Risk, Choquet Portfolios and Quantile Regression

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Joint work with Gib Bassett (UIC) and Gregory Kordas (Athens)

Outline

- Is there a useful role for pessimism in decision theory?
- A pessimistic theory of risk
- How to be pessimistic?

St. Petersburg Paradox

What would you be willing to pay to play the game:

$$G = \{\text{pay: } p, \quad \text{win: } 2^n \text{ with probability } 2^{-n}, \quad n = 1, 2, \ldots\}$$

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Daniel Bernoulli ($\sim$ 1728) observed that even though the expected payoff was infinite, the gambler who maximized logarithmic utility would pay only a finite value to play. For example, given initial wealth 100,000 Roubles, our gambler would be willing to pay only 17 Roubles and 55 kopecks. If initial wealth were only 1000 Roubles, then the value of the game is only about 11 Roubles.

Expected Utility

To decide between two real valued gambles

$$X \sim F \quad \text{and} \quad Y \sim G$$

we choose X over Y if

$$Eu(X) = \int u(x)dF(x) \geqslant \int u(y)dG(y) = Eu(Y)$$





On Axiomatics

Suppose we have acts $P, Q, R, \ldots$ in a space $\mathcal{P}$, which admits enough convex structure to allow us to consider mixtures,

$$\alpha P + (1 - \alpha) Q \in \mathcal{P} \quad \alpha \in (0, 1)$$

Think of P, Q, R as probability measures on some underlying outcome/event space, $\mathcal{X}$.
Or better, view P, Q, R as acts mapping a space $\mathcal{S}$ of soon-to-be-revealed "states of nature" to the space of probability measures on the outcome space, $\mathcal{X}$.

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(A.3) (continuity) For all $P, Q, R \in \mathcal{P}$, if $P \succ Q$ and $Q \succ R$, then there exist α and $\beta \in (0,1)$, such that, $\alpha P + (1-\alpha)R \succ \beta Q(1-\beta)R$.

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Then there exists a linear function u on $\mathcal{P}$ such that for all $P, Q \in \mathcal{P}$, $P \succ Q$ if and only if $u(P) > u(Q)$.

Weakening the Independence Axiom

The independence axiom seems quite innocuous, but it is extremely powerful. We will consider a weaker form of independence due to Schmeidler (1989).

(A.2') (comonotonic independence) For all pairwise comonotonic $P, Q, R \in \mathcal{P}$ and $\alpha \in (0,1)$ $P \succ Q \Rightarrow \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$,

Definition Two acts P and Q in $\mathcal{P}$ are comonotonic, or similarly ordered, if for no s and t in $\mathcal{S}$,

$$P(\{t\}) \succ P(\{s\}) \quad \text{and} \quad Q(\{s\}) \succ Q(\{t\}).$$

"If P is better in state t than state s, then Q is also better in t than s."

On Comonotonicity

Definition The two functions $X, Y : \Omega \to \mathbb{R}$ are comonotonic if there exists a third function $Z : \Omega \to \Re$ and increasing functions f and g such that $X = f(Z)$ and $Y = g(Z)$.

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From our point of view the crucial property of comonotonic random variables is the behavior of quantile functions of their sums. For comonotonic random variables X, Y, we have

$$F_{X+Y}^{-1}(u) = F_X^{-1}(u) + F_Y^{-1}(u)$$

By comonotonicity we have a $U \sim U[0, 1]$ such that $Z = g(U) = F_X^{-1}(U) + F_Y^{-1}(U)$ where g is left continuous and increasing, so by monotone invariance, $F_{g(U)}^{-1} = g \circ F_U^{-1} = F_X^{-1} + F_Y^{-1}$.
Comonotonic random variables are maximally dependent *a la* Fréchet

Choquet Expected Utility

Among the many proposals offered to extend expected utility theory the most attractive (to us) replaces

$$E_F u(X) = \int_0^1 u(F^{-1}(t))dt \geqslant \int_0^1 u(G^{-1}(t))dt = E_G u(Y)$$

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The measure ν permits distortion of the probability assessments after ordering the outcomes. This rank dependent form of expected utility has been pioneered by Quiggin (1981), Yaari (1987), Schmeidler (1989), Wakker (1989) and Dennenberg (1990).

Pessimism

By relaxing the independence axiom we obtain a larger class of preferences representable as Choquet capacities and introducing pessimism. The simplest form of Choquet expected utility is based on the "distortion"

$$\nu_\alpha(t) = \min\{t/\alpha, 1\}$$

so

$$E_{\nu_\alpha, F} u(X) = \alpha^{-1} \int_0^\alpha u(F^{-1}(t))dt$$

This exaggerates the probability of the proportion α of least favorable events, and totally discounts the probability of the $1 - \alpha$ most favorable events.

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Expect the worst – and you won't be disappointed.

A Smoother example

A simple, yet intriguing, one-parameter family of pessimistic Choquet distortions is the measure:

$$\nu_\theta(t) = 1 - (1-t)^\theta \qquad \theta \geqslant 1$$

Note that, changing variables, $t \to F_X(u)$, we have,

$$E_{\nu_\theta} X = \int_0^1 F_X^{-1}(t) d\nu(t) = \int_{-\infty}^{\infty} u d(1 - (1 - F_X(u))^\theta)$$

The pessimist imagines that he gets not a single draw from X but θ draws, and from these he always gets the worst. The parameter θ is a natural "measure of pessimism," and need not be an integer.

Savage on Pessimism

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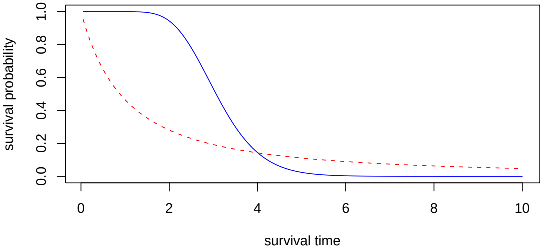
Pessimistic Medical Decision Making?



Survival Functions for a hypothetical medical treatment: The Lehmann quantile treatment effect (QTE) is the horizontal distance between the survival curves. In this example consideration of the mean treatment effect would slightly favor the (dotted) treatment curve, but the pessimistic patient might favor the (solid) placebo curve. Only the luckiest 15% actually do better under the treatment.

Risk as Pessimism?

In expected utility theory risk is entirely an attribute of the utility function:

Risk Neutrality	$\Rightarrow$	$u(x) \sim$ affine
Risk Aversion	$\Rightarrow$	$u(x) \sim$ concave
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Locally, the risk premium, i.e. the amount one is willing to pay to accept a zero mean risk, X, is

$$\pi(w, X) = \frac{1}{2} A(w) V(X)$$

where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt coefficient of absolute risk aversion and $V(X)$ is the variance of X.

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where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt coefficient of absolute risk aversion and $V(X)$ is the variance of X. Why is variance a reasonable measure of risk?

A Little Risk Aversion is a Dangerous Thing

Would you accept the gamble:

$$G_1 \qquad\qquad 50-50 \left\langle \begin{array}{l} \text{win } \$110 \\ \text{lose } \$100 \end{array} \right.$$

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Moral: A little local risk aversion over small gambles implies implausibly large risk aversion over large gambles. Reference: Rabin (2000)

Are Swiss Bicycle Messengers Risk Averse?

When Veloblitz and Flash bicycle messengers from Zurich were confronted with the bet:

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More than half (54%) rejected the bet.
Reference: Fehr and Götte (2002)

Coherent Risk

Definition (Artzner, Delbaen, Eber and Heath (1999)) For real valued random variables $X \in \mathcal{X}$ on $(\Omega, \mathcal{A})$ a mapping $\rho : \mathcal{X} \to \mathcal{R}$ is called a coherent risk measure if,

1. Monotone: $X, Y \in \mathcal{X}$, with $X \leqslant Y \Rightarrow \rho(X) \geqslant \rho(Y)$.
2. Subadditive: $X, Y, X + Y \in \mathcal{X}, \Rightarrow \rho(X + Y) \leqslant \rho(X) + \rho(Y)$.
3. Linearly Homogeneous: For all $\lambda \geqslant 0$ and $X \in \mathcal{X}$, $\rho(\lambda X) = \lambda\rho(X)$.
4. Translation Invariant: For all $\lambda \in \mathcal{R}$ and $X \in \mathcal{X}$, $\rho(\lambda + X) = \rho(X) - \lambda$.

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Many conventional measures of risks including those based on standard deviation are ruled out by these requirements. So are quantile based measures like "value at risk."

Choquet α-Risk

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Note that $\rho_{\nu_\alpha}(X) = -E_{\nu_\alpha, F}(X)$, so Choquet α-risk is just negative Choquet expected utility with the distortion function ν_α.

Pessimistic Risk Measures

Definition A risk measure ρ will be called pessimistic if, for some probability measure φ on [0, 1]

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By Fubini

$$\begin{aligned}
\rho(X) &= -\int_0^1 \alpha^{-1} \int_0^\alpha F^{-1}(t) dt d\varphi(\alpha) \\
&= -\int_0^1 F^{-1}(t) \int_t^1 \alpha^{-1} d\varphi(\alpha) dt \\
&\equiv -\int_0^1 F^{-1}(t) d\nu(t)
\end{aligned}$$

Approximating General Pessimistic Risk Measures

We can approximate any pessimistic risk measure by taking

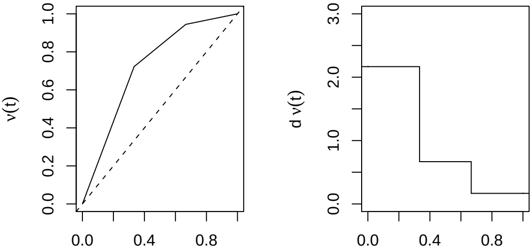
$$d\varphi(t) = \sum \varphi_i \delta_{\tau_i}(t)$$

where δ_τ denotes (Dirac) point mass 1 at τ. Then

$$\rho(X) = -\varphi_0 F^{-1}(0) - \int_0^1 F^{-1}(t)\gamma(t)dt$$

where $\gamma(t) = \sum \varphi_i \tau_i^{-1} I(t < \tau_i)$ and $\varphi_i > 0$, with $\sum \varphi_i = 1$.

An Example



$$d\varphi(t) = \frac{1}{2}\delta_{1/3}(t) + \frac{1}{3}\delta_{2/3}(t) + \frac{1}{6}\delta_1(t)$$

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- Pessimistic Choquet risk measures correspond to *concave* ν, i.e., *monotone decreasing* $d\nu$.
- Probability assessments are distorted to accentuate the probability of the least favorable events.
- The crucial coherence requirement is subadditivity, or submodularity, or 2-alternatingness in the terminology of Choquet capacities.

An Example

Samuelson (1963) describes asking a colleague at lunch whether he would be willing to make a

$$50 - 50 \text{ bet} \quad \left\langle \begin{array}{l} \text{win } 200 \\ \text{lose } 100 \end{array} \right.$$

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The colleague (later revealed to be E. Cary Brown) responded

"no, but I *would* be willing to make 100 such bets."

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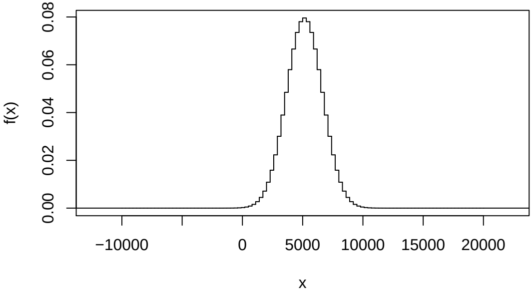
"no, but I *would* be willing to make 100 such bets."

This response has been interpreted not only as reflecting a basic confusion about how to maximize expected utility but also as a fundamental misunderstanding of the law of large numbers.

Payoff Density of 100 Samuelson trials



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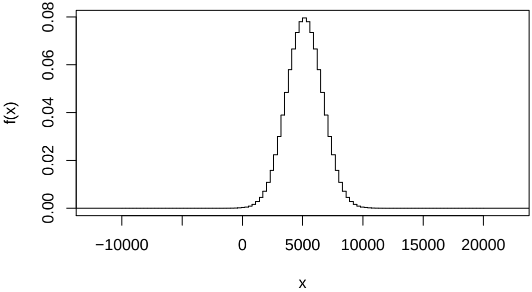

Odds of losing money on the 100 trial bet is 1 chance in 2300.

Was Brown really irrational?

Suppose, for the sake of simplicity that

$$d\varphi(t) = \tfrac{1}{2}\delta_{1/2}(t) + \tfrac{1}{2}\delta_1(t)$$

so for one Samuelson coin flip we have the unfavorable evaluation,

$$E_{\nu,F}(X) = \tfrac{1}{2}(-100) + \tfrac{1}{2}(50) = -25$$

but for $S = \sum_{i=1}^{100} X_i \sim \mathcal{B}in(.5, 100)$ we have the favorable evaluation,

$$\begin{aligned} E_{\nu,F}(S) &= \tfrac{1}{2}2\int_0^{1/2} F_S^{-1}(t)dt + \tfrac{1}{2}(5000) \\ &= 1704.11 + 2500 \\ &= 4204.11 \end{aligned}$$

How to be Pessimistic

Theorem Let X be a real-valued random variable with $EX = \mu < \infty$, and $\rho_\alpha(u) = u(\alpha - I(u < 0))$. Then

$$\min_{\xi \in \mathcal{R}} E\rho_\alpha(X - \xi) = \alpha\mu + \rho_{\nu_\alpha}(X)$$

So α risk can be estimated by the sample analogue

$$\hat{\rho}_{\nu_\alpha}(x) = (n\alpha)^{-1} \min_{\xi} \sum \rho_\alpha(x_i - \xi) - \hat{\mu}_n$$

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 I knew it! Eventually everything looks like quantile regression to this guy!

Pessimistic Portfolios

Now let $X = (X_1, \ldots, X_p)$ denote a vector of potential portfolio asset returns and $Y = X^\top \pi$, the returns on the portfolio with weights π. Consider

$$\min_{\pi} \rho_{\nu_\alpha}(Y) - \lambda\mu(Y)$$

Minimize α-risk subject to a constraint on mean return.

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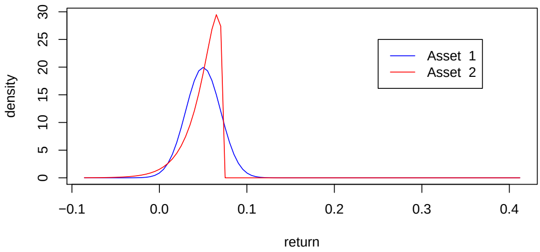
Minimize α-risk subject to a constraint on mean return.
This problem can be formulated as a linear quantile regression problem

$$\min_{(\beta,\xi)\in\mathcal{R}^p} \sum_{i=1}^{n} \rho_\alpha(x_{i1} - \sum_{j=2}^{p}(x_{i1} - x_{ij})\beta_j - \xi) \quad \text{s.t.} \quad \bar{x}^\top \pi(\beta) = \mu_0,$$

where $\pi(\beta) = (1 - \sum_{j=2}^{p} \beta_j, \beta^\top)^\top$.

An Example



Two asset return densities with identical mean and variance.

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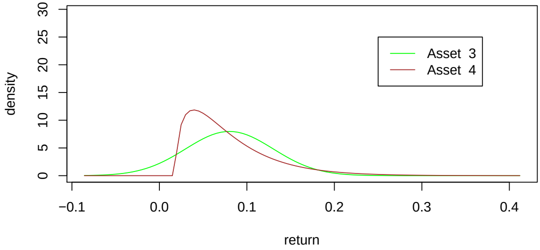
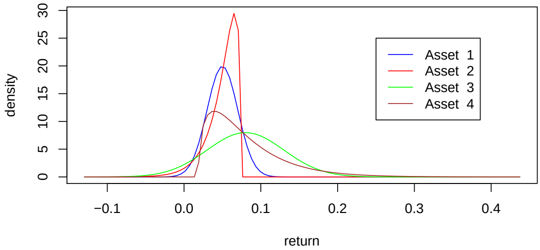


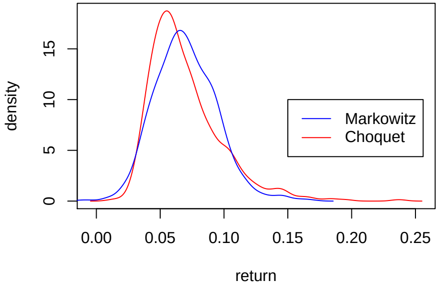
Two more asset return densities with identical mean and variance.

An Example



Two pairs of asset return densities with identical mean and variance.

Optimal Choquet and Markowitz Portfolio Returns



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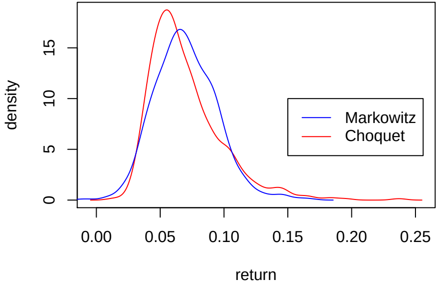
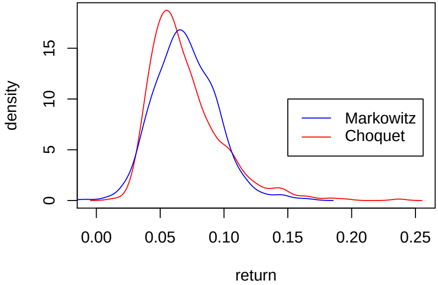


Markowitz portfolio minimizes the standard deviation of returns subject to mean return $\mu = .07$. The Choquet portfolio minimizes Choquet risk (for $\alpha = .10$) subject to earning the same mean return. The Choquet portfolio has better performance in both tails than mean-variance Markowitz portfolio.

Optimal Choquet and Markowitz Portfolio Returns



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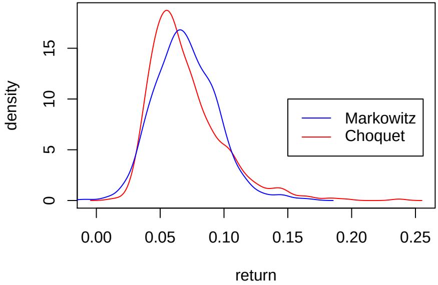

Now, the Markowitz portfolio minimizes the standard deviation of returns subject to mean return $\mu = .07$. The Choquet portfolio maximizes expected return subject to achieving the same Choquet risk (for $\alpha = .10$) as the Markowitz portfolio. Choquet portfolio has expected return $\mu = .08$ a full percentage point higher than the Markowitz portfolio.

A Unified Theory of Pessimism

Any pessimistic risk measure may be approximated by

$$\rho_\nu(X) = \sum_{k=1}^{m} \varphi_k \rho_{\nu_{\alpha_k}}(X)$$

where $\varphi_k > 0$ for $k = 1, 2, ..., m$ and $\sum \varphi_k = 1$.
Portfolio weights can be estimated for these risk measures by solving linear programs that are weighted sums of quantile regression problems:

$$\min_{(\beta,\xi)\in\mathcal{R}^p} \sum_{k=1}^{m} \sum_{i=1}^{n} \nu_k \rho_{\alpha_k}(x_{i1} - \sum_{j=2}^{p} (x_{i1} - x_{ij})\beta_j - \xi_k) \quad s.t. \quad \bar{x}^\top \pi(\beta) = \mu_0,$$

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Software in R is available on from my web pages.

Conclusions

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- Choquet (non-additive, rank dependent) expected utility provides a simple, tractable alternative.
- Mean-variance Portfolio allocation is also unsatisfactory since it relies on unpalatable assumptions of Gaussian returns, or quadratic utility.
- Choquet portfolio optimization can be formulated as a quantile regression problem thus providing an attractive practical alternative to the dominant mean-variance approach of Markowitz (1952).