TABLE B.6
Critical Values for the Phillips-Perron Z_t Test and for the Dickey-Fuller Test Based on Estimated *OLS t* Statistic

Sample size T	*Probability that $(\hat{\rho} - 1)/\hat{\sigma}_{\hat{\rho}}$ is less than entry*							
	0.01	*0.025*	*0.05*	*0.10*	*0.90*	*0.95*	*0.975*	*0.99*
				Case 1				
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
				Case 2				
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60
				Case 4				
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

The probability shown at the head of the column is the area in the left-hand tail.

Source: Wayne A. Fuller, *Introduction to Statistical Time Series*, Wiley, New York, 1976, p. 373.

TABLE B.9

Critical Values for the Phillips Z_t Statistic or the Dickey-Fuller t Statistic When Applied to Residuals from Spurious Cointegrating Regression

Number of right-hand variables in regression, excluding trend or constant $(n-1)$	*Sample size* (T)	*Probability that* $(\hat{\rho}-1)/\hat{\sigma}_{\hat{\rho}}$ *is less than entry* 0.010	0.025	0.050	0.075	0.100	0.125	0.150
				Case 1				
1	500	−3.39	−3.05	−2.76	−2.58	−2.45	−2.35	−2.26
2	500	−3.84	−3.55	−3.27	−3.11	−2.99	−2.88	−2.79
3	500	−4.30	−3.99	−3.74	−3.57	−3.44	−3.35	−3.26
4	500	−4.67	−4.38	−4.13	−3.95	−3.81	−3.71	−3.61
5	500	−4.99	−4.67	−4.40	−4.25	−4.14	−4.04	−3.94
				Case 2				
1	500	−3.96	−3.64	−3.37	−3.20	−3.07	−2.96	−2.86
2	500	−4.31	−4.02	−3.77	−3.58	−3.45	−3.35	−3.26
3	500	−4.73	−4.37	−4.11	−3.96	−3.83	−3.73	−3.65
4	500	−5.07	−4.71	−4.45	−4.29	−4.16	−4.05	−3.96
5	500	−5.28	−4.98	−4.71	−4.56	−4.43	−4.33	−4.24
				Case 3				
1	500	−3.98	−3.68	−3.42	—	−3.13	—	—
2	500	−4.36	−4.07	−3.80	−3.65	−3.52	−3.42	−3.33
3	500	−4.65	−4.39	−4.16	−3.98	−3.84	−3.74	−3.66
4	500	−5.04	−4.77	−4.49	−4.32	−4.20	−4.08	−4.00
5	500	−5.36	−5.02	−4.74	−4.58	−4.46	−4.36	−4.28

The probability shown at the head of the column is the area in the left-hand tail.

Source: P. C. B. Phillips and S. Ouliaris, "Asymptotic Properties of Residual Based Tests for Cointegration," *Econometrica* 58 (1990), p. 190. Also Wayne A. Fuller, *Introduction to Statistical Time Series*, Wiley, New York, 1976, p. 373.

TABLE B.10
Critical Values for Johansen's Likelihood Ratio Test of the Null Hypothesis of h Cointegrating Relations Against the Alternative of No Restrictions

Number of random walks ($g = n - h$) (g)	Sample size (T)	Probability that $2(\mathcal{L}_A - \mathcal{L}_0)$ is greater than entry					
		0.500	*0.200*	*0.100*	*0.050*	*0.025*	*0.001*
				Case 1			
1	400	0.58	1.82	2.86	3.84	4.93	6.51
2	400	5.42	8.45	10.47	12.53	14.43	16.31
3	400	14.30	18.83	21.63	24.31	26.64	29.75
4	400	27.10	33.16	36.58	39.89	42.30	45.58
5	400	43.79	51.13	55.44	59.46	62.91	66.52
				Case 2			
1	400	2.415	4.905	6.691	8.083	9.658	11.576
2	400	9.335	13.038	15.583	17.844	19.611	21.962
3	400	20.188	25.445	28.436	31.256	34.062	37.291
4	400	34.873	41.623	45.248	48.419	51.801	55.551
5	400	53.373	61.566	65.956	69.977	73.031	77.911
				Case 3			
1	400	0.447	1.699	2.816	3.962	5.332	6.936
2	400	7.638	11.164	13.338	15.197	17.299	19.310
3	400	18.759	23.868	26.791	29.509	32.313	35.397
4	400	33.672	40.250	43.964	47.181	50.424	53.792
5	400	52.588	60.215	65.063	68.905	72.140	76.955

The probability shown at the head of the column is the area in the right-hand tail. The number of random walks under the null hypothesis (g) is given by the number of variables described by the vector autoregression (n) minus the number of cointegrating relations under the null hypothesis (h). In each case the alternative is that $g = 0$.

Source: Michael Osterwald-Lenum, "A Note with Quantiles of the Asymptotic Distribution of the Maximum Likelihood Cointegration Rank Test Statistics," *Oxford Bulletin of Economics and Statistics* 54 (1992), p. 462; and Søren Johansen and Katarina Juselius, "Maximum Likelihood Estimation and Inference on Cointegration—with Applications to the Demand for Money," *Oxford Bulletin of Economics and Statistics* 52 (1990), p. 208.

Example 17.8

The following model was estimated by *OLS* for the interest rate data described in Example 17.3 (standard errors in parentheses):

$$\begin{aligned} i_t = \underset{(0.0788)}{0.335}\ \Delta i_{t-1} - \underset{(0.0808)}{0.388}\ \Delta i_{t-2} + \underset{(0.0800)}{0.276}\ \Delta i_{t-3} \\ - \underset{(0.0794)}{0.107}\ \Delta i_{t-4} + \underset{(0.109)}{0.195} + \underset{(0.018604)}{0.96904}\ i_{t-1}. \end{aligned}$$

Dates t = 1948:II through 1989:I were used for estimation, so in this case the sample size is $T = 164$. For these estimates, the augmented Dickey-Fuller ρ test [17.7.35] would be

$$\frac{164}{1 - 0.335 + 0.388 - 0.276 + 0.107}(0.96904 - 1) = -5.74.$$

Since $-5.74 > -13.8$, the null hypothesis that the Treasury bill rate follows a fifth-order autoregression with no constant term, and a single unit root, is accepted at the 5% level. The *OLS t* test for this same hypothesis is

$$(0.96904 - 1)/(0.018604) = -1.66.$$

Since $-1.66 > -2.89$, the null hypothesis of a unit root is accepted by the augmented Dickey-Fuller t test as well. Finally, the *OLS F* test of the joint null hypothesis that $\rho = 1$ and $\alpha = 0$ is 1.65. Since this is less than 4.68, the null hypothesis is again accepted.

The null hypothesis that the autoregression in levels requires only four lags is based on the *OLS t* test of $\zeta_4 = 0$:

$$-0.107/0.0794 = -1.35.$$

From Table B.3, the 5% two-sided critical value for a t variable with 158 degrees of freedom is -1.98. Since $-1.35 > -1.98$, the null hypothesis that only four lags are needed for the autoregression in levels is accepted.

Asymptotic Results for Other Autoregressions

Up to this point in this section, we have considered an autoregression that is a generalization of case 2 of Section 17.4—a constant is included in the estimated regression, though the population process is presumed to exhibit no drift. Parallel generalizations for cases 1, 3, and 4 can be obtained in the same fashion. The reader is invited to derive these generalizations in exercises at the end of the chapter. The key results are summarized in Table 17.3.

TABLE 17.3
Summary of Asymptotic Results for Autoregressions Containing a Unit Root

Case 1:

Estimated regression:

$$y_t = \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \cdots + \zeta_{p-1} \Delta y_{t-p+1} + \rho y_{t-1} + \varepsilon_t$$

True process: same specification as estimated regression with $\rho = 1$

Any t or F test involving $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ can be compared with the usual t or F tables for an asymptotically valid test.

TABLE 17.3 (continued)

Z_{DF} has the same asymptotic distribution as the variable described under the heading Case 1 in Table B.5.

OLS t test of $\rho = 1$ has the same asymptotic distribution as the variable described under Case 1 in Table B.6.

Case 2:

Estimated regression:

$$y_t = \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \cdots + \zeta_{p-1} \Delta y_{t-p+1} + \alpha + \rho y_{t-1} + \varepsilon_t$$

True process: same specification as estimated regression with $\alpha = 0$ and $\rho = 1$

Any t or F test involving $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ can be compared with the usual t or F tables for an asymptotically valid test.

Z_{DF} has the same asymptotic distribution as the variable described under Case 2 in Table B.5.

OLS t test of $\rho = 1$ has the same asymptotic distribution as the variable described under Case 2 in Table B.6.

OLS F test of joint hypothesis that $\alpha = 0$ and $\rho = 1$ has the same asymptotic distribution as the variable described under Case 2 in Table B.7.

Case 3:

Estimated regression:

$$y_t = \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \cdots + \zeta_{p-1} \Delta y_{t-p+1} + \alpha + \rho y_{t-1} + \varepsilon_t$$

True process: same specification as estimated regression with $\alpha \neq 0$ and $\rho = 1$

$\hat{\rho}_T$ converges at rate $T^{3/2}$ to a Gaussian variable; all other estimated coefficients converge at rate $T^{1/2}$ to Gaussian variables.

Any t or F test involving any coefficients from the regression can be compared with the usual t or F tables for an asymptotically valid test.

Case 4:

Estimated regression:

$$y_t = \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \cdots + \zeta_{p-1} \Delta y_{t-p+1} + \alpha + \rho y_{t-1} + \delta t + \varepsilon_t$$

True process: same specification as estimated regression with α any value, $\rho = 1$, and $\delta = 0$

Any t or F test involving $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ can be compared with the usual t or F tables for an asymptotically valid test.

Z_{DF} has the same asymptotic distribution as the variable described under Case 4 in Table B.5.

OLS t test of $\rho = 1$ has the same asymptotic distribution as the variable described under Case 4 in Table B.6.

OLS F test of joint hypothesis that $\rho = 1$ and $\delta = 0$ has the same asymptotic distribution as the variable described under Case 4 in Table B.7.

Notes to Table 17.3

Estimated regression indicates the form in which the regression is estimated, using observations $t = 1, 2, \ldots, T$ and conditioning on observations $t = 0, -1, \ldots, -p + 1$.

True process describes the null hypothesis under which the distribution is calculated. In each case it is assumed that roots of

$$(1 - \zeta_1 z - \zeta_2 z^2 - \cdots - \zeta_{p-1} z^{p-1}) = 0$$

are all outside the unit circle and that ε_t is i.i.d. with mean zero, variance σ^2, and finite fourth moment.

Z_{DF} in each case is the following statistic:

$$Z_{DF} = T(\hat{\rho}_T - 1)/(1 - \hat{\zeta}_{1,T} - \hat{\zeta}_{2,T} - \cdots - \hat{\zeta}_{p-1,T}),$$

where $\hat{\rho}_T, \hat{\zeta}_{1,T}, \hat{\zeta}_{2,T}, \ldots, \hat{\zeta}_{p-1,T}$ are the *OLS* estimates from the indicated regression.

OLS t test of $\rho = 1$ is $(\hat{\rho}_T - 1)/\hat{\sigma}_{\hat{\rho}_T}$, where $\hat{\sigma}_{\hat{\rho}_T}$ is the *OLS* standard error of $\hat{\rho}_T$.

OLS F test of a hypothesis involving two restrictions is given by expression [17.7.39].