University of Illinois — Department of Economics
Fall 2001 — Roger Koenker

Economics 478
Lecture 3
Predictable Variation

1. Predictable Variation in Discrete Time

Suppose $(X_t, \mathcal{F}_t)$ is a 0 mean martingale with $EX_t^2 < \infty$. Then $(X_t^2, \mathcal{F}_t)$ is a submartingale and

$$Z_t = X_t^2 - A_t$$

is a zero mean martingale adapted to $\mathcal{F}_t$ with $A_0 = EX_0^2 = V(X_0)$ and

$$A_t = \sum_{k=1}^{t} E((\Delta X_k)^2 | \mathcal{F}_{k-1}) + A_0$$

is the conditional variance of the X_t process.

Note

$$\begin{aligned} A_t &= \sum (E(X_k^2 | \mathcal{F}_{k-1}) - X_{k-1}^2) + EX_0^2 \\ &= \sum E((X_k - X_{k-1})^2 | \mathcal{F}_{k-1}) + EX_0^2 \\ &= \sum (E(\Delta X_k)^2 | \mathcal{F}_{k-1}) + EX_0^2 \end{aligned}$$

We write $<X_t> \equiv <X_t, X_t> = A_t$. And

$$EX_t^2 = EA_t$$

2. Martingale Transforms

Suppose H_t is a predicatable process wrt $\mathcal{F}_t$, for any process (X_t) we define the H-transform of X, by

$$(H \cdot X)_t \equiv \sum_{k=1}^{t} H_k (X_k - X_{k-1}) + H_0 X_0$$

Thm. Suppose $(X_t, \mathcal{F}_t)$ is a (sub)martingale and $(H_t, \mathcal{F}_t)$ is predictable with $0 < H_t < \infty$, then $\{H \cdot X, \mathcal{F}_t\}$ is a (sub)martingale.

Pf. To gain confidence in computation

$$\begin{aligned} E((H \cdot X)_{t+1} | \mathcal{F}_t) &= (H \cdot X)_t + E(H_{t+1}(X_{t+1} - X_t) | \mathcal{F}_t) \\ &= (H \cdot X)_t + H_{t+1} E(\Delta X_{t+1} | \mathcal{F}_t) \\ &\begin{matrix} = \\ \geq \end{matrix} (H \cdot X)_t \left\{ \begin{matrix} \text{martingale} \\ \text{submartingale} \end{matrix} \right\} \end{aligned}$$

3. Predictable Variation for Counting Process Martingales

In continuous time computing the predictable variation $<M>$ of the martingale M is somewhat more esoteric, but yields results easily rationalized by comparison with the discrete case. Consider

$$\begin{aligned} d <M> (t) &\equiv \mathbb{E}(dM^2(t)|\mathcal{F}_{t-}) \\ &= \mathbb{E}\{M(t-)dM(t) + M(t)dM(t)|\mathcal{F}_{t-}\} \\ &= \mathbb{E}\{2M(t-)dM(t) + (dM(t))^2|\mathcal{F}_{t-}\} \\ &= 2M(t-)\cdot 0 + \mathbb{E}\{(dM(t))^2|\mathcal{F}_{t-1}\} \\ &= \mathbb{E}\{(dM(t))^2|\mathcal{F}_{t-}\} \end{aligned}$$

as in discrete case. Now applying this to the counting process example where $M = N - A$, we have that the quadratic variation of M

$$\begin{aligned} d <M> (t) &= \mathbb{E}\{(dN(t) + dA(t))^2|\mathcal{F}_{t-}\} \\ &= \mathbb{E}\{(dN(t))^2 - 2dA(t)dN(t) + (dA(t))^2|\mathcal{F}_{t-}\} \\ &= \mathbb{E}\{(dN(t))^2|\mathcal{F}_{t-}\} - 2(dA(t))^2 + (dA(t))^2 \end{aligned}$$

but for the counting process $(dN(t))^2 = dN(t)$, this is just a consequence of the discrete nature of the process, either it jumps 1 at t, or not. So, we have

$$\begin{aligned} d <M> (t) &= dA(t) - (dA(t))^2 \\ &= (1 - \Delta A(t))dA(t) \end{aligned}$$

where $\Delta A(t) = A(t) - A(t-)$ so we may write, finally,

$$<M> (t) = \int_0^t (1 - \Delta A(s))dA(s)$$

Now consider the continuous time version of the Martingale transform,

$$W(t) = \int_0^t H(s)dM(s)$$

We have that $W(t)$ is a martingale provided $E|W(t)| < \infty$, since

$$\mathbb{E}(dW(t)|\mathcal{F}_t) = \mathbb{E}H(t)dM(t)|\mathcal{F}_t = H(t)\mathbb{E}dM(t)|\mathcal{F}_t = 0$$

Its predictable variation is

$$\begin{aligned} d <W> (t) &= \mathbb{E}\{(dW(t))^2|\mathcal{F}_{t-}\} \\ &= \mathbb{E}\{(H(t)dM(t))^2|\mathcal{F}_{t-1}\} \\ &= H^2(t)\mathbb{E}\{(dM(t))^2|\mathcal{F}_{t-1}\} \\ &= H^2(t)d <M> (t) \end{aligned}$$

and consequently, we infer

$$<W> (t) = \int_0^t H^2(s)d <M> (t)$$

This also suggests, provided that

$$D(t) = W^2(t) - <W> (t)$$

is a martingale with $D(0) = 0$, so $ED(t) = 0$.
This yields the grafito

$$\int H(t)dM(t) = W(t)$$

$$\int (\text{predictable})d(\text{martingale}) = \text{martingale}$$

gambling strategies are predictable, i.e. bets have to be down before the roulette wheel spins and this yields a martingale return process.

Now suppose that we have a sequence of martingales, M_n, whose increments satisfy a Lindeberg condition. From the martingale condition we have seen that

$$\begin{aligned}
\operatorname{Cov}(M(t) - M(s), M(s)) &= \lim_{n\to\infty} \mathbb{E}((M_n(t) - M_n(s))M_n(s)) \\
&= \lim_{n\to\infty} \mathbb{E}M_n(s)\mathbb{E}\{M_n(t) - M_n(s)|\mathcal{F}_s\} \\
&= 0
\end{aligned}$$

If the limiting process were normal, then this uncorrelated increments condition would imply independent increments. The limiting variance process of M is

$$\begin{aligned}
\mathbb{E}M^2(t) &= \lim \mathbb{E}M_n^2(t) \\
&= \lim E < M_n > (t)
\end{aligned}$$

so if we have a Lindeberg condition on increments and

$$\lim \mathbb{E} < M_n > (t) \to V(t)$$

with $V(t) \nearrow$, right continuous and $V(0) = 0$, then we can anticipate a limiting Guassian process. This is (roughly) Robolledo's CLT.

4. Optional Sampling, Stopping Times, and XXX

Given a discrete process $X_1, X_2, \ldots$ we often would like to consider a transformed process derived from the original one by deciding *after observing* $X_1, \ldots, X_n$ whether X_n will be the first value of the transformed process.

Def. Let $X_1, X_2, \ldots$ be an arbitrary random process, the sequence $m_1, m_2, \ldots$ of integer valued rv's are called *sampling variables* if

(i) $1 \leq m_2 \leq m_2 \leq \ldots$
(ii) $\{m_k = j\} \in \mathcal{F}(X_1, \ldots, X_j)$

The process $\tilde{X}_n = x_{m_n}$ is called the process derived by optional sampling from $X_1, X_2, \ldots$.

Thm. Suppose $X_1, X_2, \ldots$ is a martingale (submartingale) and m_1, m_2 sampling variables and $\tilde{X}_n$ the derived process. If

(i) $E|\tilde{X}_k| < \infty$ all k
(ii) $\underline{\lim} \inf_{m_n < N} |X_N| dP = 0$ all n

then $\tilde{X}_1, \tilde{X}_2, \ldots$ is a martingale (submartingale).

An application of the foregoing result given by Breiman ( ) is the following.

Prop. If $X_1, \ldots, X_n$ is a submartingale, then for any $x > 0$,

(i) $P(\max_{1 \le j \le k} X_j > x) \le \frac{1}{x} \mathbb{E}|X_k|$
(ii) $P(\min_{1 \le j \le k} X_j < -x) \le \frac{1}{x}(\mathbb{E}|X_k| - EX_1)$

Pf.(i) Define sampling variables

$$\begin{aligned} m_1 &= \left\{ \begin{array}{l} \text{first } j \le k \text{ such that } X_j > x, \text{ or} \\ k \text{ if no such } j \text{ exits} \end{array} \right\} \\ m_n &= k \qquad n \ge 2 \end{aligned}$$

Now the conditions of the previous theorem can be verified since

$$\{m_n > N\} = \phi \quad \text{ for } N \ge k$$

and

$$\mathbb{E}|\tilde{X}_n| \le \sum_{j=1}^{k} \mathbb{E}|X_j| < \infty$$

5. Martingale Convergence

Thm. Let $X_1, X_2, \ldots$ be a submartingale and $\limsup E|X_n| < \infty$, then $\exists\ X$ such that $X_n \to X$ and $E|X| < \infty$.

Pf. Use sampling variables to show that for $b > a$

$$P(X_n \le a \quad i.o. \text{ and } X_n \ge b \quad i.o.) = 0$$

more specifically it is shown that

$$P(U\{\liminf X_n \le a < b \le \limsup X_n\}) = 0$$

where the U is over all rational a, b. Thus, only possible cases are there is $con.v.X$

$$X_n \to X \qquad \text{or} \qquad |X_n| \to \infty$$

but the latter case is ruled out by Fatou's lemma which says

$$\int \liminf |X_n| dP \le \liminf \int |X_n| dP$$

so

$$\int |X| dP \le \liminf \int |X_n| dP < \infty.$$

By construction m_n can't exceed k, and the bound in the second line is trivial since $\sum \mathbb{E}|X_j|$ includes $\mathbb{E}|\tilde{X}_n|$ as one of the summands. Now, note that

$$P(\max_{1 \le j \le k} X_j > x) = P(X_{m_1} > x) \le \frac{1}{x} \int_{\{X_{m_1} > x\}} X_{m_1} dP$$

and by the prior theorem,

$$\int_{\{X_{m_1}>x\}} X_{m_1} dP \leq \int_{\{X_{m_1}>x\}} X_k dP \leq E|X_k|$$

[Why? 2nd inequality is obvious since $x > 0$, but first one comes from submartingale property. Why?]

Under such conditions we typically have

$$V(t) = \lim \mathbb{E} <M_n> (t) = \lim \mathbb{E} M_n^2(t) = \mathbb{E} M^2(t)$$

Finally, to come back to the counting process example, suppose we have $N_i(t)$ $i = 1, \dots, n$ with

$$\begin{aligned} M_i(t) &= N_i(t) - A_i(t) \\ <M_i> (t) &= \int_0^t (1 - \Delta A_i(s)) dA_i(s) \end{aligned}$$

Consider for $\mathcal{F}_{t-}$ measurable functions $c_i(\cdot)$

$$\mathbb{M}_n(t) \equiv \sum_{i=1}^{n} c_i(t) M_i(t)$$

Clearly $\mathbb{M}_n$ is a martingale, and

$$\begin{aligned} d <\mathbb{M}_n> (t) &= \mathbb{E}\{(d\mathbb{M}_n(t))^2 | \mathcal{F}_{t-}\} \\ &= \sum c_i^2(t) \mathbb{E}\{(dM_i(t))^2 | \mathcal{F}_{t-}\} + \sum_{i \neq j} \sum c_i^2(t) \mathbb{E}\{(dM_i(t))^2 | \mathcal{F}_{t-}\} \\ &= \sum c_i^2(t) <M_i> (t) \end{aligned}$$