METHOD OF QUANTILES

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1. Introduction

The underlying idea of Reich, Fuentes, and Dunson (2009) to use the asymptotic normal approximation of the quantile regression estimator as a "substitute" likelihood can be regarded as a convenient dumbing-down of the Jeffrey's idea elaborated by Lavine (1995) and Dunson and Taylor (2005). The obvious disadvantage of the original Jeffrey's suggestion is that it is difficult to compute/update the binomial proposal, whereas the normal approximation is made for Bayes rule. One way to enter the dungeons and dragons of Reich, Fuentes, and Dunson (2009) is to peel away all the Bayesian prior layers and focus on the basic model in its simplest setting. This puts us in the realm of MoQ, or method of quantiles.[1]

Essentially, MoQ is the much beloved MoM, with moments replaced by quantiles. Quantiles are moments too, of a sort. Suppose we have a parametric model: $Y_i \sim f(y, \theta)$, having quantiles $q(\tau, \theta)$ for $\tau \in (0, 1)$. A vector of sample quantiles $\hat{q}_n = (\hat{q}_n(\tau_i)$ based on a random sample of size n, is asymptotically Gaussian with mean $q(\theta) = (q(\tau_i, \theta))$ and covariance matrix

$$V = \frac{\tau_i \wedge \tau_j - \tau_i \tau_j}{f(q(\tau_i, \theta)) f(q(\tau_j, \theta))}$$

Thus, a natural estimator of θ is

$$\hat{\theta} = \text{argmin}_\theta \{(\hat{q} - q(\theta))^\top V^{-1} (\hat{q} - q(\theta))\}.$$

Usually, V needs to be estimated, but several approaches are already available.

Example. A particularly simple example of MoQ assumes that we can express,

$$q(\tau, \theta) = \sum_{j=1}^{m} \varphi_j(\tau) \theta_j.$$

For such linear basis expansion models we can immediately write the asymptotically optimal estimator of θ as,

$$\hat{\theta} = (X^\top V^{-1} X)^{-1} X^\top V^{-1} \hat{q}$$

where $X = (\varphi_j(\tau_i))$ is an p by m design matrix describing the basis expansion. For the sake of definiteness we can consider the Bernstein basis illustrated in Figure 1. Obviously, any

Version: September 7, 2010. R code to reproduce any or all of this will be available as a compressed tarball on the reading group website.

[1] For some general background see Koenker (2005) Section 4.8. The acronym MoQ should not be confused with the metaphysics of quality, an offshoot of Robert Pirsig's motorcycle zen zeitgeist.

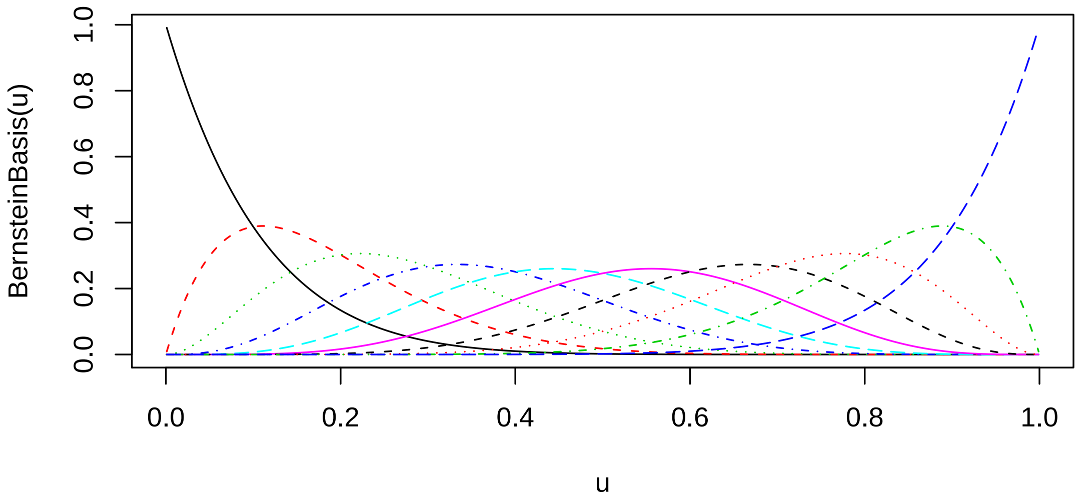

FIGURE 1. Bernstein basis functions for m = 10.

choice of a basis determines a family of distributions especially well suited to that basis in the sense that this family is well approximated by a small number of the basis functions. In general there is likely to be some approximation error, or bias, due to truncation of the basis expansion. To illustrate this, consider the example taken from Reich, Fuentes, and Dunson (2009) with quantile function,

$$q(\tau) = 1 + (\tau + 1)\Phi^{-1}(\tau) + \frac{5}{4}\tau^2$$

To get a visual impression of the skewed form of this distribution we plot its quantile and density function in contrast to the standard Gaussian in Figure 2.

To investigate the adequacy of the Bernstein basis model for this test case we have conducted a small simulation experiment. In Figure 3 we illustrate a typical realization of the experiment. The solid (red) line represents the true quantile curve, the sample quantiles, $\{0.05, 0.10, \cdots, 0.95\}$ appear as open circles with associated (pointwise) confidence intervals as vertical error bars. The dashed (blue) curve is the MoQ curve as described above, the dotted (green) curve is a constrained form of the MoQ curve estimated as in equation (6) of Reich, Fuentes, and Dunson (2009). The constraint imposed has two aspects: one requires that the first differences of the Bernstein coefficients be non-negative, this is a sufficient, but not necessary condition for monotonicity. The second shrinks these first differences toward zero. The amount of shrinkage imposed by the latter constraint is controlled by a parameter λ that we have taken to be one in this figure. However, $\lambda = 1$ appears to be too stringent, as is evident in the figure, so for the simulation we have reduced this value

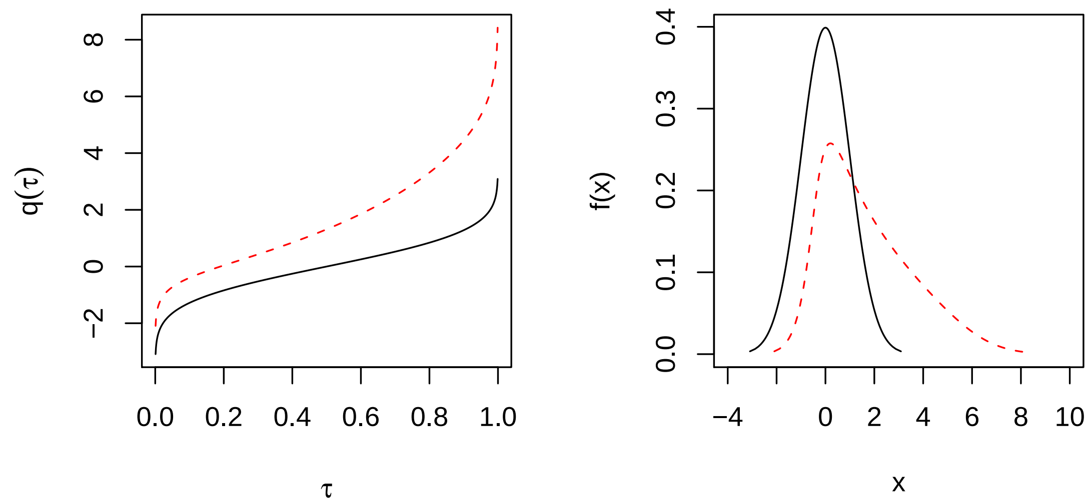

FIGURE 2. Test quantile and density function contrasted with standard Gaussian.

and selected $\lambda = 0.1$, with this value the constrained and unconstrained MoQ estimates are typically quite similar.

In Figure 4 we illustrate root mean square error performance of the three estimators based on 10000 simulation replications. The lower curve represents the root mean squared error at the 19 selected quantiles of the ordinary sample quantiles. The upper curves representing RMSE performance of the MoQ and constrained MoQ estimators aren't too much worse, but this is about the best that can be said about them. Not surprisingly the main component of the differences in RMSE is the bias component that is depicted in Figure 5. Clearly, the constrained MoQ estimator has problems in both tails of the distribution. The simpler MoQ estimator does somewhat better, but still the restrictions imposed by the Bernstein basis appear to create some difficulties.

References

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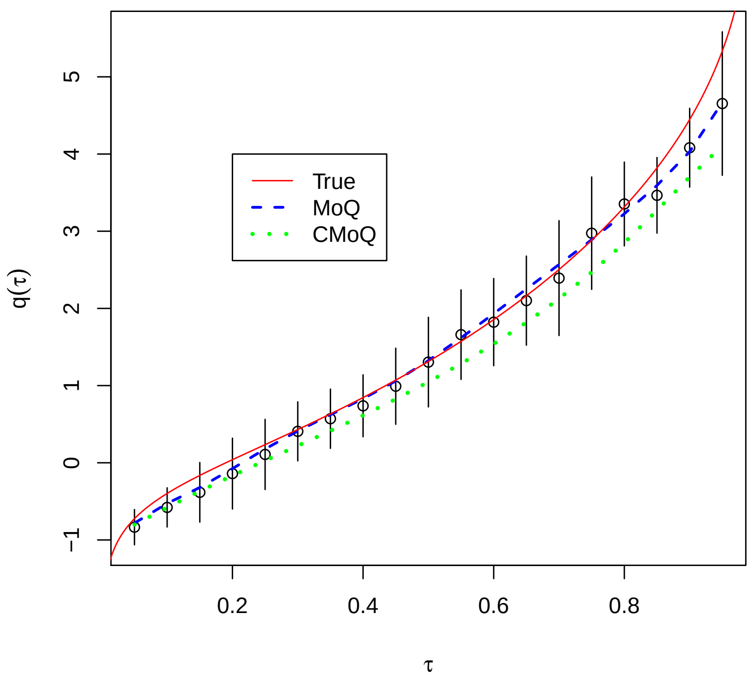

Figure 3. Method of quantiles estimates of the test function.

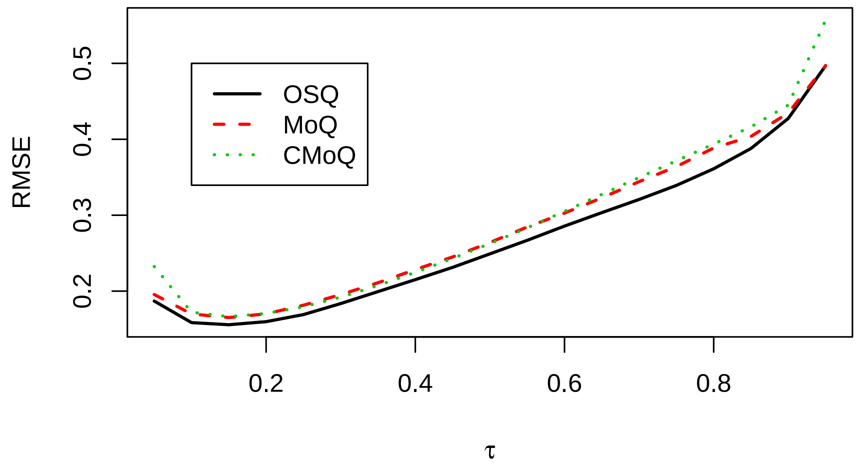

FIGURE 4. Root mean squared error of three competing estimators of the sample quantiles based on 10000 replications.

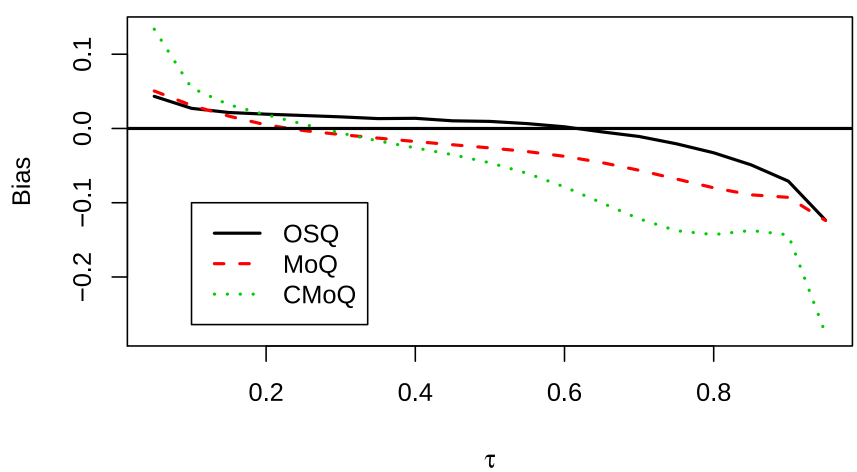

FIGURE 5. Bias of three competing estimators of the sample quantiles based on 10000 replications.