Risk, Choquet Portfolios and Quantile Regression

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5th RMetrics Workshop, Meielisalp: 28 June 2011

Joint work with Gib Bassett (UIC) and Gregory Kordas (Athens)

Risk as Pessimism?

In expected utility theory risk is entirely an attribute of the utility function:

Risk Neutrality	$\Rightarrow$	$u(x) \sim$ affine
Risk Aversion	$\Rightarrow$	$u(x) \sim$ concave
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Locally, the risk premium, i.e. the amount one is willing to pay to accept a zero mean risk, X, is

$$\pi(w, X) = \frac{1}{2} A(w) V(X)$$

where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt coefficient of absolute risk aversion and $V(X)$ is the variance of X.

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where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt coefficient of absolute risk aversion and $V(X)$ is the variance of X. Why is variance a reasonable measure of risk?

A Little Risk Aversion is a Dangerous Thing

Would you accept the gamble:

$$G_1 \qquad 50-50 \left\langle \begin{array}{l} \text{win } \$110 \\ \text{lose } \$100 \end{array} \right.$$

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Moral: A little local risk aversion over small gambles implies implausibly large risk aversion over large gambles. Reference: Rabin (2000)

Are Swiss Bicycle Messengers Risk Averse?

When Veloblitz and Flash bicycle messengers from Zurich were confronted with the bet:

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More than half (54%) rejected the bet.
Reference: Fehr and Götte (2002)

Coherent Risk

Definition (Artzner, Delbaen, Eber and Heath (1999)) For real valued random variables $X \in \mathcal{X}$ on $(\Omega, \mathcal{A})$ a mapping $\rho : \mathcal{X} \rightarrow \mathcal{R}$ is called a coherent risk measure if,

1. Monotone: $X, Y \in \mathcal{X}$, with $X \leqslant Y \Rightarrow \rho(X) \geqslant \rho(Y)$.
2. Subadditive: $X, Y, X+Y \in \mathcal{X}, \Rightarrow \rho(X+Y) \leqslant \rho(X) + \rho(Y)$.
3. Linearly Homogeneous: For all $\lambda \geqslant 0$ and $X \in \mathcal{X}$, $\rho(\lambda X) = \lambda\rho(X)$.
4. Translation Invariant: For all $\lambda \in \mathcal{R}$ and $X \in \mathcal{X}$, $\rho(\lambda + X) = \rho(X) - \lambda$.

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Many conventional measures of risks including those based on standard deviation are ruled out by these requirements. So are quantile based measures like "value at risk."

Choquet α-Risk

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Note that $\rho_{\nu_\alpha}(X) = -E_{\nu_\alpha,F}(X)$, so Choquet α-risk is just negative Choquet expected utility with the distortion function ν_α.

Pessimistic Risk Measures

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By Fubini

$$\begin{aligned}
\rho(X) &= -\int_0^1 \alpha^{-1} \int_0^\alpha F^{-1}(t) dt d\varphi(\alpha) \\
&= -\int_0^1 F^{-1}(t) \int_t^1 \alpha^{-1} d\varphi(\alpha) dt \\
&\equiv -\int_0^1 F^{-1}(t) d\nu(t)
\end{aligned}$$

Approximating General Pessimistic Risk Measures

We can approximate any pessimistic risk measure by taking

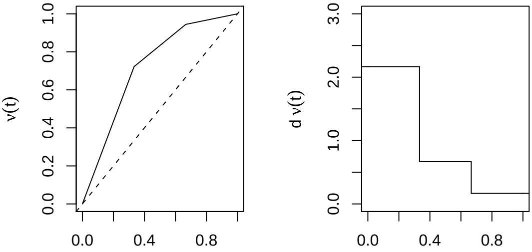
$$d\varphi(t) = \sum \varphi_i \delta_{\tau_i}(t)$$

where δ_τ denotes (Dirac) point mass 1 at τ. Then

$$\rho(X) = -\varphi_0 F^{-1}(0) - \int_0^1 F^{-1}(t)\gamma(t)dt$$

where $\gamma(t) = \sum \varphi_i \tau_i^{-1} I(t < \tau_i)$ and $\varphi_i > 0$, with $\sum \varphi_i = 1$.

An Example



$$d\varphi(t) = \frac{1}{2}\delta_{1/3}(t) + \frac{1}{3}\delta_{2/3}(t) + \frac{1}{6}\delta_1(t)$$

A Theorem

Theorem (Kusuoka (2001)) A regular risk measure is *coherent* in the sense of Artzner *et. al.* if and only if it is *pessimistic*.

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- Pessimistic Choquet risk measures correspond to *concave* ν, i.e., *monotone decreasing* $d\nu$.
- Probability assessments are distorted to accentuate the probability of the least favorable events.
- The crucial coherence requirement is subadditivity, or submodularity, or 2-alternatingness in the terminology of Choquet capacities.

An Example

Samuelson (1963) describes asking a colleague at lunch whether he would be willing to make a

$$50 - 50 \text{ bet} \quad \left\langle \begin{array}{l} \text{win } 200 \\ \text{lose } 100 \end{array} \right.$$

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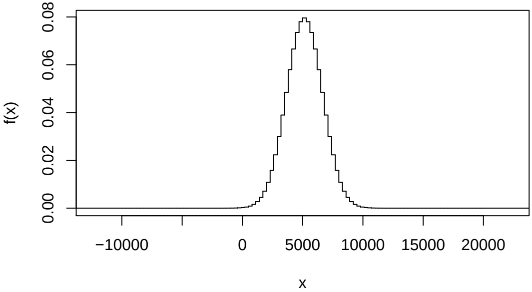
"no, but I *would* be willing to make 100 such bets."

This response has been interpreted not only as reflecting a basic confusion about how to maximize expected utility but also as a fundamental misunderstanding of the law of large numbers.

Payoff Density of 100 Samuelson trials



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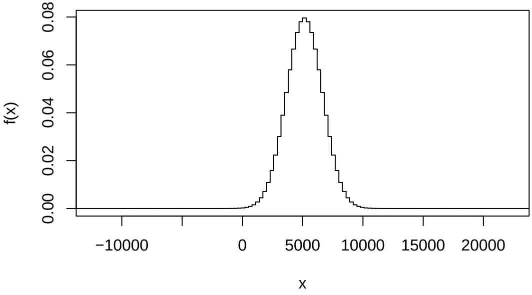

Odds of losing money on the 100 trial bet is 1 chance in 2300.

Was Brown really irrational?

Suppose, for the sake of simplicity that

$$d\varphi(t) = \tfrac{1}{2}\delta_{1/2}(t) + \tfrac{1}{2}\delta_1(t)$$

so for one Samuelson coin flip we have the unfavorable evaluation,

$$E_{\nu,F}(X) = \tfrac{1}{2}(-100) + \tfrac{1}{2}(50) = -25$$

but for $S = \sum_{i=1}^{100} X_i \sim \mathcal{B}in(.5, 100)$ we have the favorable evaluation,

$$\begin{aligned} E_{\nu,F}(S) &= \tfrac{1}{2}2\int_0^{1/2} F_S^{-1}(t)dt + \tfrac{1}{2}(5000) \\ &= 1704.11 + 2500 \\ &= 4204.11 \end{aligned}$$

How to be Pessimistic

Theorem Let X be a real-valued random variable with $EX = \mu < \infty$, and $\rho_\alpha(u) = u(\alpha - I(u < 0))$. Then

$$\min_{\xi \in \mathcal{R}} E\rho_\alpha(X - \xi) = \alpha\mu + \rho_{\nu_\alpha}(X)$$

So α risk can be estimated by the sample analogue

$$\hat{\rho}_{\nu_\alpha}(x) = (n\alpha)^{-1} \min_\xi \sum \rho_\alpha(x_i - \xi) - \hat{\mu}_n$$

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 I knew it! Eventually everything looks like quantile regression to this guy!

Pessimistic Portfolios

Now let $X = (X_1, \ldots, X_p)$ denote a vector of potential portfolio asset returns and $Y = X^\top \pi$, the returns on the portfolio with weights π. Consider

$$\min_{\pi} \rho_{\nu_\alpha}(Y) - \lambda \mu(Y)$$

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Minimize α-risk subject to a constraint on mean return.
This problem can be formulated as a linear quantile regression problem

$$\min_{(\beta,\xi)\in\mathcal{R}^p} \sum_{i=1}^{n} \rho_\alpha(x_{i1} - \sum_{j=2}^{p}(x_{i1} - x_{ij})\beta_j - \xi) \quad \text{s.t.} \quad \bar{x}^\top \pi(\beta) = \mu_0,$$

where $\pi(\beta) = (1 - \sum_{j=2}^{p} \beta_j, \beta^\top)^\top$.

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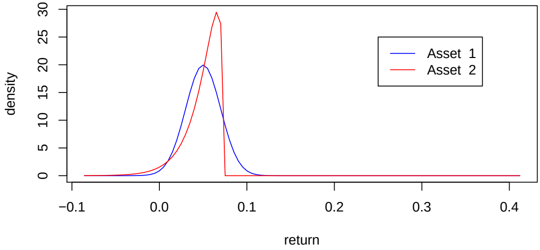

Two asset return densities with identical mean and variance.

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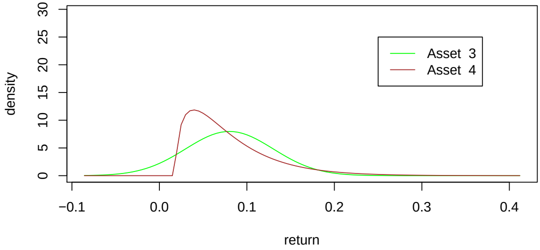
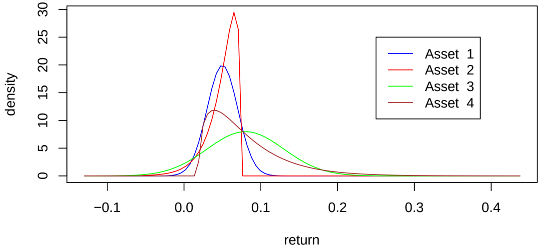


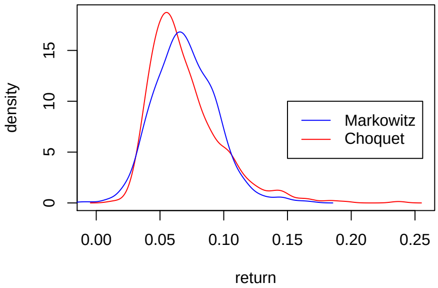
Two more asset return densities with identical mean and variance.

An Example



Two pairs of asset return densities with identical mean and variance.

Optimal Choquet and Markowitz Portfolio Returns



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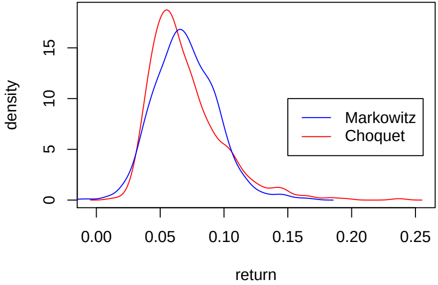

Markowitz portfolio minimizes the standard deviation of returns subject to mean return $\mu = .07$. The Choquet portfolio minimizes Choquet risk (for $\alpha = .10$) subject to earning the same mean return. The Choquet portfolio has better performance in both tails than mean-variance Markowitz portfolio.

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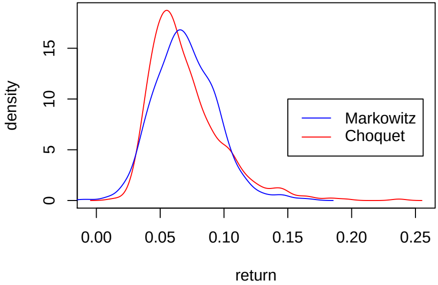

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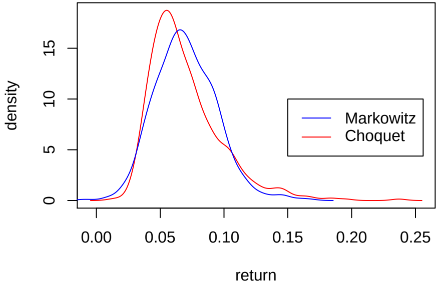

Now, the Markowitz portfolio minimizes the standard deviation of returns subject to mean return $\mu = .07$. The Choquet portfolio maximizes expected return subject to achieving the same Choquet risk (for $\alpha = .10$) as the Markowitz portfolio. Choquet portfolio has expected return $\mu = .08$ a full percentage point higher than the Markowitz portfolio.

A Unified Theory of Pessimism

Any pessimistic risk measure may be approximated by

$$\rho_\nu(X) = \sum_{k=1}^{m} \varphi_k \rho_{\nu_{\alpha_k}}(X)$$

where $\varphi_k > 0$ for $k = 1, 2, \ldots, m$ and $\sum \varphi_k = 1$.
Portfolio weights can be estimated for these risk measures by solving linear programs that are weighted sums of quantile regression problems:

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Software in R is available on from my web pages.

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- Choquet (non-additive, rank dependent) expected utility provides a simple, tractable alternative.
- Mean-variance Portfolio allocation is also unsatisfactory since it relies on unpalatable assumptions of Gaussian returns, or quadratic utility.
- Choquet portfolio optimization can be formulated as a quantile regression problem thus providing an attractive practical alternative to the dominant mean-variance approach of Markowitz (1952).