Quantile Regression: A Gentle Introduction

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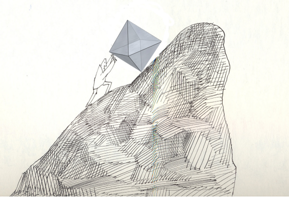


Preview

Least squares methods of estimating conditional mean functions

- were developed for, and
- promote the view that,

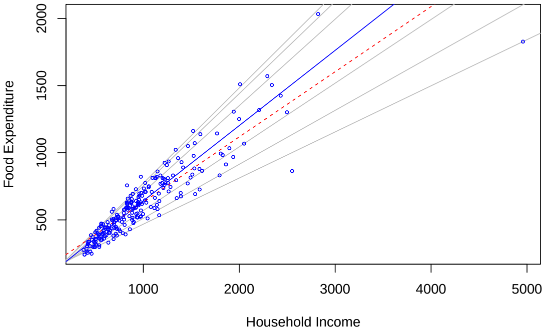
$$\text{Response} = \text{Signal} + \text{iid Measurement Error}$$

When we write,

$$y_i = x_i^\top \beta + u_i$$

we are (often implicitly) endorsing this view. Covariates exert a pure location shift effect on the response.
In fact the world is rarely this simple. Quantile regression permits covariate effects to "grow up" to become distributional objects.

Engel's Food Expenditure Data



Engel Curves for Food: This figure plots data taken from Engel's (1857) study of the dependence of households' food expenditure on household income. Seven estimated quantile regression lines for $\tau \in \{.05, .1, .25, .5, .75, .9, .95\}$ are superimposed on the scatterplot. The median $\tau = .5$ fit is indicated by the blue solid line; the least squares estimate of the conditional mean function is indicated by the red dashed line.

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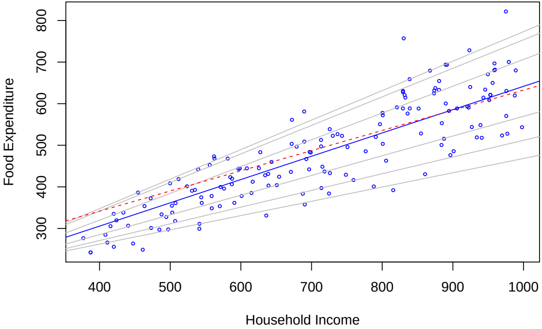

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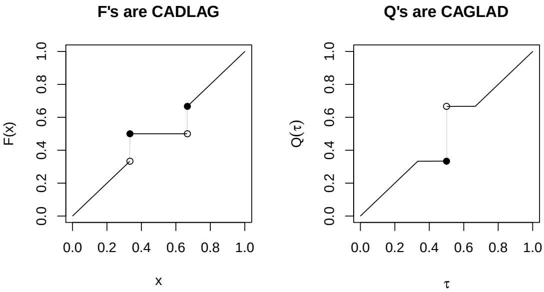
Univariate Quantiles

Given a real-valued random variable, X, with distribution function F, we can define the τth quantile of X as

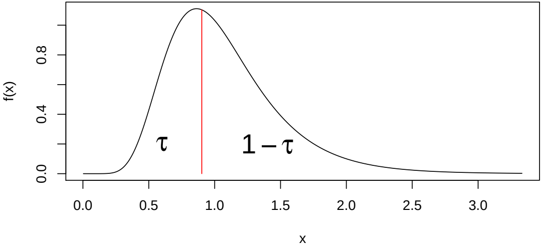
$$Q_X(\tau) = F_X^{-1}(\tau) = \inf\{x \mid F(x) \geqslant \tau\}.$$

This definition follows the usual convention that F is CADLAG, and Q is CAGLAD as illustrated in the following pair of pictures.



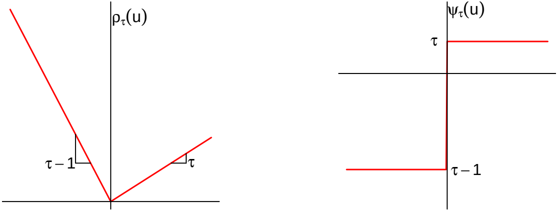
Univariate Quantiles

Viewed from the perspective of densities, the τth quantile splits the area under the density into two parts: one with area τ below the τth quantile and the other with area $1 - \tau$ above it:



Two Bits of Convex Analysis

A convex function ρ and its subgradient ψ:



The subgradient of a convex function $\rho_\tau(u)$ at a point u consists of all the possible "tangents."

Population Quantiles as Optimizers

Quantiles solve a simple optimization problem:

$$\alpha(\tau) = \text{argmin}_a \, \mathbb{E} \, \rho_\tau(Y - a)$$

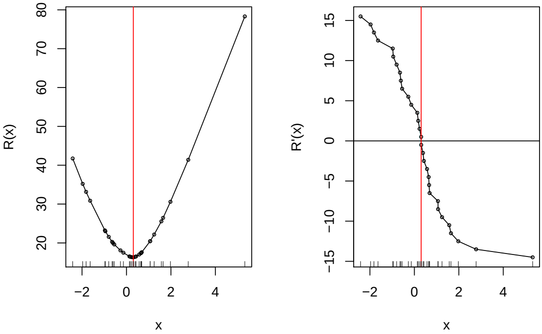
Proof: Let $\psi_\tau(u) = \rho_\tau'(u)$, so differentiating wrt to a:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \psi_\tau(y - a) dF(y) \\ &= (\tau - 1) \int_{-\infty}^{a} dF(y) + \tau \int_{a}^{\infty} dF(y) \\ &= (\tau - 1)F(a) + \tau(1 - F(a)) \end{aligned}$$

implying $\tau = F(a)$ and thus solving: $\alpha(\tau) = F^{-1}(\tau)$.

Sample Quantiles as Optimizers

For sample quantiles replace F by $\hat{F}$, the empirical distribution function. The objective function becomes a polyhedral convex function whose derivative is monotone decreasing, in effect the gradient simply counts observations above and below and weights the counts by τ and $\tau - 1$.



Conditional Quantiles: The Least Squares Meta-Model

The unconditional mean solves

$$\mu = \text{argmin}_m \mathbb{E}(Y - m)^2$$

The conditional mean $\mu(x) = E(Y|X = x)$ solves

$$\mu(x) = \text{argmin}_m \mathbb{E}_{Y|X=x}(Y - m(X))^2.$$

Similarly, the unconditional τth quantile solves

$$\alpha_\tau = \text{argmin}_a \mathbb{E}\rho_\tau(Y - a)$$

and the conditional τth quantile solves

$$\alpha_\tau(x) = \text{argmin}_a \mathbb{E}_{Y|X=x}\rho_\tau(Y - a(X))$$

Computation of Linear Regression Quantiles

Primal Formulation as a Linear Program:

$$\min\{\tau 1^\top u + (1-\tau)1^\top v | y = Xb + u - v, (b, u, v) \in \mathbb{R}^p \times \mathbb{R}_+^{2n}\}$$

Dual Formulation as a Linear Program:

$$\max\{y'd | X^\top d = (1-\tau)X^\top 1, d \in [0,1]^n\}$$

Solutions are characterized by an exact fit to p observations.

Let $h \in \mathcal{H}$ index p-element subsets of $\{1, 2, ..., n\}$ then primal solutions take the form:

$$\hat{\beta}(\tau) = \hat{\beta}(h) = X(h)^{-1}y(h)$$

These solutions may be viewed as p-dimensional analogues of the order statistics for the linear regression model.

Least Squares from the p-subset Perspective

OLS is a weighted average of these $\hat{\beta}(h)$'s:

$$\hat{\beta}_{OLS} = (X^\top X)^{-1} X^\top y = \sum_{h \in \mathcal{H}} w(h) \hat{\beta}(h),$$

$$w(h) = |X(h)|^2 / \sum_{h \in \mathcal{H}} |X(h)|^2$$

The determinants $|X(h)|$ are the (signed) volumes of the parallelipipeds formed by the columns of the the matrices $X(h)$. In the simplest bivariate case, we have,

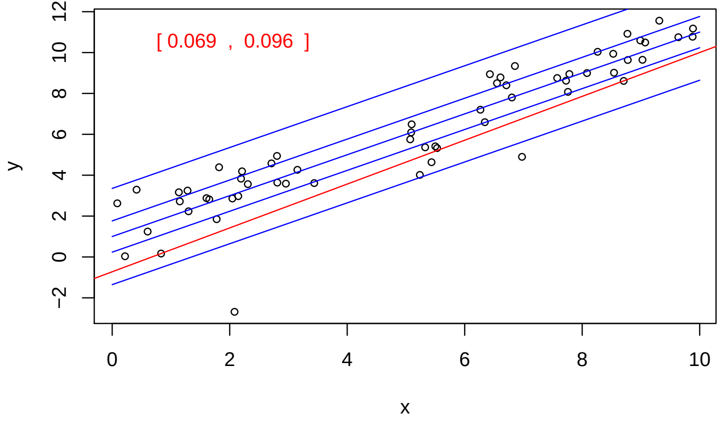
$$|X(h)|^2 = \begin{vmatrix} 1 & x_i \\ 1 & x_j \end{vmatrix}^2 = (x_j - x_i)^2$$

so pairs of observations that are far apart are given more weight. There are $\binom{n}{p}$ of these subsets, but only roughly $n \log n$ distinct quantile regression solutions for $\tau \in (0, 1)$.

Quantile Regression: The Movie

- Bivariate linear model with iid Student t errors
- Conditional quantile functions are parallel in blue
- 100 observations indicated in blue
- Fitted quantile regression lines in red.
- Intervals for $\tau \in (0, 1)$ for which the solution is optimal.

Quantile Regression in the iid Error Model



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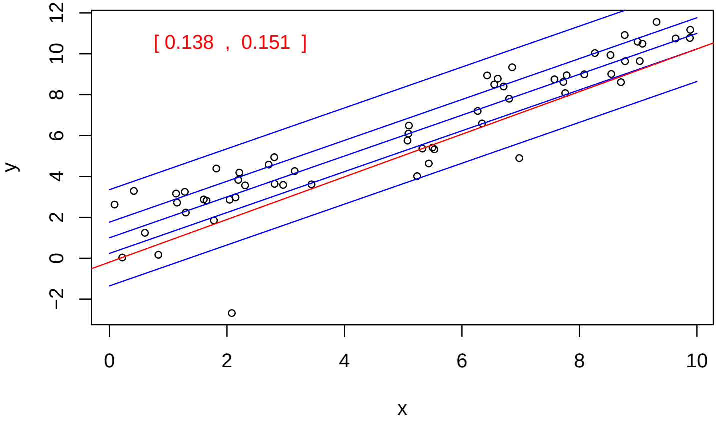

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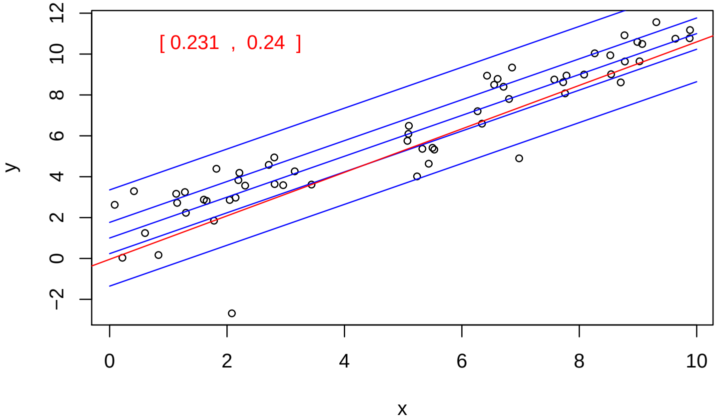

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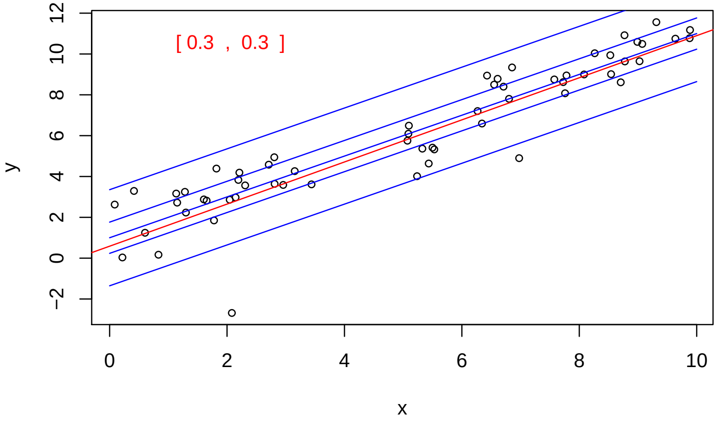

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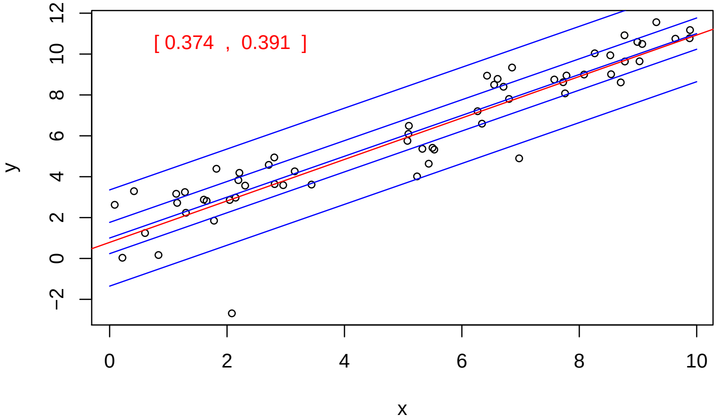

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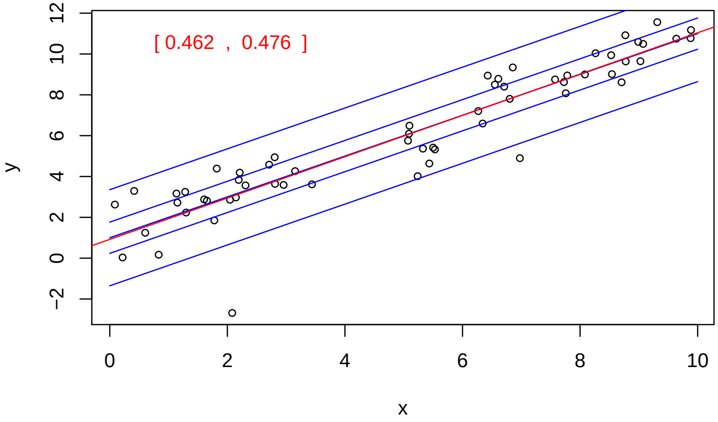

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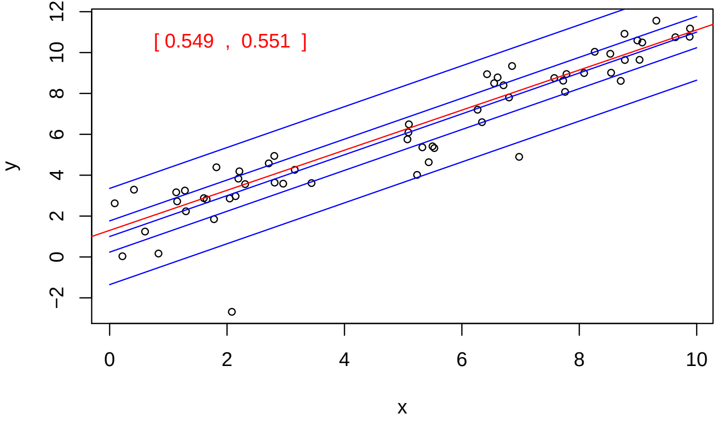

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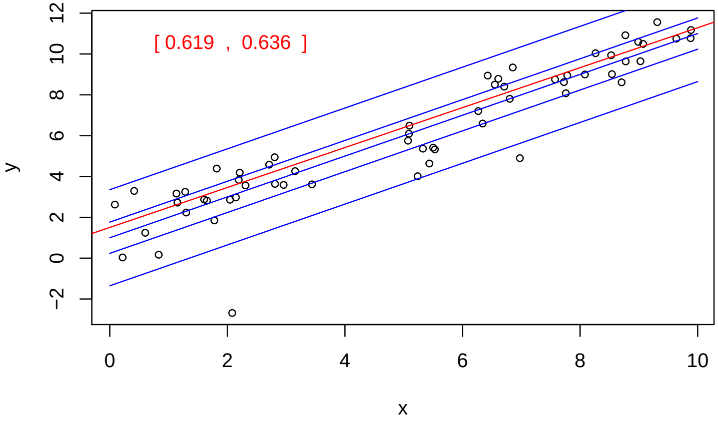

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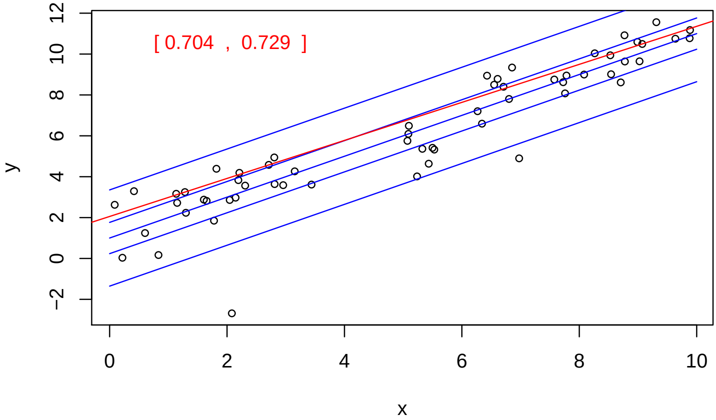

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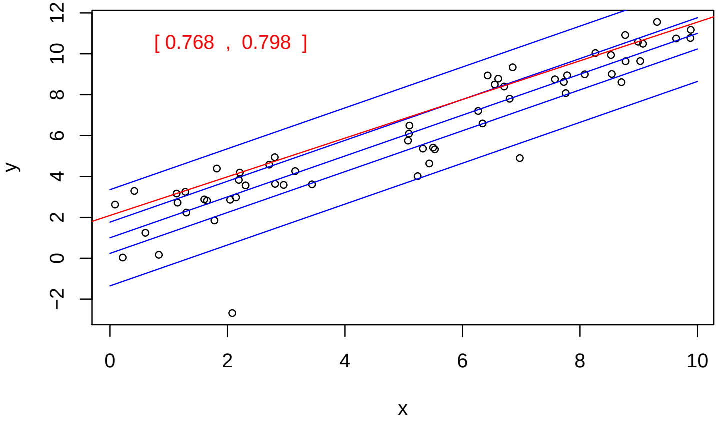

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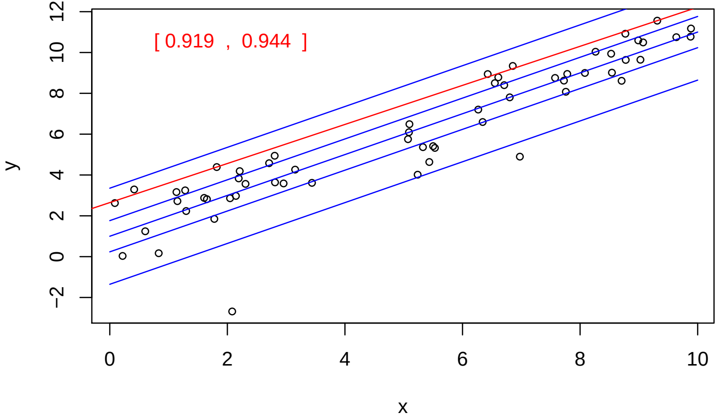
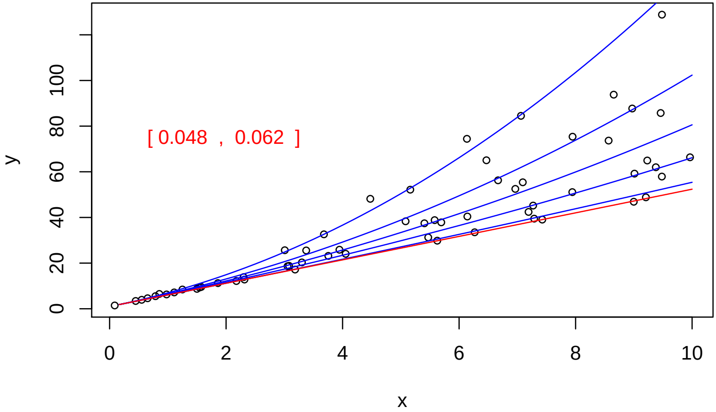


Quantile Regression: The Sequel

- Bivariate quadratic model with Heteroscedastic χ^2 errors
- Conditional quantile functions drawn in blue
- 100 observations indicated in blue
- Fitted quadratic quantile regression lines in red
- Intervals of optimality for $\tau \in (0, 1)$.

Quantile Regression in the Heteroscedastic Error Model



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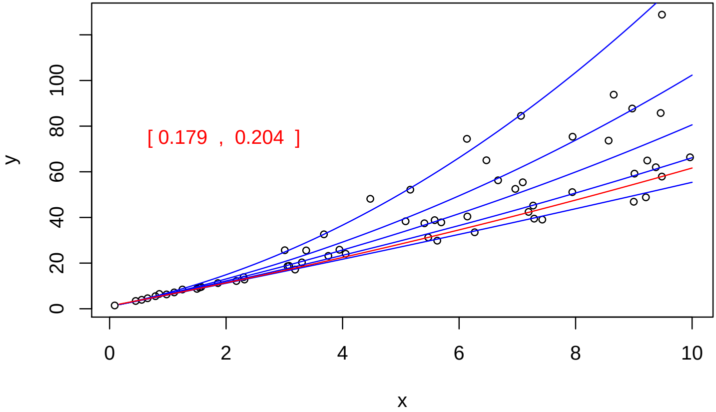

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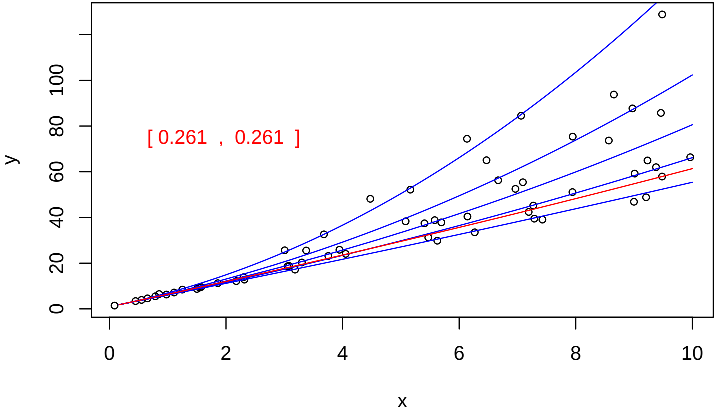

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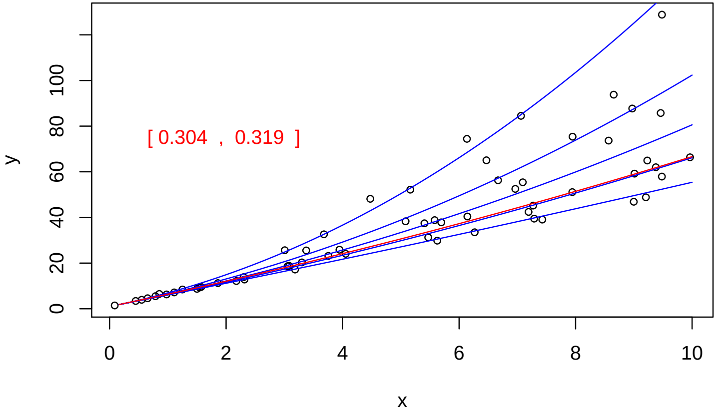

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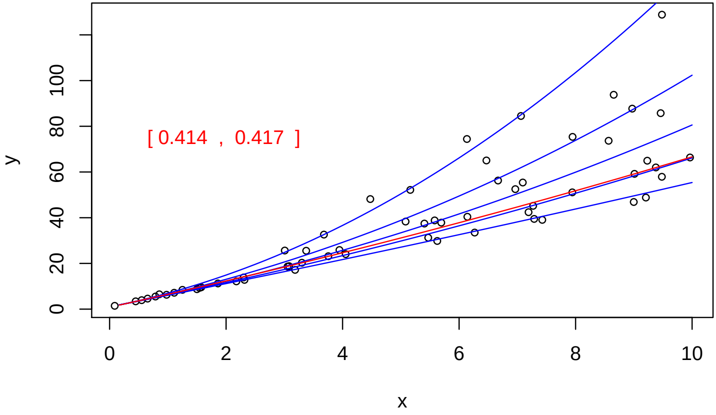

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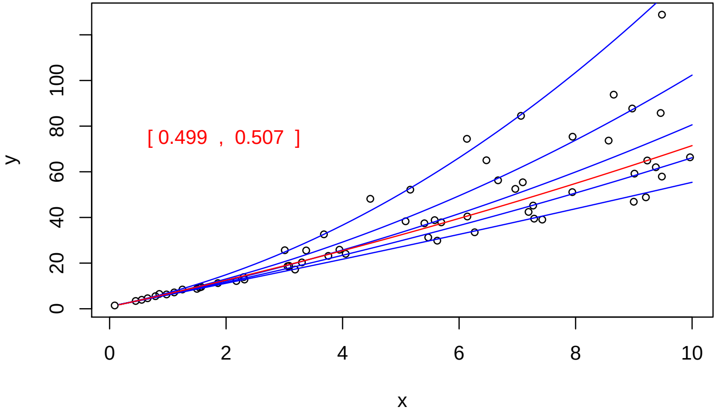

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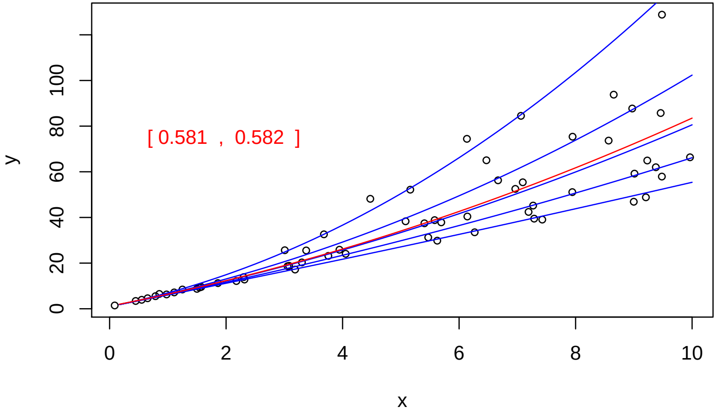

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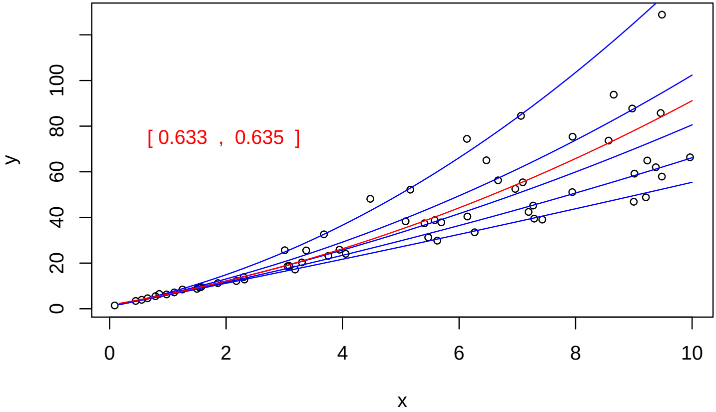

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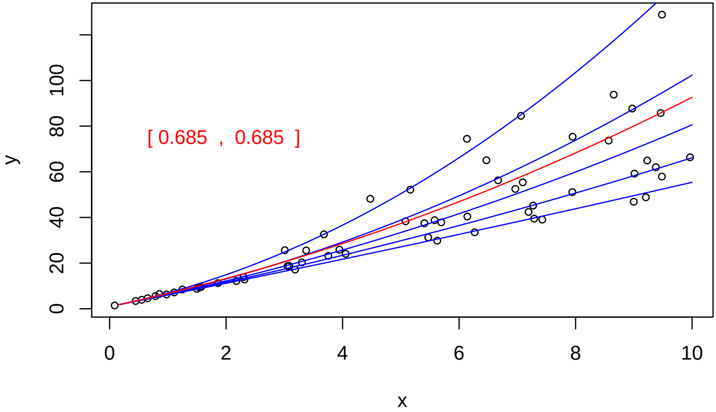

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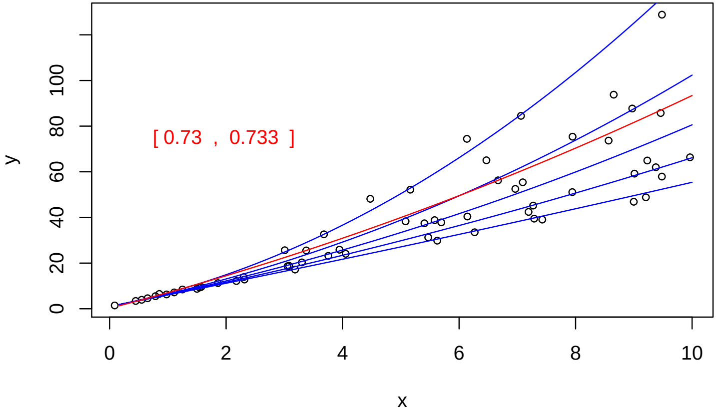

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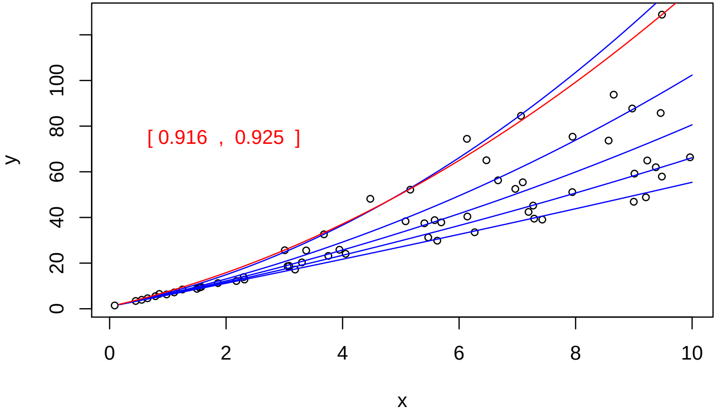
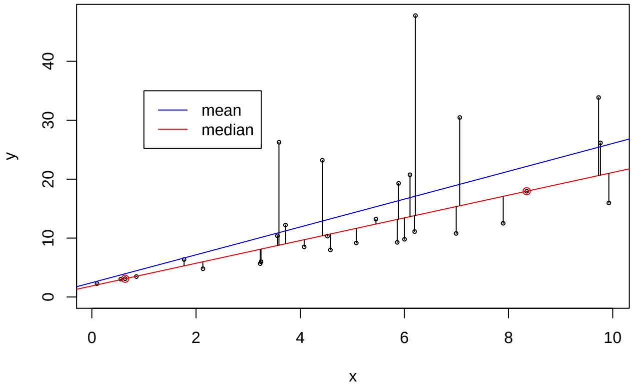


Conditional Means vs. Medians



Minimizing absolute errors for median regression can yield something quite different from the least squares fit for mean regression.

Equivariance of Regression Quantiles

- Scale Equivariance: For any $a > 0$, $\hat{\beta}(\tau; ay, X) = a\hat{\beta}(\tau; y, X)$ and $\hat{\beta}(\tau; -ay, X) = a\hat{\beta}(1 - \tau; y, X)$
- Regression Shift: For any $\gamma \in \mathbb{R}^p$ $\hat{\beta}(\tau; y + X\gamma, X) = \hat{\beta}(\tau; y, X) + \gamma$
- Reparameterization of Design: For any $|A| \neq 0$, $\hat{\beta}(\tau; y, XA) = A^{-1}\hat{\beta}(\tau; yX)$
- Robustness: For any diagonal matrix D with nonnegative elements. $\hat{\beta}(\tau; y, X) = \hat{\beta}(\tau, y + D\hat{u}, X)$

Equivariance to Monotone Transformations

For any monotone function h, conditional quantile functions $Q_Y(\tau|x)$ are equivariant in the sense that

$$Q_{h(Y)|X}(\tau|x) = h(Q_{Y|X}(\tau|x))$$

In contrast to conditional mean functions for which, generally,

$$E(h(Y)|X) \neq h(EY|X)$$

Examples:
$h(y) = \min\{0, y\}$, Powell's (1985) censored regression estimator.
$h(y) = \text{sgn}\{y\}$ Rosenblatt's (1957) perceptron, Manski's (1975) maximum score estimator. estimator.

Beyond Average Treatment Effects

Lehmann (1974) proposed the following general model of treatment response:

> *"Suppose the treatment adds the amount $\Delta(x)$ when the response of the untreated subject would be x. Then the distribution G of the treatment responses is that of the random variable $X + \Delta(X)$ where X is distributed according to F."*

Lehmann QTE as a QQ-Plot

Doksum (1974) defines $\Delta(x)$ as the "horizontal distance" between F and G at x, *i.e.*

$$F(x) = G(x + \Delta(x)).$$

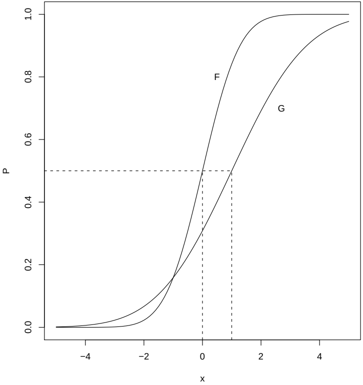
Then $\Delta(x)$ is uniquely defined as

$$\Delta(x) = G^{-1}(F(x)) - x.$$

This is the essence of the conventional QQ-plot. Changing variables so $\tau = F(x)$ we have the quantile treatment effect (QTE):

$$\delta(\tau) = \Delta(F^{-1}(\tau)) = G^{-1}(\tau) - F^{-1}(\tau).$$

Lehmann-Doksum QTE



QTE via Quantile Regression

The Lehmann QTE is naturally estimable by

$$\hat{\delta}(\tau) = \hat{G}_n^{-1}(\tau) - \hat{F}_m^{-1}(\tau)$$

where $\hat{G}_n$ and $\hat{F}_m$ denote the empirical distribution functions of the treatment and control observations, Consider the quantile regression model

$$Q_{Y_i}(\tau|D_i) = \alpha(\tau) + \delta(\tau)D_i$$

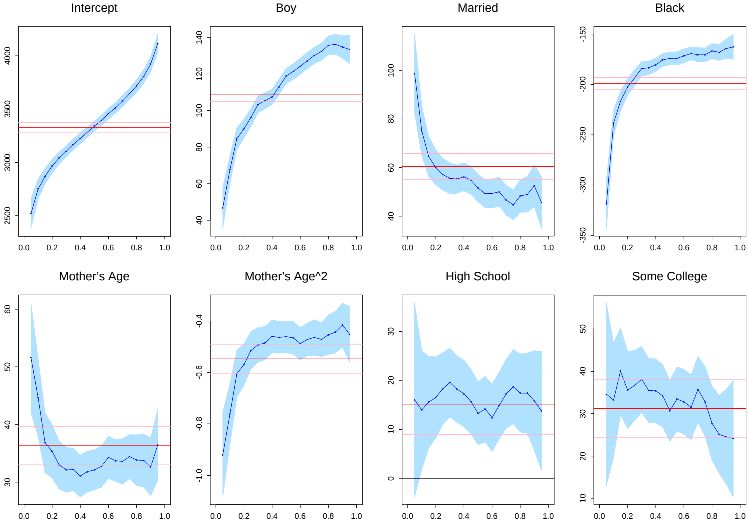
where D_i denotes the treatment indicator, and $Y_i = h(T_i)$, *e.g.* $Y_i = \log T_i$, which can be estimated by solving,

$$\min \sum_{i=1}^{n} \rho_\tau(y_i - \alpha - \delta D_i)$$

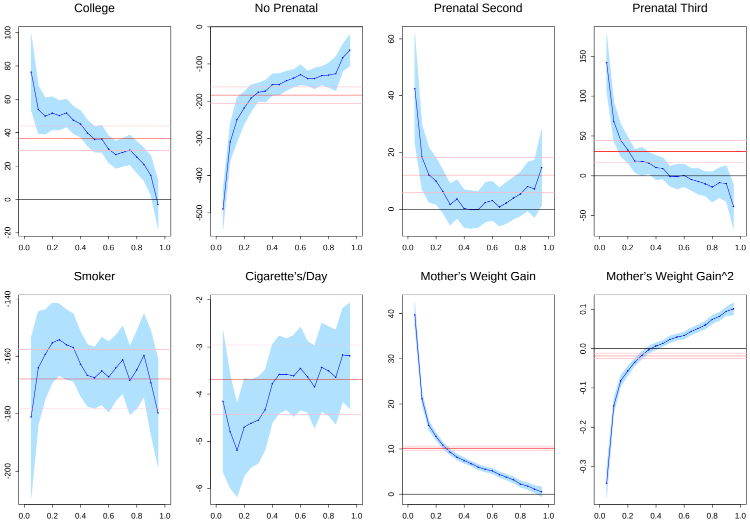
A Model of Infant Birthweight

- Reference: Abrevaya (2001), Koenker and Hallock (2001)
- Data: June, 1997, Detailed Natality Data of the US. Live, singleton births, with mothers recorded as either black or white, between 18-45, and residing in the U.S. Sample size: 198,377.
- Response: Infant Birthweight (in grams)
- Covariates:
  - Mother's Education
  - Mother's Prenatal Care
  - Mother's Smoking
  - Mother's Age
  - Mother's Weight Gain

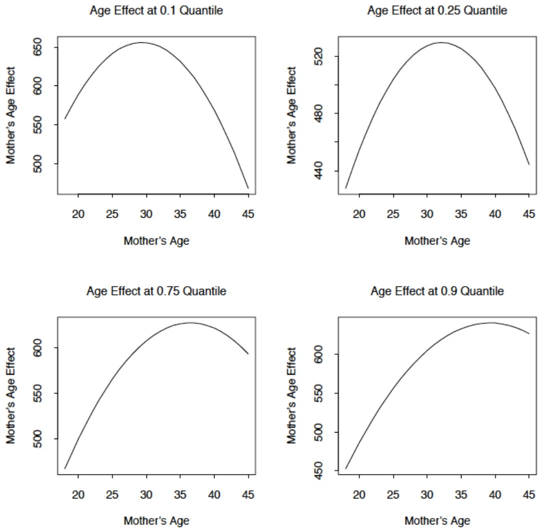
Quantile Regression Birthweight Model I



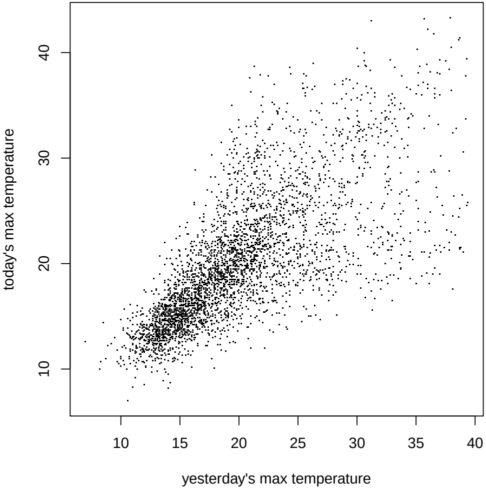
Quantile Regression Birthweight Model II



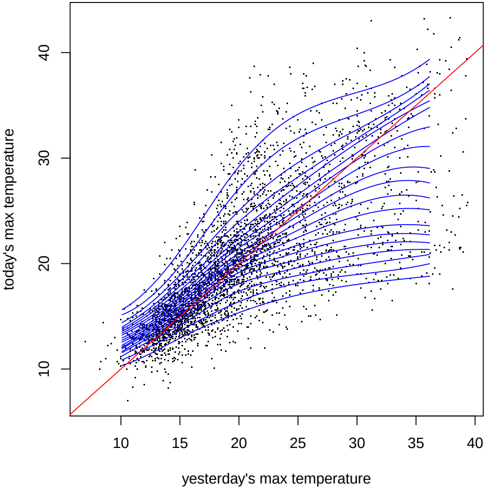
Marginal Effect of Mother's Age



Daily Temperature in Melbourne: AR(1) Scatterplot



Daily Temperature in Melbourne: Nonlinear QAR(1) Fit



Conditional Densities of Melbourne Daily Temperature