University of Illinois | Department of Economics
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# Lecture 1
# An Introduction to Empirical Demand Analysis

This introductory lecture will review some basic consumer demand theory stressing empirical implementation. The theory is intended to provide a basis for analyzing the data in the Giffen good problem set.

1. Fundamentals. Given some rather mild regularity conditions on preferences it can be shown that the preference relation $x \succeq z$ which we read as "the bundle $x$ is weakly preferred to the bundle $z$" can be rationalized by a utility function $u(\cdot)$, that is, there exists $u : \Re^n \to \Re$ such that,

$$x \succeq z \Leftrightarrow u(x) \geq u(z).$$

Note that $u(\cdot)$ need not convey any cardinal information about "utility" since any monotonic transformation of $u(x)$, e.g., $\log(u(x))$, yields the same ordering of bundles.

Given such a utility function representing preferences we may formulate "consumer behavior" as the problem

$$\max u(x) \qquad \text{s.t.} \qquad p^\top x = y \tag{1}$$

where $p$ denotes a vector of prices of the coordinate commodities of $x$, and $y$ denotes income. We abstract from various complications: nonlinear pricing, intertemporal decision making, conspicuous consumption, etc., etc.

The solution to the problem formulated in (1) may be written as

$$x = g(p, y)$$

called the "Marshallian demands". Substituting these demands into the utility function we obtain the indirect utility function:

$$v(p, y) \equiv u(g(p, y))$$

which expresses the maximum achievable utility under price-income regime $(p, y)$.

An alternative approach which yields equivalent results is to formulate the dual problem

$$\min p^\top x \qquad \text{s.t.} \qquad u(x) = u \tag{2}$$

for the sake of definiteness we could think of $u$ as $u_0$, the level of utility acheived by solving (1). The problem (2) has solution,

$$x = h(p, u)$$

which we call the "Hicksian demands" since they represent optimal consumption behavior as a function of the utility level, $u$, in contrast to the Marshallian formulation which is in terms of the (*observable*!) level of income.

Again substituting, we have

$$c(p,u) \equiv p^\top h(p,u)$$

which is usually called the cost or expenditure function, and plays an important role in welfare economics.

Yet a third variant of the foregoing is the so-called Frisch demands. Recall that solving the primal problem yields the first order conditions

$$\frac{\partial u(x)}{\partial x_i} = \lambda p_i \qquad i = 1, \ldots, n \tag{3}$$

where $\lambda$ denotes the marginal utility of income, a Lagrange multiplier on the budget constraint. Thus, we can interpret $r = 1/\lambda$ as the marginal cost of utility at current prices. *When utility is additive*, i.e.,

$$u(x) = \sum_{i=1}^{n} u_i(x_i)$$

then (3) becomes

$$u_i'(x_i) = p_i/r$$

and inverting, recall $u_i(q_i)$ must be monotonic by non-satiation, we have

$$x_i = f(p_i/r),$$

so demand for the $i$th good depends only on $r$ and the $i$th price. In general, this doesn't make any sense but under additivity, e.g., in many intertemporal models, it is quite useful.

*Example*: An important example of the foregoing theory is the linear expenditure system which was developed by Stone, Gorman, Samuelson and others. It takes the additive form

$$u(x) = \sum \beta_i \log(x_i - \alpha_i)$$

with $\sum \beta_i = 1$, so

$$u_i'(x_i) = \frac{\beta_i}{x_i - \alpha_i} = \lambda p_i \qquad i = 1, \ldots, n$$

and using the $\beta$ constraint and the budget constraint,

$$1 = \sum \beta_i = \lambda \sum p_i(x_i - \alpha_i)$$

so

$$\lambda = (y - \sum p_i \alpha_i)^{-1}$$

and

$$x_i = \alpha_i + \frac{\beta_i}{p_i}[y - \sum p_i \alpha_i].$$

This formulation has a convenient economic interpretation:

$\alpha_i$ is the "committed quantity" of consumption of $x_i$, purchased irregardless of currently prevailing prices, and
$\beta_i$ is the marginal budget share of $x_i$, i.e. how the share changes with $y$.

Thus we may view the consumer as utility deciding to purchase $\alpha_i$ of $x_i$, then computing his remaining, "excess" income $y - \sum p_i\alpha_i$ and then allocating that according to the $\beta_i$'s.

Substituting this $g(p, y)$ into $u(x)$ as above, we obtain,

$$v(p, y) = \sum \beta_i \log\left(\frac{\beta_i}{p_i}(y - \sum p_i\alpha_i)\right)$$

Taking exponentials, we have the equivalent form,

$$\nu(p, y) = e^{v(p,y)} = (y - \sum p_i\alpha_i)\prod_{i=1}^{n}(\beta_i/p_i)^{\beta_i}.$$

Solving for $y$ we obtain the cost function,

$$c(p, u) = u\prod(p_i/\beta_i)^{\beta_i} + \sum p_i\alpha_i.$$

This provides a very convenient way to compute true cost of living indices or exact compensating variations required to change prices from $p^0$ to $p^1$. $\square$

*Theorem*: If $u(x)$ is continuous, strictly quasi-concave and non-satiated, then the associated cost (expenditure) function $c(p, u)$ is homogeneous of degree 1 in $p$, concave, strictly increasing in $u$, and has partial derivatives which are the compensated (Hicksian) demand functions.

*Proof* Each property is treated in turn.

(i) $H1^\circ$ Note $c(p, u) = \min\{p^\top q | u(x) = u\}$ so $c(\theta p, u) = \theta c(p, u)$.

(ii) *Concavity.* Take any two price vectors $p^0, p^1$. Set $p^\theta = \theta p^0 + (1-\theta)p^1$ for $\theta \in (0, 1)$. Let $x^\theta$ be optimal for $p^\theta$ and $u$, i.e., $x^\theta = h(p^\theta, u)$, then

$$c(p^\theta, u) = p^{\theta\prime}x^\theta = \theta x^{\theta\prime}p^0 + (1-\theta)x^{\theta\prime}p^1$$

but $x^\theta$ isn't optimal at $p^0$ or $p^1$, i.e.,

$$c(p^i, u) \le x^{\theta\prime}p^1 \qquad i = 1, 2$$

so the result follows.

(iii) $\nearrow$ *in* $u$. This follows from nonsatiation since more $u$ requires more $x$ in at least one coordinate.

(iv) *Derivatives.* Existence of derivatives follows from standard convexity arguments. Let

$$Z(p) = p^\top x^0 - c(p, u)$$

where $x^0 = h(p^0, u)$. Now $Z(p) \ge 0$ by construction, since $p^\top x^0$ is always bigger than (or equal to) the minimal cost way of achieving $u$, under prices $p$. Think of it this way: $x^0$ is one way to achieve utility level, $u$, but it isn't necessarily the best way unless $p = p^0$. But $Z(p)$ is known to achieve a minimum of 0 when $p = p^0$, so $Z(p)$ is stationary at $p^0$, i.e.,

$$\frac{\partial Z(p)}{\partial p_i}|_{p=p^0} = x_i^0 - \frac{\partial c(p,u)}{\partial p_i}|_{p=p^0} = 0$$

Note that this equality depends on strict convexity of preferences. Thus,

$$\frac{\partial c(p,u)}{\partial p_i} = h_i(p,u)$$

as was asserted. [In the theory of production this result is called Shephard's Lemma.] □

Given Hicksian demand functions it is straightforward to obtain Marshallian demands, by simply substituting for $u$ the indirect utility function,

$$x = h(p,u) = h(p, \nu(p,y)) = g(p,y)$$

and obviously this works in reverse as well,

$$x = g(p,y) = g(p, c(p,u)) = h(p,u).$$

Finally, we may observe that Marshallian demands may be obtained from the indirect utility function by differentiating the identity,

$$\nu(c(p,u), p) = u$$

to obtain

$$\frac{\partial \nu}{\partial y}\frac{\partial c}{\partial p_i} + \frac{\partial \nu}{\partial p_i} = 0$$

which gives us Roy's identity,

$$x_i = \frac{\partial c}{\partial p_i} = -\frac{\partial \nu / \partial p_i}{\partial \nu / \partial y} = g_i(p,y)$$

Thus, to summarize, by differentiating the expenditure function with respect to prices we obtain the Hicksian demand functions via Shephard's Lemma, while by differentiating the indirect utility function we get via Roy's identity the Marshallian demands. Thus it is often convenient for empirical purposes to start with either $c(p,u)$ or $\nu(p,y)$ rather than the more conventional $u(x)$, as a parametric specification since the former allows us to derive the precise form of the demand equations by elementary differentiation.

*Theorem.* Hicksian and Marshallian demands satisfy the following conditions:

| | | |
|---|---|---|
| (i) | Adding-up: | $p^\top h(p,u) = p^\top g(p,y) = y$ |
| (ii) | Homogeneity: | $h(\theta p, u) = h(p,u)$ and $g(\theta p, \theta y) = g(p,y)$ |
| (iii) | Symmetry: | $\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_i}$ |
| (iv) | Negative-Semi-Definiteness: | $\sum \sum \xi_i \xi_j \partial h_i(p,u)/\partial p_j \leq 0$ for any $\xi \in \Re^n$, and $\sum p_i \partial h_j(p,u)/\partial p_i = 0$ |

*Proof.*

(i) This is simply nonsatiation which implies that optimal demands exhaust the available income.

(ii) Since $c(p,u)$ is homogeneous of degree one, its derivatives, $h(p,u)$, are homogeneous of degree zero. [I.e., $f(\theta x) = \theta f(x) \Rightarrow \nabla_x f(\theta x)\theta = \theta \nabla_x f(x) \Rightarrow \nabla_x f(\theta x) = \nabla_x f(x)$]. Note that this is also intuitively obvious if we observe that optimal consumption bundles are determined by *relative* prices, while for Marshallian demands $(\theta p, \theta y)$ determines the same budget (feasible) set as $(p, y)$.

(iii) Symmetry is a trivial consequence of the fact that the order of differentiation in the cross partial $\partial c(p,u)/\partial p_i \partial p_j$ doesn't matter.

(iv) Negative semidefiniteness is a consequence of the concavity of $c(p,u)$ in $p$, and singularity constraint is a consequence of $(i)$.

The Jacobian matrix of the compensated demands, or Hessian matrix of the expenditure function, with respect to $p$,

$$S = \left( \frac{\partial h_i(p,u)}{\partial p_j} \right) = \nabla_p h(p,u) = \nabla_p^2 c(p,u)$$

is extremely important, and is usually called the Slutsky matrix. It formulates the demand response to changes in price *holding utility constant.* It is useful to have an expression for $S$ in terms of Marshallian demands, since they can be estimated – Hicksian demands cannot since $u$ is unobservable. This can be done as follows:

$$g_i(p, c(p,u)) = h_i(p,u) \qquad i = 1, \ldots, n$$

so

$$\frac{\partial g_i}{\partial y} \, \frac{\partial c}{\partial p_j} + \frac{\partial g_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} = s_{ij}$$

and thus

$$s_{ij} = \frac{\partial g_i}{\partial y} x_j + \frac{\partial g_i}{\partial p_j}$$

which is the classical Slutsky (1915) decomposition. In elasticity form this is somewhat more conveniently expressed by multiplying through by $p_j/x_i$ to obtain

$$\eta_{ij}^h = \frac{\partial g_i}{\partial y} \frac{y}{x_i} \frac{p_j x_j}{y} + \frac{\partial g_i}{\partial p_j} \frac{p_j}{x_i} = \eta_{iy}\theta_j + \eta_{ij}^g$$

where $\eta_{ij}^h$ and $\eta_{ij}^g$ are the Hicksian and Marshallian price elasticities respectively, $\eta_{iy}$ is the income elasticity of commodity $i$ and $\theta_j = p_j x_j / y$ is the expenditure share of commodity $j$. Note that the fact that $s_{ij} = s_{ji}$ doesn't imply that the corresponding elasticities are symmetric unless the budget shares of the two goods, $x_i$ and $x_j$ are the same.

The own-Slutsky effect, $s_{ii}$, is necessarily negative since it is the demand response along an indifference curve to a change in a good's own price. Formally, to see this set $\xi_i = e_i$ in part (iv) of the preceding theorem. But note that, notoriously, the derivative of the Marshallian demand

$$\frac{\partial g_i}{\partial p_i} = s_{ii} - x_i \frac{\partial g_i}{\partial y}$$

can be positive if the last term is sufficiently negative. This is the well known Giffen effect which Marshall introduced. This phenomenon has a curious history in empirical economics, and the first problem set concerns an ancient attempt of mine to find a convincing real example, Koenker (1977). A more recent empirical study by Jensen and Miller (2008) provides a somewhat more convincing case. Obviously, Giffenness can only happen when $x_i$ is "inferior" i.e., when $\partial g_i/\partial y < 0$. This is atypical since we expect that most goods have positive income effects, hence the usage "normal" goods to describe cases in which $\partial g_i/\partial y > 0$.

As an important last matter of terminology we have the following conventions for cross derivatives introduced by Hicks:

| | | |
|---|---|---|
| $\partial h_i/\partial p_j > 0$ | $\Rightarrow$ | $x_i, x_j$ are substitutes |
| $\partial h_i/\partial p_j < 0$ | $\Rightarrow$ | $x_i, x_j$ are complements |
| $\partial g_i/\partial p_j > 0$ | $\Rightarrow$ | $x_i, x_j$ are gross substitutes |
| $\partial g_i/\partial p_j < 0$ | $\Rightarrow$ | $x_i, x_j$ are gross complements |

And finally for the geometrically inclined we can envisage the classical picture in which we can interpret the "large" increase in price represented by the shift of the budget line from $\alpha\beta$ to $\alpha\varepsilon$ shifting consumption from point A to point D. Drawing this picture is a useful review that will keep you prepared to teach intermediate micro. These finite change can be approximated by the Marshallian demand derivative, and can in turn can be decomposed either in the Hicksian formulation

(i) as a change from B to D as a result of a pure price effect which moves the consumer along the new indifference curve, and then

(ii) as a shift from A to B as a result in a pure income effect.

Or alternatively we may decompose the effect in Marshallian terms into:

(i) a shift from C to D which holds income constant while changing prices, and

(ii) a shift from A to C which is a pure income effect.

The connection between these two decompositions is provided by the Slutsky equation.

## References


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