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Lecture 3

Positive Theory of Asset Pricing

In the previous lecture we described what may be called a "normative theory" of how the individual *should* behave. Now, we try to infer how asset prices *would* behave assuming that the normative theory was adhered to by investors.

In somewhat the same spirit as the previous lecture, assume that we have, initially, two assets; one risk free, R_0, with mean return μ_0 and standard deviation, $\sigma_0 = 0$, and the other consisting of a representative portfolio of *all* existing assets, R_m, with mean return μ_m and standard deviation σ_m. Portfolios comprised of these two assets have mean and standard deviation

$$\begin{aligned}\mu_p &= p\mu_0 + (1-p)\mu_m \\ \sigma_p &= (1-p)\sigma_m\end{aligned}$$

Now consider a new asset, R_i, with mean μ_i and σ_i. If we combine R_i and R_m we have as before

$$\begin{aligned}\mu_p &= p\mu_i + (1-p)\mu_m \\ \sigma_p^2 &= p^2\sigma_i^2 + (1-p)^2\sigma_m^2 + 2p(1-p)\sigma_{im}\end{aligned}$$

where $\sigma_{im} = Cov(R_i, R_m) = E(R_i - \mu_i)(R_m - \mu_m)$.

We would like to ask how μ_p and σ_p^2 change as p changes, in particular, how do they change as we introduce just a tiny bit of R_i. This is an exercise in calculus:

$$\begin{aligned}\left.\frac{d\mu_p}{dp}\right|_{p=0} &= \mu_i - \mu_m \\ \left.\frac{d\sigma_p^2}{dp}\right|_{p=0} &= 2p\sigma_i^2 - 2(1-p)\sigma_m^2 + 2(1-p)\sigma_{im} - 2p\sigma_{im}\big|_{p=0} \\ &= 2(\sigma_{im} - \sigma_m^2)\end{aligned}$$

But we really want $d\sigma_p/dp$ not $d\sigma_p^2/dp$ so by the chain rule

$$\begin{aligned}\left.\frac{d(\sigma_p^2)^{1/2}}{dp}\right|_{p=0} &= \frac{1}{2}\ \frac{1}{(\sigma_p^2)^{1/2}} \cdot \left.\frac{d\sigma_p^2}{dp}\right|_{p=0} \\ &= \frac{1}{2}\frac{2(\sigma_{im} - \sigma_m^2}{\sigma_m} \\ &= \frac{\sigma_{im} - \sigma_m^2}{\sigma_m}\end{aligned}$$

Now, if our new asset R_i is going to viable we need that its risk-return tradeoff is just like the risk-return relationship prevailing between R_m and R_0, i.e., we need that

$$\frac{\mu_i - \mu_m}{(\sigma_{im} - \sigma_m^2)/\sigma_m} = \frac{\mu_m - \mu_0}{\sigma_m}$$

or

$$\mu_i - \mu_m = (\mu_m - \mu_0)\left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m^2}\right)$$

Subtract $\mu_m - \mu_0$ to both sides to obtain,

$$\mu_i - \mu_0 = (\mu_m - \mu_0)\left(\frac{\sigma_{im}}{\sigma_m^2}\right)$$

This is a fundamental relationship – what does it say? Very simply it says that expected excess returns of asset $i, \mu_i - \mu_0$, is proportional to the expected excess returns on the market portfolio, $\mu_m - \mu_0$, with factor of proportionality σ_{im}/σ_m^2. The latter factor is a regression coefficient.

Suppose we have the classical bivariate linear regression model

$$y_i = \alpha + \beta x_i + u_i$$

and we estimate α and β by minimizing the sum of squared deviations of the y_i's from the estimated line, that is we solve

$$\min_{\alpha,\beta} \sum (y_i - \alpha - \beta x_i)^2$$

The solution is

$$\begin{aligned} \hat{\beta} &= \frac{\sum(y_i - \bar{y})(x_i - \bar{x})}{\sum(x_i - \bar{x})^2} \\ \hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} \end{aligned}$$

we can interpret $\hat{\beta}$ as an estimate that simply a ratio of estimates of the covariance of x and y to the estimate of the variance of x.

Looking back at our fundamental CAPM relationship we see that this precisely what our factor σ_{im}/σ_m^2 is – the ratio of covariance of R_i and R_m to the variance of R_m. This yields the basic model,

$$(*) \qquad R_{i_t} - R_{0_t} = \alpha + \beta(R_{m_t} - R_{0_t}) + u_t$$

where β now represents σ_{im}/σ_m^2. Of course in the model we derived $\alpha = 0$, so we need to consider this a bit further. But the basic message is simple: Expected returns of R_i given market returns follows a simple, linear regression model.

The regression formulation of our CAPM relationship yields a variety of new insights. The expected excess return, $\mu_i - \mu_0$, is sometimes called the risk premium of asset R_i since it measures how much above the risk free rate of return the expected return on R_i needs to be to justify its uncertain return. Similarly, $\mu_m - \mu_0$ is the risk premium for the market portfolio. The coefficient β can then be viewed as establishing a connection between the risk premium of asset, R_i, and the risk premium of the market as a whole. If β is greater (less) than one, then R_i has a larger (smaller) risk premium than the market risk premium.

The regression formulation $(*)$ allows us to decompose the variance of R_i into a portion attributed to variation in the market and a portion independent of the market,

$$\sigma_R^2 = \beta^2\sigma_M^2 + \sigma_u^2.$$

So if we take variance as a measure of risk we can view the first component as risk associated with the market that can't be avoided, and the second component as risk that can be avoided by diversification. To see how the latter works imagine a large number, n, of stocks that have random contributions $u_{1t}, u_{2t}, \ldots, u_{nt}$

independent of the market return. For simplicity suppose that we form a portfolio with equal portfolio weights, $1/n$, then

$$V(\bar{u}_t) = V(\frac{1}{n}\sum_{i=1}^{n} u_{it}) = \frac{1}{n^2}\sum \sigma_i^2$$

As long as the σ_i^2 are all roughly the same magnitude, say $\sigma_i^2 \leq \sigma_0^2$ for all i, then

$$V(\bar{u}_t) \leq \sigma_0^2/n$$

so for large n the contribution of these components to the portfolio variance is small.

On the other hand, the portion of the variance associated with the market can't be diversified away. Assets with $\beta > 1$ have variability that magnifies the market variation and they – according to the CAPM theory – can expect to earn a large risk premium to compensate. Assets with $\beta < 1$ vary less violently than the market as a whole and therefore earn less than the market rate of return.

What if $\beta \leq 0$? If $\beta = 0$, the model predicts that $ER_i = \mu_0$, that is that the asset should earn the risk free rate of return; this is true because in this case there is only diversifiable risk. When $\beta < 0$ the situation is even more extreme: when the market goes up the expectation is that such assets go down and vice-versa. This makes them ideal diversification investments: when the market declines they tend to soften the blow. Unfortunately such assets are hard to find. In some periods gold has shown some tendency to exhibit a negative β. This is explored in the last part of the first problem set.

Finally, a word about α in the regression model $(*)$. As derived, α should take the value zero. If when we estimate the model we see an $\hat{\alpha} > 0$ it means that over the estimation period the asset in question had returns that exceeded that predicted by the CAPM, and likewise if $\hat{\alpha} < 0$ its performance was worse than predicted. Of course, when estimating such models from historical data, $\hat{\alpha}$'s are always non zero, what is more important is to ask: how strong is the evidence that $\alpha \neq 0$. Typically this evidence is very weak.