	0.086	0.082	0.104	0.193	0.251
$n = 200$	0.070	0.059	0.069	0.137	0.179
BayesQ					
$n = 25$	0.239	0.145	0.171	0.235	0.295
$n = 50$	0.133	0.103	0.135	0.210	0.255
$n = 100$	0.087	0.077	0.097	0.157	0.197
$n = 200$	0.066	0.054	0.065	0.123	0.154

Table 2. Root Mean Square Errors: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 3	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.099	0.101	0.228	0.462	0.456
$n = 50$	0.072	0.068	0.158	0.314	0.352
$n = 100$	0.051	0.046	0.095	0.219	0.267
$n = 200$	0.038	0.032	0.060	0.145	0.173
BayesQ					
$n = 25$	0.325	0.104	0.231	0.342	0.293
$n = 50$	0.164	0.062	0.175	0.276	0.263
$n = 100$	0.051	0.045	0.124	0.215	0.211
$n = 200$	0.033	0.031	0.069	0.155	0.154

Table 3. Root Mean Square Errors: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 1	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.312	0.259	0.241	0.259	0.332
$n = 50$	0.223	0.183	0.161	0.201	0.229
$n = 100$	0.156	0.129	0.114	0.135	0.163
$n = 200$	0.115	0.094	0.091	0.107	0.131
BayesQ					
$n = 25$	0.204	0.185	0.194	0.196	0.230
$n = 50$	0.188	0.158	0.160	0.175	0.181
$n = 100$	0.140	0.117	0.107	0.115	0.140
$n = 200$	0.107	0.090	0.093	0.098	0.120

Table 4. Root Mean Square Errors: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 2	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.194	0.181	0.215	0.367	0.447
$n = 50$	0.139	0.112	0.144	0.269	0.388
$n = 100$	0.099	0.087	0.106	0.211	0.260
$n = 200$	0.068	0.060	0.074	0.132	0.181
BayesQ					
$n = 25$	0.265	0.140	0.186	0.233	0.278
$n = 50$	0.148	0.098	0.131	0.212	0.272
$n = 100$	0.092	0.079	0.099	0.176	0.208
$n = 200$	0.063	0.054	0.071	0.118	0.160

Table 5. Root Mean Square Errors: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 3	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.091	0.091	0.230	0.416	0.497
$n = 50$	0.069	0.065	0.115	0.311	0.371
$n = 100$	0.052	0.052	0.095	0.217	0.264
$n = 200$	0.034	0.031	0.060	0.148	0.168
BayesQ					
$n = 25$	0.324	0.092	0.215	0.308	0.310
$n = 50$	0.159	0.066	0.157	0.280	0.263
$n = 100$	0.047	0.050	0.123	0.212	0.207
$n = 200$	0.032	0.031	0.071	0.144	0.147

Table 6. Root Mean Square Errors: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 1	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.327	0.270	0.230	0.261	0.334
$n = 50$	0.250	0.200	0.182	0.202	0.243
$n = 100$	0.174	0.133	0.132	0.147	0.176
$n = 200$	0.113	0.093	0.080	0.096	0.124
BayesQ					
$n = 25$	4.419	4.164	3.852	1.720	1.031
$n = 50$	4.103	3.293	0.835	0.627	0.570
$n = 100$	2.618	0.526	0.351	0.311	0.320
$n = 200$	0.220	0.163	0.148	0.160	0.170

Table 7. Root Mean Square Errors: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 2	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.188	0.163	0.211	0.375	0.494
$n = 50$	0.144	0.125	0.157	0.271	0.404
$n = 100$	0.102	0.082	0.106	0.203	0.269
$n = 200$	0.069	0.062	0.073	0.128	0.182
BayesQ					
$n = 25$	4.298	3.797	3.156	1.385	1.389
$n = 50$	3.959	2.712	0.457	0.610	0.873
$n = 100$	1.676	0.225	0.203	0.363	0.534
$n = 200$	0.112	0.093	0.110	0.224	0.313

Table 8. Root Mean Square Errors: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 3	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.092	0.100	0.279	0.442	0.465
$n = 50$	0.072	0.063	0.136	0.323	0.364
$n = 100$	0.051	0.042	0.080	0.212	0.272
$n = 200$	0.038	0.032	0.054	0.143	0.178
BayesQ					
$n = 25$	3.815	3.160	2.327	1.213	1.388
$n = 50$	2.760	1.269	0.224	0.838	0.846
$n = 100$	0.615	0.089	0.112	0.547	0.454
$n = 200$	0.047	0.043	0.068	0.322	0.292

Table 9. Root Mean Square Errors: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 1	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.331	0.249	0.233	0.271	0.361
$n = 50$	0.208	0.181	0.162	0.199	0.252
$n = 100$	0.160	0.141	0.125	0.138	0.156
$n = 200$	0.127	0.108	0.083	0.089	0.115
BayesQ					
$n = 25$	0.225	0.186	0.184	0.193	0.231
$n = 50$	0.169	0.157	0.162	0.167	0.191
$n = 100$	0.143	0.126	0.125	0.125	0.141
$n = 200$	0.112	0.095	0.083	0.086	0.111

Table 10. Root Mean Square Errors: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 2	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.195	0.150	0.188	0.359	0.483
$n = 50$	0.126	0.118	0.153	0.302	0.364
$n = 100$	0.104	0.086	0.106	0.208	0.258
$n = 200$	0.068	0.055	0.069	0.134	0.174
BayesQ					
$n = 25$	0.254	0.119	0.151	0.215	0.284
$n = 50$	0.131	0.097	0.139	0.234	0.255
$n = 100$	0.094	0.080	0.099	0.181	0.206
$n = 200$	0.067	0.053	0.060	0.121	0.148

Table 11. Root Mean Square Errors: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distribution 3	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.9$
SampleQ					
$n = 25$	0.106	0.094	0.216	0.437	0.471
$n = 50$	0.068	0.066	0.161	0.281	0.318
$n = 100$	0.050	0.046	0.071	0.223	0.263
$n = 200$	0.038	0.034	0.055	0.150	0.173
BayesQ					
$n = 25$	0.401	0.098	0.206	0.335	0.296
$n = 50$	0.115	0.063	0.177	0.265	0.240
$n = 100$	0.076	0.047	0.108	0.216	0.198
$n = 200$	0.037	0.032	0.064	0.159	0.153

Table 12. Root Mean Square Errors: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

the first table. The RMSE is 0.114. The efficient estimator of the median in this case is the sample mean, which would have RMSE of $1/\sqrt{100} = 0.100$, and the sample median would be expected to have RMSE approximately equal to $\sqrt{\pi/200} \approx 0.125$ based on its asymptotic theory. In the process of investigating this anomaly, it was discovered that computing sample quantiles with `rq(x ~ 1, taus)` when the solution is non-unique, that is when $n\tau$ is an integer, tends to choose the order statistic closest to zero.[2] Thus, when the target density has median zero, this produces a desirable "shrinkage effect." Whether this slight shrinkage is responsible to a significant degree for the unexpectedly good performance in Table 1 is unclear.

In the other tables we see that the Bayes estimates are also performing quite well, but in a few cases there seems to be a serious enough bias problem to dominate the variance reduction advantage that they offer. To explore these findings further, three new variants of the experiment were run with results presented in Tables 4-12. Tables 4-6 are simply a reproduction of the first set of experiments with exactly the same settings, but a new random number seed. Tables 7-9 centers the target distribution at $x = 5$, but leaves the prior unchanged. And Tables 10-12 return the centering of the target densities to zero, but increase the prior variance from 1 to 10. Shifting the location of the target densities has a disastrous effect on the Bayes estimator, underlining the need for a prior that doesn't conflict dramatically with reality. Finally, increasing the dispersion of the prior attentuates the shrinkage advantage exhibited in Tables 1-6 for the Bayes estimators, but not as much as one might have expected. It is worth noting that the performance of the sample median for $n = 100$ in Table 10 is much closer to asymptotic expectations. Since the seeds in Tables 4 and 10 are identical it may seem troubling that the sample medians have different performance, but further reflection reveals that different priors for the Bayes estimators implies different MCMC sequences and therefore different data for the whole simulation. This suggests that the differences in the performance observed across the 4 sets of tables in the sample quantiles reflects the inherent variability of the simulation dictated by the relatively small number (200) of replications. Since each table requires almost 3 hours of cpu time for the MCMC one is reluctant to increase the number of replications in such circumstances. This is one of the most troubling aspects of the MCMC revolution: computing effort is often so substantial that thorough evaluation of the sensitivity of results to the plethora of tuning parameters is quite difficult.

A more stringent test of the success of the Bayesian approach involves examination of the credible intervals produced by the MCMC iterations. To examine this we computed both 0.90 and 0.95 intervals for each of the quantiles for all of the experiments described above. Tables 13-24 report coverage frequencies for these experiments.

The 0.95 intervals are quite good, but the 0.90 intervals are quite horrible. Recall that the third group of tables corresponds to the shifted location version of the model, and there both sets of intervals are awful. It would be interesting to know whether increasing the length of the MCMC chain would improve the performance of the 0.90 intervals.

[2] The simplex algorithm used by `rq` must choose one of the two central order statistics when computing the median from an even sample of distinct observations.

Distn 1	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.355	0.020	0.635	0.030	0.255	0.990	1.000	1.000	1.000	1.000
$n = 50$	0.100	0.050	0.925	0.040	0.050	0.975	1.000	1.000	1.000	0.985
$n = 100$	0.000	0.035	1.000	0.025	0.000	0.980	1.000	1.000	1.000	0.995
$n = 200$	0.000	0.050	1.000	0.070	0.000	0.980	1.000	1.000	1.000	0.975

Table 13. Credible Interval Coverage: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 2	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.495	0.025	0.645	0.020	0.130	0.985	1.000	1.000	1.000	0.970
$n = 50$	0.120	0.040	0.900	0.025	0.030	1.000	1.000	1.000	1.000	1.000
$n = 100$	0.005	0.050	1.000	0.025	0.000	0.985	1.000	1.000	1.000	0.990
$n = 200$	0.000	0.065	1.000	0.030	0.000	0.995	1.000	1.000	1.000	0.980

Table 14. Credible Interval Coverage: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 3	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.690	0.020	0.655	0.010	0.075	0.995	1.000	1.000	1.000	0.985
$n = 50$	0.160	0.055	0.935	0.020	0.045	0.995	1.000	1.000	1.000	0.985
$n = 100$	0.005	0.035	1.000	0.075	0.000	0.980	1.000	1.000	1.000	0.995
$n = 200$	0.000	0.040	1.000	0.035	0.000	0.985	1.000	1.000	1.000	0.985

Table 15. Credible Interval Coverage: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 1	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.290	0.015	0.600	0.015	0.285	0.990	1.000	1.000	1.000	0.990
$n = 50$	0.100	0.025	0.950	0.045	0.070	0.995	1.000	1.000	1.000	0.985
$n = 100$	0.000	0.040	1.000	0.030	0.000	0.995	1.000	1.000	1.000	0.995
$n = 200$	0.000	0.050	1.000	0.055	0.000	0.985	1.000	1.000	1.000	0.985

Table 16. Credible Interval Coverage: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 2	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.520	0.035	0.580	0.030	0.125	0.975	1.000	1.000	1.000	0.985
$n = 50$	0.115	0.040	0.945	0.045	0.040	0.995	1.000	1.000	1.000	0.965
$n = 100$	0.000	0.035	0.995	0.025	0.005	0.985	1.000	1.000	1.000	0.975
$n = 200$	0.000	0.030	1.000	0.035	0.000	0.985	1.000	1.000	1.000	0.965

Table 17. Credible Interval Coverage: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 3	**Credibility 0.9**					**Credibility 0.95**				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.775	0.055	0.595	0.015	0.090	1.000	1.000	1.000	0.995	0.975
$n = 50$	0.235	0.060	0.945	0.015	0.045	1.000	1.000	1.000	1.000	0.975
$n = 100$	0.005	0.055	1.000	0.055	0.000	0.995	1.000	1.000	1.000	0.985
$n = 200$	0.000	0.030	1.000	0.040	0.000	0.985	1.000	1.000	1.000	0.990

Table 18. Credible Interval Coverage: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 1	**Credibility 0.9**					**Credibility 0.95**				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.145	0.000	0.000	0.825	0.970	0.935	0.955	0.070	0.650	0.980
$n = 50$	0.030	0.000	0.000	0.000	0.000	0.905	0.950	0.005	0.120	0.940
$n = 100$	0.000	0.000	0.000	0.000	0.000	0.490	0.580	0.000	0.000	0.000
$n = 200$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 19. Credible Interval Coverage: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 2	**Credibility 0.9**					**Credibility 0.95**				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.355	0.000	0.000	0.650	0.785	0.805	0.840	0.015	0.295	0.925
$n = 50$	0.060	0.000	0.000	0.000	0.000	0.765	0.800	0.005	0.060	0.735
$n = 100$	0.000	0.000	0.000	0.000	0.000	0.150	0.170	0.000	0.000	0.000
$n = 200$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 20. Credible Interval Coverage: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 3	**Credibility 0.9**					**Credibility 0.95**				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.270	0.000	0.000	0.380	0.460	0.575	0.585	0.010	0.245	0.735
$n = 50$	0.010	0.000	0.000	0.000	0.000	0.240	0.270	0.000	0.020	0.225
$n = 100$	0.000	0.000	0.000	0.000	0.000	0.010	0.005	0.000	0.000	0.000
$n = 200$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 21. Credible Interval Coverage: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 1	**Credibility 0.9**					**Credibility 0.95**				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.295	0.030	0.575	0.010	0.305	0.990	1.000	1.000	1.000	0.975
$n = 50$	0.060	0.060	0.935	0.025	0.080	0.990	1.000	1.000	1.000	0.990
$n = 100$	0.005	0.020	1.000	0.055	0.005	0.995	1.000	1.000	1.000	0.985
$n = 200$	0.000	0.040	1.000	0.040	0.000	0.980	1.000	1.000	1.000	0.990

Table 22. Credible Interval Coverage: Normal Mixture 1 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 2	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.505	0.030	0.710	0.010	0.095	0.985	1.000	1.000	1.000	0.980
$n = 50$	0.115	0.050	0.955	0.050	0.055	0.995	1.000	1.000	1.000	0.985
$n = 100$	0.005	0.050	1.000	0.040	0.005	0.990	1.000	1.000	1.000	0.980
$n = 200$	0.000	0.050	1.000	0.030	0.000	0.995	1.000	1.000	1.000	0.975

Table 23. Credible Interval Coverage: Normal Mixture 2 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

Distn 3	Credibility 0.9					Credibility 0.95				
	0.1	0.25	0.5	0.75	0.9	0.1	0.25	0.5	0.75	0.9
$n = 25$	0.715	0.050	0.620	0.010	0.080	0.995	1.000	1.000	1.000	0.985
$n = 50$	0.240	0.045	0.935	0.050	0.030	0.990	1.000	1.000	1.000	1.000
$n = 100$	0.015	0.070	1.000	0.030	0.000	0.990	1.000	1.000	1.000	0.965
$n = 200$	0.000	0.065	1.000	0.050	0.000	0.995	1.000	1.000	1.000	0.965

Table 24. Credible Interval Coverage: Normal Mixture 3 , 200 replications, 5000 Metropolis-Hastings steps with Jeffreys substitute Likelihood

References

Casella, G., and E. George (1992): "Explaining the Gibbs sampler," *American Statistician*, 46, 167–174.

Dunson, D., and J. Taylor (2005): "Approximate Bayesian inference for quantiles," *Journal of Nonparametric Statistics*, 17(3), 385–400.

Jeffreys, S. (1939): *Theory of probability*. Oxford University Press.

Lavine, M. (1995): "On an approximate likelihood for quantiles," *Biometrika*, 82(1), 220–222.

Neelon, B., and D. Dunson (2004): "Bayesian isotonic regression and trend analysis," *Biometrics*, 60, 398–406.

Segal, E. (1970): *Love Story*. HarperTorch.