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Lecture 3

Introduction to Dynamic Demand Models in Econometrics

In Problem Set 2 we will investigate a number of simple dynamic models for US gasoline demand since 1947. A typical model takes the form,

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + x_t^\top \beta_0 + x_{t-1}^\top \beta_1 + u_t \tag{1}$$

where y_t is per-capita U.S. gasoline consumption and x_t is a vector of exogenous variables, e.g., $x_t = (1, p_t, z_t)$ where p_t is price per gallon and z_t is income per capita.

An indispensable notational device for exploring models of this type is the lag operator, L, which has the property that

$$\begin{aligned} y_{t-1} &= Ly_t \\ y_{t-2} &= Ly_{t-1} = L^2 y_t \\ \cdots & \text{ etc} \end{aligned}$$

So we may write model (1) as

$$(1 - \alpha_1 L - \alpha_2 L^2) y_t = \alpha_0 + (\beta_0 + \beta_1 L)^\top x_t + u_t$$

or even more compactly as,

$$A(L) y_t = \alpha_0 + B(L) x_t + u_t \tag{2}$$

where $A(\cdot)$ and $B(\cdot)$ are viewed as polynomials in the lag operator L.

It is tempting to "solve" (2) by writing

$$y_t = A(L)^{-1} \alpha_0 + A(L)^{-1} B(L)^\top x_t + A(L)^{-1} u_t. \tag{3}$$

This is often called a "linear transfer function model" due to its roots in the electrical engineering literature. We would like to interpret (3), explaining the rather mysterious $A(L)^{-1}$ notation and relating (3) to the crucial notion of equilibrium forms of the model (2).

Digression on stability in linear difference equations

We begin with a very simplistic introduction to deterministic linear difference equations which illustrates some basic aspects of "equilibrium behavior" for models like (1).

Example 1: Consider the simplest possible case

$$X_t = aX_{t-1}$$

Clearly, by repeated substitution we have,

$$X_t = aX_{t-1} = a^2 X_{t-2} = \cdots = a^t X_0$$

where X_0 denotes an initial condition. For $|a| < 1$, note that $X_t \to 0$, for $|a| > 1, X_t$ diverges and for $|a| = 1$ we get either $X_t \equiv X_0$ or $X_t = \pm X_0$.

Example 2: Next, consider the second order difference equation,

$$X_t = a_1 X_{t-1} + a_2 X_{t-2},$$

Suppose, and this is really wishful thinking at this point based on Example 1, that solutions take the form,

$$X_t = A_1 \theta_1^t + A_2 \theta_2^t$$

where A_1 and A_2 denote parameters which are determined by initial conditions and the θ's are dependent in some way on the a's. Substituting this proposed solution into our equation yields,

$$A_1 \theta_1^t + A_2 \theta_2^t = a_1(A_1 \theta_1^{t-1} + A_2 \theta_2^{t-1}) + a_2(A_1 \theta_1^{t-2} + A_2 \theta_2^{t-2})$$

or

$$0 = A_1 \theta_1^t (1 - a_1 \theta_1^{-1} - a_2 \theta_1^{-2}) + A_2 \theta_2^t (1 - a_1 \theta_2^{-1} - a_2 \theta_2^{-2})$$

This looks quite promising. Suppose we find the roots of the quadratic equation

$$1 - a_1 z - a_2 z^2 = 0$$

and call these roots θ_1^{-1} and θ_2^{-1}, then we have solved the original problem. Why?

Note that the examples generalize immediately to higher order difference equations. What is needed for stability in this case? Suppose for a moment that the roots are real (recall that this needn't be the case), then again we have the requirement that both θ_1 and θ_2 must be less than one in absolute value, in order for the solution not to blow up as $t \to \infty$.

What about complex roots? When θ is complex we have something like

$$\theta = \lambda_1 + \lambda_2 i$$

where $i^2 = -1$, so this is a "number" which we can plot in the complex plane Note that we can represent θ in polar coordinates as

$$\theta = r(\cos(\varphi) + i\sin(\varphi))$$

where $r = (\lambda_1^2 + \lambda_2^2)^{1/2}$, $\cos(\varphi) = \lambda_1/r$, and $\sin(\varphi) = \lambda_2/r$, and we see that as long as θ is *inside* the unit circle θ^t stays inside the unit circle since $r^t < r$.[1] However, when $r > 1$ so θ is outside the unit circle we have θ^t explosive. For fun and possible enlightenment try this in R: `plot( (0.5 + 0.8i)^(1:100),type = 'l')`, or for a smoother version: `plot( (0.5 + 0.8i)^(1:1000/10),type = 'l')`. Now, play around with the coefficients, try for example $(0.8 + 0.8i)$.

You might be wondering, how do I find these roots when the order of the polynomial is greater than two? If you didn't learn this in high-school, I certainly didn't, you could check wikipedia for cubics, or quartics, but after that there is no nice algebraic solution strategy, so it is easier to simply fire up R and let it do the heavy work. Suppose we have the characteristic equation:

$$1 + 0.8z + 0.5z^2 + 0.25z^3 = 0$$

then we can do

```
> roots <- polyroot(c(1,.8,.5,.2))
> roots
[1] -0.342860+1.624303i -1.814281-0.000000i -0.342860-1.624303i
> Mod(roots)
[1] 1.660094 1.814281 1.660094
```

Note that the complex roots come in pairs, just as we were led to expect by the simple quadratic case, and since all the roots have modulus bigger than 1, we are safe.

Thus, by analogy with the scalar case it is necessary that the roots of the equation

$$1 - a_1 z - a_2 z^2 = 0$$

[1] Recall (!) for example that $\theta^2 = r^2(\cos(2\varphi) + i\sin(2\varphi))$ using the trignometric identities, $\cos^2(\varphi) - \sin^2(\varphi) = \cos(2\varphi)$ and $2\cos(\varphi)\sin(\varphi) = \sin(2\varphi)$. Recall also that the reciprocal of a complex number $1/(a+bi) = (a^2+b^2)^{-1}(a-bi) = (a^2+b^2)^{-1}a - (a^2+b^2)^{-1}bi$.

have roots *outside* the unit circle. Don't forget that these roots are θ_i^{-1} which accounts for the flip of inside/outside, this is potentially confusing so beware!

We shall see that the existence of roots to an equation like this play a fundamental role in determining the stability of linear time series models of the form (2.). Roots outside the unit circle are good in the sense that they imply stability of the model, while roots inside imply explosive behavior. Roots on the unit circle are more difficult and will require a separate discussion at a later moment.

Impulse Response Functions

The preceeding discussion has suggested a way of evaluating the stability of linear time-series models. We now turn to the question of interpreting the expression

$$D(L) = A(L)^{-1}B(L)$$

Consider,

$$B(L) = A(L)D(L)$$

or

$$\beta_0 + \beta_1 L + \cdots + \beta_s L^s = (1 - \alpha_1 L - \cdots - \alpha_r L^r)(\delta_0 + \delta_1 L + \cdots)$$

so, clearly for $j \leq s$, equating coefficients we have,

$$\begin{aligned} \beta_0 &= \delta_0 \\ \beta_1 &= -\delta_0\alpha_1 + \delta_1 \\ \beta_2 &= -\delta_0\alpha_2 - \delta_1\alpha_1 + \delta_2 \\ &\vdots \\ \beta_j &= -\delta_0\alpha_j - \cdots - \delta_{j-1}\alpha_1 + \delta_j \end{aligned}$$

a system which can be solved recursively, given the α, β's for the δ's. That is, given that we have estimated a model in the form (1) we can then compute the coefficients corresponding to the form (3).

More generally, see e.g., Harvey, (p. 234), we can write, denoting $j \wedge s = min\{j, s\}$

$$\delta_j = \begin{cases} \sum_{i=1}^{j\wedge r} \alpha_i \delta_{j-i} + \beta_j & j \leq s \\ \sum_{i=1}^{j\wedge r} \alpha_i \delta_{j-i} & j > s \end{cases}$$

The function, defined on the integers, of cumulative sums of the δ's,

$$\Delta(j) = \sum_{i=1}^{j} \delta_i$$

is usually called the impulse response function. It may be interpreted as providing a complete picture of the time path of the response of y to a once-and-for-all unit shock in x:

So, in the simplest case, imagine a thought experiment in which there is a single exogenous variable x, which has taken the value x_0 for a long time so y is randomly fluctuating around an equilibrium value of y_0. Now, x changes to x_1 and stays there, what happens to y?

In the first period we get the "impact" effect δ_0, and in subsequent periods this effect is gradually modified until (presuming stability in the process) we get a new value of y which corresponds to the "equilibrium" value of y corresponding to $x = x_1$. From (3), write

$$\begin{aligned} E\Delta y_t &= A(L)^{-1}B(L)\Delta x_t \\ &= D(L)\Delta x_t \\ &\to D(1)\Delta x \qquad \text{since for all } t\ \Delta x \equiv \Delta x_t \\ &= \sum_{i=1}^{\infty}\delta_i\Delta x. \end{aligned}$$

Interpretation: If there is a new equilibrium, then the change is just the accumulation of the short run impulse responses.

Heuristic: If there is a new equilibrium we can find it by setting $y_t = y_e$ and $x_t = x_e$ and solving. Obviously, this works only if there is an equilibrium, i.e. if the model is stable.

Caveat: Note that if the roots of the $A(z) = 0$ polynomial lie outside the unit circle we are ok, but otherwise we have problems with the existence of equilibrium.

Multipliers: The coefficients of the cumulative impulse response are often referred to, in deference to the associated macro literature, as impact, interim and long-run *multipliers*.

Inference: An interesting issue which has attracted considerable recent research is how to do inference on the δ's. We will not address this here, except to invoke the principle: *Every good estimate deserves a standard error*. This question will arise a little later in connection with the bootstrap.

Lag Distributions: It is often useful to have some way to characterize or compare lag distributions or shapes of the impulse response function. Two simple ideas in this direction are

Mean Lag: Think of δ_i's as a *pdf* and compute

$$\pi_i = \delta_i / \sum \delta_i$$

that is as a proportion of the total effect which is attributable to lag i, then

$$\mu = \sum i\pi_i = \frac{\sum i\delta_i}{\sum \delta_i}$$

Note that if the variables are in logs then this proportion is nicely interpreted in percentage terms. A useful trick in this regard is,

$$D'(L) = \delta_1 + 2\delta_2 L + 3\delta_3 L^2 + \cdots$$

so

$$\mu = \frac{D'(1)}{D(1)} = \frac{A(1)B'(1) - A'(1)B(1)}{D(1)A^2(1)} = \frac{B'(1)}{B(1)} - \frac{A'(1)}{A(1)}$$

Median Lag:

$$\nu = \min\{i | \sum \pi_i \geq .5\}$$

Caveat: Note that if π_i's can be negative, and they frequently are in practice, then the analogy with *pdf*'s is rather silly, and practically useless.

Error Correction Form:

Consider the simple dynamic model

$$y_t = \alpha_1 y_{t-1} + \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + u_t$$

In equilibrium with $x_t \equiv x_e$ we have

$$y_e = \frac{\alpha_0}{1-\alpha_1} + \frac{\beta_0 + \beta_1}{1-\alpha_1} x_e + \frac{1}{1-\alpha_1} u_t$$

It is sometimes useful to embed this equilibrium version of the model in the dynamic formulation itself. To do this, subtract y_{t-1} from both sides of model and then add and subtract $\beta_0 x_{t-1}$ to get

$$\Delta y_t = (\alpha_1 - 1) y_{t-1} + \alpha_0 + \beta_0 \Delta x_t + (\beta_0 + \beta_1) x_{t-1} + u_t$$

or

$$\Delta y_t = \beta_0 \Delta x_t + (\alpha_1 - 1)[y_{t-1} - \frac{\alpha_0}{1-\alpha_1} - \frac{\beta_0 + \beta_1}{1-\alpha_1} x_{t-1}] + u_t$$

This is called the error-correction form of the model since changes in y are decomposed into two natural pieces. (i) changes induced directly by changes in x, and (ii) changes induced indirectly because the previous period's y is out of equilibrium. The approach used here can be easily generalized to more complicated models and plays an important role in the discussion of cointegrated econometric models, a topic considered somewhat later in the course.