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Economics 478
Lecture 5
Stopping Times and Local Martingales

We now significantly expand the domain of relevant martingale results by considering local martingales.

Def. A non-negative random variable τ is called a *stopping time* with respect to a filtration $\{\mathcal{F}_t : t \in T\}$ if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Typically we will view τ as the time that an event occurs, and thus τ will be a stopping time (aka Markov time) if the information in $\mathcal{F}_t$ allows us to determine whether the event has occurred by time t.

Def. An increasing ($\nearrow$) sequence of times $\tau_n, \quad n = 1, 2, \ldots$ is called a *localizing sequence* wrt to $\mathcal{F}_t$ if (i) Each τ_n is a stopping time wrt $\mathcal{F}_t$ (ii) $\lim_{n\to\infty} \tau_n = \infty \;\; a.s.$

Def. A process $M = \{M(t) : t \in T\}$ is a local martingale (or local submartingale) wrt $\mathcal{F}_t$ if there exists a localizing sequence $\{\tau_n\}$ such that for each n,

$$M_n = \{M(t \wedge \tau_n) : 0 \leq t < \infty\}$$

is a $\mathcal{F}_t$ martingale (submartingale). Further, we say that M is a local square integrable martingale if M_n is a square integrable martingale for each n, and M is locally bounded if M_n is bounded for each n.

We can choose the localizing sequence $\tau_n = \sup\{t : \sup_{0\leq s\leq t} |X(s)| < n\} \wedge n, \;\; n = 1, 2, \ldots$ and the stopped process $X_n(t) = X(t \wedge \tau_n)$

Thm (Optional Sampling) Let $\{X(t) : t \in T\}$ be a right continuous process adapted to $\{\mathcal{F}_t : t \in T\}$ $T = [0, \infty)$, and let τ and τ^* be any $\mathcal{F}_t$ stopping times with $P(\tau < \tau^*) = 1$. If $X(t)$ is a martingale (submartingale), then

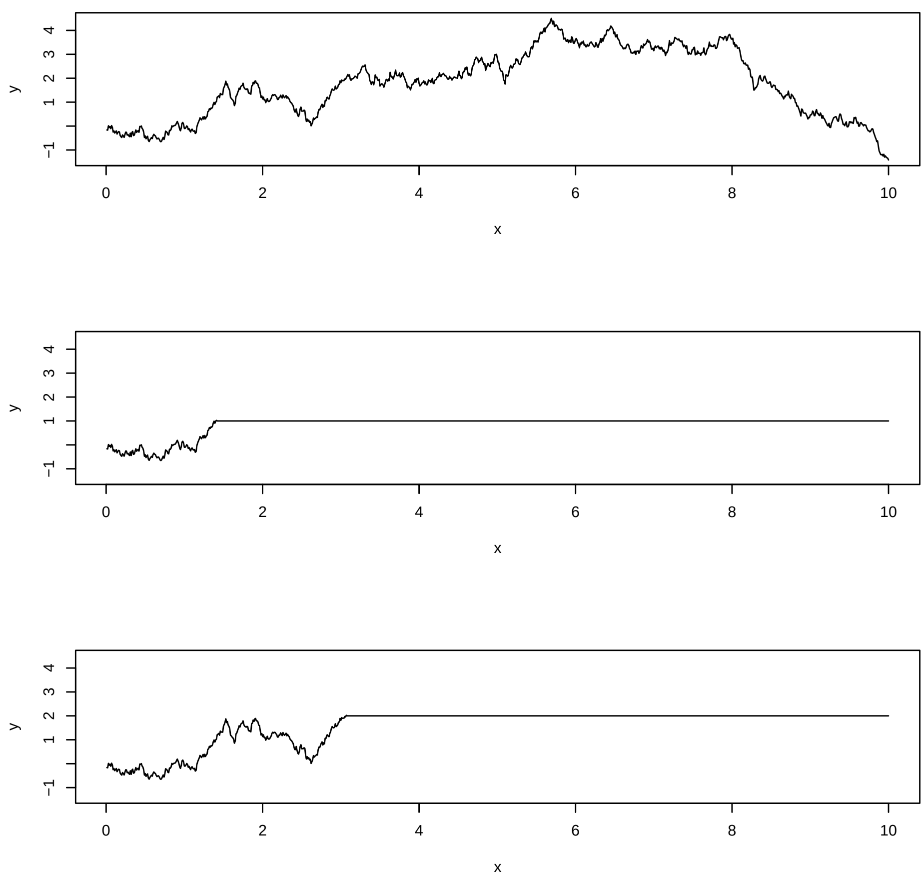
$$E(X(\tau^*)|\mathcal{F}_\tau) \underset{(\geq)}{=} X(\tau) \qquad a.s.$$

provided there exists T_0 such that $P(\tau^* \leq T_0) = 1$. see Fleming and Harrington for extensions.

Thm (Optional Stopping) Let $\{X(t) : t \in [0, \infty)\}$ be a right continuous $\mathcal{F}_t$ martingale (submartingale) and let τ be a $\mathcal{F}_t$ stopping time. Then $\{X(t \wedge \tau) : t \in [0, \infty)\}$ is a martingale (submartingale).

Pf. we won't bother to show that
(i) $X(t \wedge \tau)$ is $\mathcal{F}_t$ measurable
(ii) for $s < t, t, t \wedge \tau, s \wedge (t \wedge \tau)$ are stopping time bounded by t

FIGURE 1. A Martingale and Two Stopped Martingales. The panels depict the martingale $X(t), X_1(t)$, and $X_2(t)$ constructed with the localizing sequence, $\tau_n = \sup\{t : \sup_{0 \le s \le t} |X(s)| < n\} \wedge n, \;\; n = 1, 2, \ldots$



but these are quite easy. By the Optional Sampling theorem for the martingale $\{X(u) : 0 \le u \le t\}$

$$\begin{aligned} E|X(t \wedge \tau)| &= E|EX(t)|\mathcal{F}_{t\wedge\tau}| \\ &\le E|X(t)| \\ &< 0 \end{aligned}$$

$$EX(t \wedge \tau)|\mathcal{F}_s = X(s \wedge \tau) \quad a.s.$$

Write,

$$X(t \wedge \tau) = I_{\{\tau < s\}} X(t \wedge \tau) + I_{\{\tau \ge s\}} X(t \wedge \tau)$$

but

$$I_{\{\tau<s\}}X(t\wedge\tau)=I_{\{\tau<s\}}X(s\wedge\tau)$$

is $\mathcal{F}_s$ measurable, so

$$E\{I_{\{t<s\}}X(t\wedge\tau)|\mathcal{F}_s\}=I_{\{t<s\}}X(s\wedge\tau)\quad a.s.$$

Now, $\tau^*=\tau\vee s$ is a stopping time and

$$I_{\{\tau\geq s\}}X(t\wedge\tau)=I_{\{\tau\geq s\}}X(t\wedge\tau^*).$$

And since $I_{\{\tau\geq s\}}$ is $\mathcal{F}_s$ measurable,

$$\begin{aligned} E\{I_{\{\tau\geq s\}}X(t\wedge\tau^*)|\mathcal{F}_s\} &= I_{\{\tau\geq s\}}E\{X(t\wedge\tau^*)|\mathcal{F}_s\} \\ &= I_{\{\tau\geq s\}}X(s) \qquad \text{Optional Sampling} \\ &= I_{\{\tau\geq s\}}X(s\wedge\tau)\quad a.s. \end{aligned}$$

□

Doob-Meyer Decomposition (Extended) Let $X=\{X(t):t\geq 0\}$ be a nonnegative right continuous $\mathcal{F}_t$-local submartingale with localizing sequence $\{\tau_n\}$. Then there exists a unique right continuous predictable process A such that $A(0)=0$ $a.s.$ $P(A(t)<\infty)=1$ for all $t>0$ and $X-A$ is a right continuous local martingale. At each $t, A(t)$ may be taken as the $a.s.$ $\lim_{n\to\infty}A_n(t)$ where $A_n(t)$ is the compensator of the stopped submartingale $X(\cdot\wedge\tau_n)$.

Pf. Fleming and Harrington, pp.58-9.

Remark

This result allows us to extend the earlier results which relied on $EN(t)<\infty$, $EM^2(t)<\infty$ and boundedness of $A(t)$ to cases where these properties hold locally for a sequence of stopped processes, $X(\cdot\wedge\tau_n)$ for a sequence of stopping times $\{\tau_n\}$.

For an arbitrary (adapted) counting process N i.e. on satisfying $P\{N(t)<\infty\}=1$ for all t is locally bounded, hence a local submartingale.

For any nonnegative local submartingale X, there is an increasing right continuous, predictable process A (with ($A(0)=0$ and $P\{A(t)<\infty\}=1$) such that

$$M(t)=X(t)-A(t)$$

is a local martingale. Let $\{I_n\}$ be any localizing sequence for X and A_n be unique compensator for the stopped submartingale $X(\cdot\wedge\tau_n)$. Then take

$$A(t)=A_n(t)\quad\text{for } t\in(\tau_{n-1},\tau_n]$$

and let $n\to\infty$.

These results also imply that quite generally the martingale associated with the DMD has a unique predictable quadratic variation and covariation so

$$M^2 - <M>$$

and

$$M_1 M_2 - <M_1, M_2>$$

are local martingales. Provided $EM_i^2(t) < \infty$ for all t these local martingale are also martingales.

Further, if H is a locally bounded predictable process, (recall $\{X(t) : t \geq 0\}$ is locally bounded if for suitable sequence of localizing constants $\{\tau_n\}, X_n = \{X(t \wedge \tau_n) : t \geq 0\}$ is bounded for each n.) and M is a local martingale with $\Delta M(0) = 0$ and locally bounded variation process, then

$$\int H dM$$

is a local (square-integrable) martingale. Then

$$E \int H dM = 0$$

so for $M = N - A$ we have

$$E \int H dN = E \int H dA$$

This usual simplifies computations considerably.

For arbitrary counting processes N_i and locally bounded predictable processes H_i if

$$(*) \qquad E \int_0^t H_i^2 d <M_i> < \infty$$

which implies $E \int_0^t H_i dM_i = 0$ and that

$$E(\int_0^t H_1 dM_1 \int_0^t H_2 dM_2) = E \int_0^t H_1 H_2 d <M_1, M_2>$$

So in effect we have replaced the very stringent $EN_i(t) < \infty, EM_i(t) < \infty$, and boundedness of H_i with the weaker condition $(*)$.

Accessible formulae for $<M_i>$ are available. We have seen that

$$d <M> (s) = E\{dM^2(s) | \mathcal{F}_{s-}\}$$

since

$$dM^2(s) = \{dM(s)\}^2 + 2M(s - ds)dM(s)$$

and $E\{dM | \mathcal{F}_{s-}\} = 0$, we have

$$\begin{aligned} d <M> &= E[\{dM(s)\}^2 | \mathcal{F}_{s-}] \\ &= \text{Var } (dM(s) | \mathcal{F}_{s-}) \end{aligned}$$

so the predictable variation $<M>(t)$ of M is the sum over $(0, t]$ of the conditional variances of $M(s)$, given $\mathcal{F}_{s-}$.

In the counting process setup, $dN(s)$ is Bernoulli with

$$E\{dN(s)|\mathcal{F}_{s-}\} = dA(s)$$

and

$$V(d\wedge|(s) - dA(s)|\mathcal{F}_{s-}) = \{1 - dA(s)\}dA(s)$$

So we anticipate,

$$<M> (t) = \int_0^t \{1 - \Delta A(s)\}dA(s)$$

and that

$$< M> (t) = A(t)$$

when A is continuous. This is the Poisson result that will prove critical later.

In the multivariate settings we would like to consider, I like to call this Wally's world, we have a multivariate counting process

$$N = (N_1(t), \dots, N_k(t))'$$

whose components are restricted so that no two can jump simultaneously. Then for $M_i = N_i - A_i$ we have

$$<M>= A_i \quad \text{and} \quad <M_i, M_j>= 0$$

so this is conditionally *Poisson-like* with rate $dA_i(t)$ and *conditionally* pairwise uncorrelated. If

$$A(t) = \int_0^t \lambda(s)ds$$

then N behaves locally, given its history as a Poisson process with rate (intensity) function $\lambda(t)$.