

Economics 471
Lecture 2
Elementary Probability, Portfolio Theory and Regression

In this lecture we review some basic concepts of probability including expectation, variance, covariance and correlation, and incidentally reveal how to get rich in the stock market.

Suppose you buy an asset today at price p_0 , it might be 100 shares of IBM stock or a painting by Matisse. A month from today it has price p_1 and in the interim it paid dividend (had net income) d . Then its nominal rate of return is,

$$R = \frac{p_1 + d - p_0}{p_0}$$

Viewed from today's perspective R is a random variable. It can take many possible values, say r_1, r_2, \dots , and each of these values could be assigned a probability π_1, π_2, \dots reflecting the relative likelihood of the r_i 's. These probabilities should satisfy two conditions (i) $\pi_i \geq 0$ (ii) $\pi_1 + \pi_2 + \dots = 1$. In classical finance theory we would like to know two important features about R : the first is its expected return, the second is some measure of the riskiness of R .

Expected return is the average value we would expect to get if we repeated the experience many times, that is,

$$ER = \sum r_i \pi_i$$

Example: In the famous 18th century St. Petersburg lottery

$$R = 2^i \text{ with probability } 2^{-i}$$

think of it as a coin flipping game in which you pay for the opportunity to flip a coin repeatedly until you get a "tails". If you flip tails immediately you get \$2, if you flip one Head and then a Tail you get \$4, and so forth:

#H	Payoff	Probability
0	2	1/2
1	4	1/4
2	8	1/8
⋮	⋮	⋮

What is the expectation?

$$\begin{aligned} ER &= 2 \frac{1}{2} + 4 \frac{1}{4} + 8 \frac{1}{8} + \dots \\ &= 1 + 1 + 1 + \dots \\ &= \infty \end{aligned}$$

What is wrong? How much would you be willing to pay for a ticket to play this lottery? □

Clearly as potential investors we are interested in more than the expected value of R . At the very least we would like to know something about the variability of R . One such measure is the variance, defined as

$$V(R) \equiv E(R - ER)^2$$

the average squared discrepancy between R and its mean value. Expanding the squared term we have

$$\begin{aligned} V(R) &= E(R^2 - 2R(ER) + (ER)^2) \\ &= ER^2 - 2(ER)^2 + (ER)^2 \\ &= ER^2 - (ER)^2 \end{aligned}$$

Exercises. Show that for μ and σ fixed constants: (i) $E(\mu + \sigma X) = \mu + \sigma EX$, (ii) $V(\mu + \sigma X) = \sigma^2 V(X)$ \square .

When we have more than one random variable then we need to carefully distinguish between the marginal distributions of the random variables and their joint distributions. Suppose for simplicity we have two random variables and

$$P(X = x_i, Y = y_j) = \pi_{ij}$$

This describes the joint distribution. The marginal distributions of X and Y are obtained by summing over the other variable

$$\begin{aligned} P(X = x_i) &= \sum_{j=1}^{n_j} \pi_{ij} = \pi_i \\ P(Y = y_j) &= \sum_{i=1}^{n_i} \pi_{ij} = \pi_j \end{aligned}$$

In the special case that X and Y are *independent*, we can factor π_{ij} into two multiplicative components that is

$$\pi_{ij} = \pi_i \pi_j$$

we will see that in this case X conveys no information about Y and vice-versa.

Expectations have the nice property that

$$E(X + Y) = EX + EY$$

but variance of $X + Y$ is more interesting

$$\begin{aligned} V(X + Y) &= E((X - EX) - (Y - EY))^2 \\ &= E(X - EX)^2 + E(Y - EY)^2 - 2E(X - EX)(Y - EY) \end{aligned}$$

the last term which is usually called the covariance of X, Y has an important role to play. If X and Y are independent, then this covariance is zero:

$$\begin{aligned} \text{Cov}(X, Y) &= E(X - EX)(Y - EY) \\ &= \sum \sum (x_i - EX)(y_j - EY) \pi_{ij} \\ &= \sum (x_i - EX) \pi_i - \sum (y_j - EY) \pi_j \\ &= 0 \end{aligned}$$

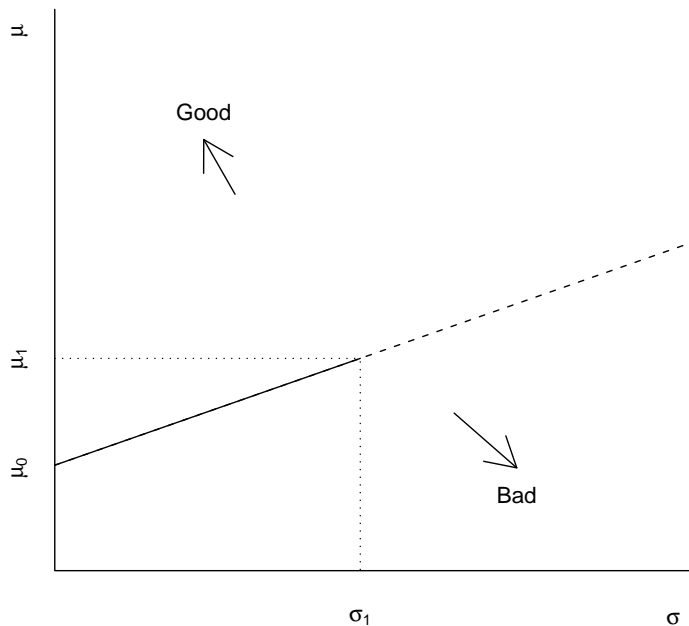


Figure 1: Portfolios composed of a single risky asset and a riskfree asset.

But in general this isn't true and the covariance is an important component.

Now let's return to our simple portfolio problem. Suppose we want to make a very simple portfolio. It will be composed of one risky asset and one risk free asset. The risk free asset earns rate of return μ_0 and has no variability, the risky asset earns expected rate of return μ_1 and has standard deviation σ_1 . We can plot the situation in Figure 1.

It is presumed that investors prefer high μ and low σ . Why?

Suppose we make a portfolio that puts proportion p of our wealth in the risk free asset and the remainder $(1 - p)$ in the risky asset. Then we can compute the mean and variance of the resulting portfolio as follows. Let R_0 be the risk free asset that pays μ_0 with no risk, $\sigma_0 = 0$, and R_1 the risky asset that pays μ_1 with standard deviation σ_1 . Then a portfolio

$$R_p = pR_0 + (1 - p)R_1$$

has mean,

$$\mu = ER = p\mu_0 + (1 - p)\mu_1$$

and variance,

$$\sigma_p^2 = (VR_p) = (1 - p)^2\sigma_1^2,$$

or standard deviation,

$$\sigma_p = (VR_p)^{(1/2)} = (1 - p)\sigma_1,$$

so changing p moves us along the line in the figure connecting $(\mu_0, 0)$ and (μ_1, σ_1) . If we assumed that we could *borrow* at the risk free rate, not just lend, then we could extend the line beyond (μ_1, σ_1) . If such “short sales” are impossible we are restricted to the line segment. The usual theory of portfolio choice would then posit preferences that choose a point on this feasible line, or line segment.

Now consider what happens when we add a third (second risky) asset. We can begin with a simple plot

The points labeled R_1 and R_2 describe the two risky assets. Clearly we could repeat the prior exercise connecting R_1 and R_2 with the risk free asset R_0 , but we can do better than that!

Consider, for a moment, a portfolio composed of only the two risky assets,

$$R_p = pR_1 + (1 - p)R_2$$

Clearly, by our prior rules

$$\mu_p = ER_p = p\mu_1 + (1 - p)\mu_2$$

but what about the risk of R_p ? Measuring risk by the variance of R_p requires a brief calculation:

$$\begin{aligned} V(R_p) &= E(R_p - \mu_p)^2 \\ &= E(p(R_1 - \mu_1) + (1 - p)(R_2 - \mu_2))^2 \\ &= p^2V(R_1) + (1 - p)^2V(R_2) + 2p(1 - p) \cdot \text{Cov}(R_1, R_2) \end{aligned}$$

where $\text{Cov}(R_1, R_2) = E(R_1 - \mu_1)(R_2 - \mu_2)$. If R_1 and R_2 are exactly linearly dependent $\text{Cov}(R_1, R_2)^2 = V(R_1)V(R_2)$. [Say $R_2 = a + bR_1$, then $V(R_2) = b^2V(R_1)$ and $\text{Cov}(R_1, R_2) = E(R_1 - \mu_1)(bR_1 - b\mu_1) = bE(R_1 - \mu_1)^2 = bV(R_1)$] But otherwise,

$$\text{Cov}(R_1, R_2)^2 \leq V(R_1)V(R_2)$$

This leads us back to the concept of correlation, you do remember correlation?

$$\rho(R_1, R_2) = \frac{\text{Cov}(R_1, R_2)}{\sqrt{V(R_1)V(R_2)}}$$

and shows that $|\rho(R_1, R_2)| \leq 1$ with equality only when R_1 and R_2 are perfectly linearly dependent. This is highly unlikely in the case of asset prices unless the two assets are essentially reflecting the value of the same thing.

Implausible or not, let’s pursue this very special case a bit further. We can write

$$V(R_p) = p^2\sigma_1^2 + (1 - p)\sigma_2^2 + 2p(1 - p)\rho\sigma_1\sigma_2$$

when $\rho = 1$ we can express $V(R_p)$ simply as

$$V(R_p) = (p\sigma_1 + (1 - p)\sigma_2)^2$$

so in this case the standard deviation of R_p would be simply,

$$\bar{\sigma}_p = p\sigma_1 + (1 - p)\sigma_2$$

Taking this together with the similar result for the mean return implies that portfolios composed of R_1 and R_2 would put us somewhere on the line segment connecting the two points on our graph. What if $\rho(R_1, R_2) < 1$? Then, clearly

$$\sigma_p < p\sigma_1 + (1 - p)\sigma_2$$

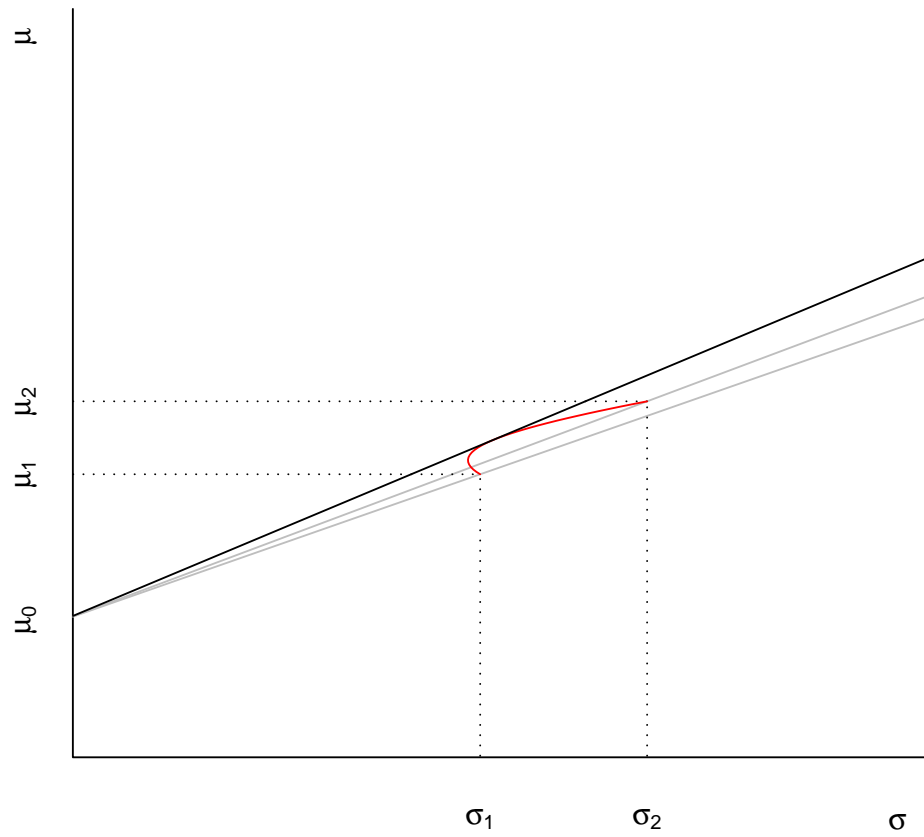


Figure 2: Portfolios with two risky assets and one riskfree asset.

so our picture looks like Figure X. As we vary p we move along the upper curve instead of the (rather boring) straight line connecting the two points.

So what? Why is the curve such a big improvement? The answer lies in the tangent line drawn through $(\mu_0, 0)$ in the figure. Combining R_1 and R_2 gives us access to better opportunities, strictly improving on what can be achieved by simply investing in R_0 and R_1 , or in R_0 and R_2 , without combining all three. These inferior portfolios are represented by the lower gray lines in the figure.

Optimal Portfolio Rule: Find p^* to determine the point of tangency, this determines an optimal proportion of the two risky assets; then find an optimal point on the tangent line by maximizing utility over the achievable points on the line.

Conclusion What have we accomplished? We have a simple way to graphically determine how to choose portfolio weights in a simple three asset world. Later in the course we will return to this problem and give a general solution for how to compute portfolio weights for an arbitrary number of assets and (possibly) explore some alternatives to variance as a measure of risk in so doing.