"AN AMUSING PROBLEM"

We observe [a single observation!] $X \sim \mathcal{N}(\xi, \sigma^2)$, with both ξ and σ unknown. Construct a confidence set for ξ, i.e.

$$\mathbb{P}_{\xi,\sigma}\{\xi \in (\underline{\xi}(X), \bar{\xi}(X)\} \geq 1 - \alpha,$$

for all (ξ, σ), with $\bar{\xi}(X) = c|X|$, and $\underline{\xi}(X) = -c|X|$, for $c > 0$.

Let $X = \xi + \sigma Y$, then

$$\begin{aligned}\mathbb{P}\{|\xi| < c|X|\} &= \mathbb{P}\{|\xi| < c|\xi + \sigma Y|\} \\ &= \mathbb{P}\{|\xi|/\sigma < c|\xi/\sigma + Y|\}\end{aligned}$$

so by symmetry,

$$\begin{aligned}\inf_{\lambda \geq 0} \mathbb{P}\{|\xi| < c|X|\} &= \inf_{\lambda \geq 0} \mathbb{P}\{\lambda < c|\lambda + Y|\} \\ &= \inf_{\lambda \geq 0} [\mathbb{P}\{Y > -\lambda, \lambda < c\lambda + Y\} + \mathbb{P}\{Y < -\lambda, \lambda < -c(\lambda + Y)\}] \\ &= \inf_{\lambda \geq 0} [\mathbb{P}\{Y > -\lambda, Y < \lambda/c - \lambda\} + \mathbb{P}\{Y < -\lambda, Y < -\frac{1+c}{c}\lambda\}] \\ &= \inf_{\lambda \geq 0} [\mathbb{P}\{Y > \lambda/c - \lambda\} + \mathbb{P}\{Y < -\lambda - \lambda/c\}]\end{aligned}$$

This is the probability of the complement of the interval of length $2\lambda/c$, centered at $-\lambda$. [Now optimize, over λ for each c, and then find c to achieve level α. Using the R function `Findc` below one obtains $c_{0.05} \approx 9.68$.]

```
Findc <- function(alpha){
    U <- function(c, a) {
       coverage <- function(z,c) 1 - pnorm(-z + z/c) + pnorm(-z - z/c)
       optimize(coverage, c(0,5), c = c)$obj - (1 - a)
       }
    uniroot(U, c(1, 20), a = alpha)$root
}
```

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