Inference on the Quantile Regression Process

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There is more to econometric life than is dreamt of in the conventional regression philosophies of location-scale shift models.

Outline

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  - ⋆ The Location-Scale Shift Hypothesis
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- Kolmogorov-Smirnov Tests and the Durbin Problem
- Khmaladze's Martingalization Approach to the Durbin Problem
- Inference on the Quantile Regression Process
- Application to the Pennsylvania Bonus Experiment

Reemployment Bonus Experiments

Can the durations of insured unemployment spells be shortened by offering cash bonuses to recipients for early reemployment?

- 1988-89 Experiment in Pennsylvania
- 6 Treatments + Control Group
  - ⋆ Two levels of bonus payment
  - ⋆ Two settings of the qualification period
- Randomized Assignment to Groups
- 13,913 Participants

Some Post-Modern Econometrics

The mean deconstructed into the quantiles:

$$\mu = \int_{-\infty}^{\infty} x dF(x) = \int_0^1 F^{-1}(t) dt$$

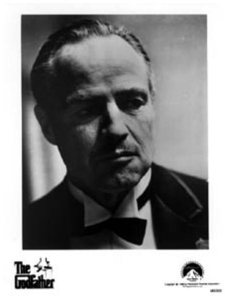
The mean treatment effect deconstructed into the quantile treatment effect:

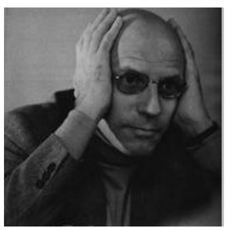
$$\delta = \mu(G) - \mu(F) = \int_0^1 (G^{-1}(t) - F^{-1}(t)) dt$$

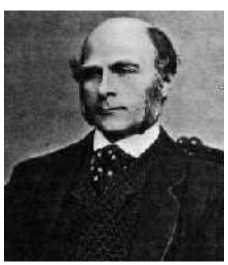
The regression mean effect deconstructed into regression quantiles:

$$E(Y|x) = \int_0^1 Q_Y(\tau|x) d\tau$$

Regression is Demeaning

 'De mean is 'de meaning.

 Regression is demeaning.

 Regression is de-meaning.

Transformation Models for Durations

Suppose

$$G^{-1}(S(t|x)) = h(t) - x^\top \beta$$

where $S(t|x)$ is the conditional survival function. For h monotone,

$$\begin{aligned} P(h(T) > t|x) &= P(T > h^{-1}(t)|x) \\ &= S(h^{-1}(t)|x) \\ &= G(t - x^\top \beta). \end{aligned}$$

We have the transformation model

$$h(T) = x^\top \beta + u$$

where u is iid from G.

Example: Cox Proportional Hazard Model

For the Cox model

$$\log \Lambda_0(T) = x^\top \beta + u$$

with $G(u) = 1 - \exp(-\exp(u))$. For Λ_0 Weibull,

$$\log \Lambda_0(t) = \gamma \log t - \alpha,$$

we obtain the accelerated failure time model,

$$\log T = x^\top \beta + u.$$

with iid u distributed as Weibull.

Quantile Regression Transformation Models

Given the transformation model the conditional quantile functions of $h(T)$, for $0 < \tau < 1$, are

$$Q_{h(T)}(\tau|x) = x^\top\beta + F_u^{-1}(\tau)$$

Since $P(h(T) \leq t) = P(T \leq h^{-1}(t))$, (monotone equivariance!)

$$Q_T(\tau|x) = h^{-1}(x^\top\beta + F_u^{-1}(\tau)).$$

Instead, we will consider,

$$Q_{h(T)}(\tau|x) = x^\top\beta(\tau),$$

for example, consider the location-scale shift model,

$$h(T_i) = x_i^\top\alpha + (x_i\gamma)u_i$$

with u_i iid from F. In this model we have a linear family of conditional quantile functions

$$Q_{h(T)}(\tau|x) = x^\top\alpha + (x^\top\gamma)F_u^{-1}(\tau) = x^\top\beta(\tau)$$

This is considerably more flexible.

An Inference Problem

We would like to test whether covariates have a pure location shift effect on the response, a location-scale shift effect, or if they have some more general effect on the response distribution:

- Location Shift Hypothesis:

$$H_0 : \beta_i(\tau) = \alpha_i \quad i = 2, ..., p.$$

- Location-Scale Shift Hypothesis:

$$H_0 : \beta_i(\tau) = \alpha_i + \gamma_i \beta_1(\tau) \quad i = 2, ..., p.$$

Tests of the Kolmogorov-Smirnov type based on the whole quantile regression process will be considered.

The Kolmogorov-Smirnov Test

Suppose $\{Y_1, \ldots, Y_n\}$ are *iid* from $df \;\; F$. We would like to test,

$$H_0 : F = F_0.$$

We want to consider the K-S statistic,

$$K_n = \sup_{x \in \mathbb{R}} \sqrt{n} |F_n(x) - F_0(x)|$$

where $F_n(x) = n^{-1} \sum I(Y_i \leq x)$.

KS Test is ADF

Classically, from Doob (1949), we know

$$U_n(x) = \sqrt{n}(F_n(x) - F_0(x))$$

or, changing variables $x \longrightarrow F_n^{-1}(\tau)$,

$$u_n(\tau) = \sqrt{n}(\tau - F_0(F_n^{-1}(\tau)))$$

converges weakly under H_0 to a Brownian Bridge process, i.e., a Gaussian process, u_0, with mean zero and covariance function $Cov(u_0(\tau_1), u_0(\tau_2)) = \tau_1 \wedge \tau_2 - \tau_1\tau_2$. so the test is *asymptotically distribution free* (ADF).

The Durbin Problem

Now suppose F_0 is known only up to parameters, e.g., $F_0(x, \theta_0) = \Phi((x - \mu_0)/\sigma_0)$, but $\theta_0 = (\mu_0, \sigma_0)$ is unknown. We are tempted to consider the process,

$$\hat{U}_n(x) = \sqrt{n}(F_n(x) - F_0(x, \hat{\theta}_n))$$

and again changing variables, so $\tau = F_0(x, \theta_0)$, setting $G(\tau, \theta_0) = \tau$,

$$\hat{u}_n(\tau) = \sqrt{n}(G_n(\tau) - G(\tau, \hat{\theta}_n))$$

Like $u_n(\tau)$, $\hat{u}_n(\tau)$ converges weakly to zero mean Gaussian process, say, $\hat{u}_n(\tau) \Rightarrow \hat{u}_0(\tau)$, but now for the mle $\hat{\theta}_n$,

$$E(\hat{u}_0(\tau_1)\hat{u}_0(\tau_2)) = \tau_1 \wedge \tau_2 - \tau_1\tau_2 - g_0(\tau_1)^\top \mathcal{J}^{-1} g_0(\tau_2)$$

where $g_0(\tau) = \partial F_0(y, \theta_0)/\partial\theta|_{y=F_0^{-1}(\tau,\theta_0)}$, and $\mathcal{J}$ is the Fisher information about θ in model F_0. Now $\hat{K}_n = \sup |\hat{u}_n(\tau)|$ depends on F_0; this is *the Durbin Problem*.

The Doob-Meyer Decomposition

The process $G_n(\tau) = F_0(F_n^{-1}(\tau))$ is Markov:

$$n\Delta G_n(\tau) = n[G_n(\tau + \Delta\tau) - G_n(\tau)] \sim \text{Bin}(n(1 - G_n(\tau)), \Delta\tau/(1 - \tau)).$$

So,

$$E[\Delta G_n(\tau)|\mathcal{F}_\tau^{G_n}] = \frac{1 - G_n(\tau)}{1 - \tau}\Delta\tau$$

and this *suggests* the representation,

$$G_n(t) = \int_0^t \frac{1 - G_n(s)}{1 - s} ds + m_n(t)$$

where $m_n(t)$ is a martingale. Now substituting from $u_n(t) = \sqrt{n}(G_n(t) - t)$ we have

$$w_n(t) = u_n(t) + \int_0^t \frac{u_n(s)}{1 - s} ds$$

where $w_n(t) = \sqrt{n} m_n(t) \Rightarrow w_0(\tau)$, is standard Brownian motion.

“Marmalade” in a Martingale

Etymology: a. Fr. martingale of obscure etymology. [First found in Rabelais in *chausses a la martingale*, men’s socks that fastened at the back of the leg. This is commonly supposed to mean literally ‘hose after the fashion of Martigues’ (in Provence).

Doob-Meyer as Recursive OLS

Let $g(t) = (t, g_1(t), \dots, g_p(t))^\top$ be a $(p+1)$-vector of real-valued functions on $[0,1]$. Suppose $\dot{g}(t) = dg(t)/dt$ are linearly independent, so

$$C(t) = \int_t^1 \dot{g}(s)\dot{g}(s)^\top ds$$

is nonsingular, and consider the transformation,

$$w_n(t) = v_n(t) - \int_0^t \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) dv_n(r) ds$$

In the Doob-Meyer case, we set $g(t) = t$ so $\dot{g}(t) = 1$, $C(s) = 1 - s$, and noting that,

$$\int_s^1 \dot{g}(r) dv_n(r) = v_n(1) - v_n(s) = -v_n(s)$$

we obtain the Doob-Meyer decomposition.

Khmaladze's Martingalization

Ingredients:

$$G(\tau, \hat{\theta}_n) = \tau + (\hat{\theta} - \theta_0)^\top g(\tau, \theta^*)$$

$$\sqrt{n}(\hat{\theta} - \theta_0) = \int_0^1 h(s, \theta_0) du_n(s) + o_p(1)$$

$$\hat{u}_n(\tau) = \sqrt{n}(G_n(\tau) - \tau + \tau - G(\tau, \hat{\theta}_n))$$

Combine and stir:

$$\hat{u}_n(\tau) \quad = u_n(\tau) - g(\tau, \theta_0)^\top \int_0^1 h(s, \theta_0) du_n(s) + o_p(1) \tag{1}$$

$$\Rightarrow u_0(\tau) - g(\tau, \theta_0)^\top \int_0^1 h(s, \theta_0) du_0(s) \tag{2}$$

but,

$$\tilde{u}_n(\tau) \quad = \hat{u}_n(\tau) - \int_0^\tau \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) d\hat{u}_n(r) ds \tag{3}$$

$$\Rightarrow w_0(\tau) \tag{4}$$

Martingalization annihilates the $g(\tau, \theta_0)$ term and restores ADF property of KS-test!

Khmaladze for the Quantile Process

Let

$$\hat{\alpha}(\tau) = \text{argmin}_{a \in \mathbb{R}} \sum \rho_\tau(y_i - a)$$

where $\{y_i\}$ are iid from $F_0((y-\mu)/\sigma)$. Consider

$$H_0 : \quad \alpha(\tau) = F_y^{-1}(\tau) = \mu + \sigma F_0^{-1}(\tau)$$

under H_0,

$$v_n(\tau) = \sqrt{n}\varphi_0(\tau)(\hat{\alpha}(\tau) - \alpha(\tau))/\sigma \Rightarrow v_0(\tau)$$

where $\varphi_0(\tau) = f_0(F_0^{-1}(\tau))$ and $v_0(\tau)$ is the Brownian Bridge process.
To test H_0, set $\tilde{\alpha}(\tau) = \xi(\tau)^\top \tilde{\theta} = (1, F_0^{-1}(\tau))\tilde{\theta}$, then

$$\hat{v}_n(t) = \sqrt{n}\varphi_0(t)(\hat{\alpha}(t) - \tilde{\alpha}(t))/\sigma \quad (5)$$

$$= \sqrt{n}\varphi_0(t)(\hat{\alpha}(t) - \alpha(t) - (\tilde{\alpha}(t) - \alpha(t)))/\sigma \quad (6)$$

$$= v_n(t) - \sqrt{n}\varphi_0(\tau)(\tilde{\theta} - \theta_0)^\top \xi(t)/\sigma \quad (7)$$

Now we apply martingalization as before.

Testing for Normality

In the typical case that θ_0 consists of a location and scale parameter we have,

$$g(\tau) = (\tau, \varphi_0(\tau)\xi(\tau)^\top)^\top$$

so,

$$\dot{g}(\tau) = (1, \dot{f}/f, 1 - F_0^{-1}(\tau)\dot{f}/f)^\top$$

where $\dot{f}/f$ is evaluated at $F^{-1}(\tau)$. In the Gaussian case, $F_0 = \Phi$, we have

$$\dot{g}(\tau) = (1, -\Phi^{-1}(\tau), 1 - \Phi^{-1}(\tau)^2)^\top$$

Inference for Quantile Regression

Now consider the quantile regression process,

$$\hat{\beta}(\tau) = \text{argmin}_{b \in \mathbb{R}^p} \sum \rho_\tau(y_i - x_i^\top b)$$

The analogue of the location scale model is

$$y_i = x_i^\top \alpha + (x_i^\top \gamma) u_i$$

with $\{u_i\}$ iid from F_0. This implies the null hypothesis,

$$H_0 : \quad \beta_i(\tau) = \alpha_i + \gamma_i F_0^{-1}(\tau) \quad i = 1, \ldots, p.$$

We would like to test, H_0, versus a general alternative. Note that, H_0 implies that all p coordinates of $\beta(\cdot)$ are affine functions of a single univariate function.

Simple Nulls

When α, γ, F_0 are all known we have, subject to some regularity conditions,

$$v_n(\tau) = \sqrt{n} J_n^{-1/2} H_n(\hat{\beta}(\tau) - \beta(\tau)) \Rightarrow v_0$$

where v_0 is now a p-variate Brownian Bridge, $J_n = n^{-1} X^\top X$, $H_n = n^{-1} X^\top \Gamma^{-1} X$, and $\Gamma = \text{diag}(x_i^\top \gamma)$.

This leads to Wald, LR and LM/rankscore tests as in Koenker and Machado (JASA, 1999), employing Bessel processes as in Kiefer(1959). But when (α, γ) are unknown, the Durbin problem arises again.

A General Linear Hypothesis

Consider the hypothesis,

$$R\beta(\tau) - r = \Psi(\tau) \qquad \tau \in \mathcal{T} \tag{8}$$

where R denotes a $q \times p$ matrix, $q \leq p, r \in \mathbb{R}^q$, and $\Psi(\tau)$ denotes a known function $\Psi : \mathcal{T} \longrightarrow \mathbb{R}^q$. and the local alternative,

$$R\beta_n(\tau) - r - \Psi(\tau) = \zeta(\tau)/\sqrt{n}.$$

Test based on:

$$v_n(\tau) = \sqrt{n}\varphi_0(\tau)(R\Omega R^\top)^{-1/2}(R\hat{\beta}(\tau) - r - \Psi(\tau))$$

where $\Omega = H_0^{-1} J_0 H_0^{-1}$ with $J_0 = \lim n^{-1} \sum x_i x_i^\top$, and $H_0 = \lim n^{-1} \sum x_i x_i^\top / \gamma^\top x_i$.

Regularity Conditions

Assumption 1. *The distribution function* F_0*, has a continuous Lebesgue density,* f_0*, with* $f_0(u) > 0$ *on* $\{u : 0 < F_0(u) < 1\}$.

Assumption 2. *The sequence of design matrices* $\{X_n\} = \{(x_i)_{i=1}^n\}$ *satisfy:*

(i) $x_{i1} \equiv 1 \qquad i = 1, 2, \ldots$
(ii) $J_n = n^{-1} X_n^\top X_n \longrightarrow J_0$*, a positive definite matrix.*
(iii) $H_n = n^{-1} X_n^\top \Gamma_n^{-1} X_n \longrightarrow H_0$*, a positive definite matrix where* $\Gamma_n =$ *diag*$(\gamma^\top x_i)$.

Assumption 3. *There exists a fixed, continuous function* $\zeta(\tau) : [0, 1] \longrightarrow \mathbb{R}^q$ *such that for samples of size* n*,*

$$R\beta_n(\tau) - r - \Psi(\tau) = \zeta(\tau)/\sqrt{n}.$$

More Regularity Conditions

Assumption 4. *There exist estimators* $\varphi_n(\tau)$ *and* Ω_n *satisfying*

i. $\sup_{\tau \in \mathcal{T}} |\varphi_n(\tau) - \varphi_0(\tau)| = o_p(1)$,
ii. $||\Omega_n - \Omega|| = o_p(1)$.

Assumption 5. *The function* $g(t)$ *satisfies:*

i $\int \| \dot{g}(t) \|^2 \, dt < \infty$,
ii $\{\dot{g}_i(t) : i = 1, \ldots, m\}$ *are linearly independent in a neighborhood of 1.*

Theorem 1. *Let $\mathcal{T}$ denote the closed interval $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0, 1/2)$. Under conditions A.1-3*

$$\begin{aligned} v_n(\tau) &= \sqrt{n}\varphi_0(\tau)(R\Omega R^\top)^{-1/2}(R\hat{\beta}(\tau) - r - \Psi(\tau)) && (9) \\ &\Rightarrow v_0(\tau) + \eta(\tau) \text{ for } \tau \in \mathcal{T} && (10) \end{aligned}$$

where $v_0(\tau)$ denotes a q-variate standard Brownian bridge process and $\eta(\tau) = \varphi_0(\tau)(R\Omega R^\top)^{-1/2}\zeta(\tau)$. Under the null hypothesis, $\zeta(\tau) = 0$, the test statistic

$$\sup_{\tau \in \mathcal{T}} \| v_n(\tau) \| \Rightarrow \sup_{\tau \in \mathcal{T}} \| v_0(\tau) \| .$$

Theorem 2. *Under conditions A.1-5, we have*

$$\begin{aligned} \hat{v}_n(\tau) &= \sqrt{n}\varphi_0(\tau)[R_n\Omega R_n^\top]^{-1/2}(R_n\hat{\beta}(\tau) - r_n - \Psi(\tau)) && (11) \\ &\Rightarrow Z_n^\top \xi(\tau) + v_0(\tau) + \eta(\tau) \text{ for } \tau \in \mathcal{T} && (12) \end{aligned}$$

where $\xi(\tau) = \varphi_0(\tau)(1, F_0^{-1}(\tau))^\top$, and $Z_n = \mathcal{O}_p(1)$, with $v_0(\tau)$ and $\eta(\tau)$ as specified in Theorem 1.

Theorem 3. *Under conditions A.1 - 6, we have*

$$\tilde{v}_n(\tau)^\top = \hat{v}_n(\tau)^\top - \int_0^\tau \dot{g}(s)^\top C^{-1}(s) \int_s^1 \dot{g}(r) d\hat{v}_n(r)^\top ds \quad (13)$$

$$\Rightarrow w_0(\tau) + \tilde{\eta}(\tau) \text{ for } \tau \in \mathcal{T} \quad (14)$$

where $w_0(\tau)$ denotes a q-variate standard Brownian motion, and under the null hypothesis, $\zeta(\tau) = 0$,

$$\sup_{\tau \in \mathcal{T}} \| \tilde{v}_n(\tau) \| \Rightarrow \sup_{\tau \in \mathcal{T}} \| w_0(\tau) \| .$$

Pennsylvania Bonus Experiment

Table 1: Treatment Groups

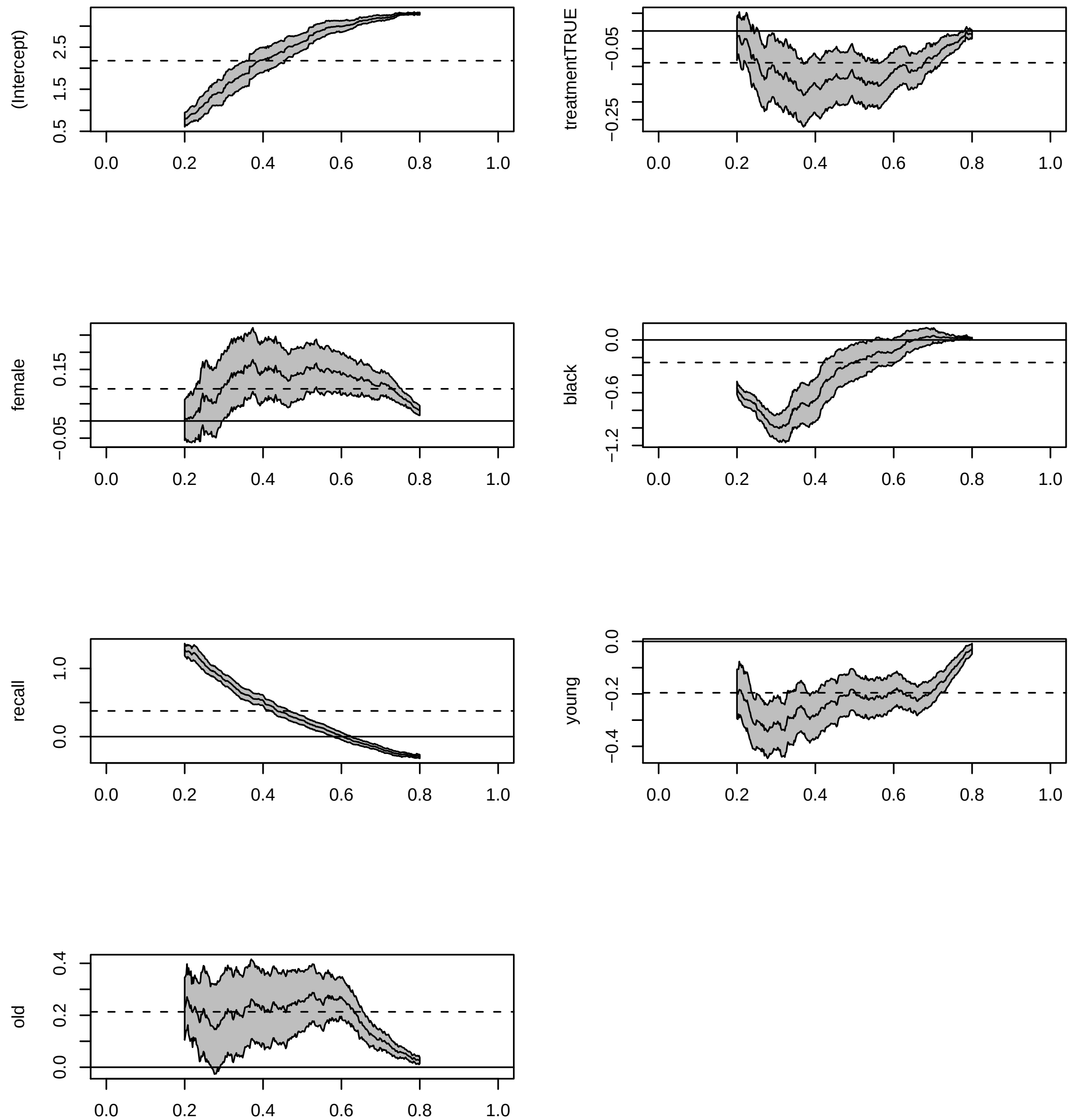
Group	Bonus Amount	Qualification Period	Workshop Offer
Controls	0	0	No
Treatment 1	Low	Short	Yes
Treatment 2	Low	Long	Yes
Treatment 3	High	Short	Yes
Treatment 4	High	Long	Yes
Treatment 5	Declining	Long	Yes
Treatment 6	High	Long	No

Note: The low benefit was 3 times UI weekly benefit amount, the high benefit was 6 times this amount. The declining bonus declined from 6 times the weekly benefit to zero, over a 12 week period. The short qualification period was 6 weeks, and the long period was 12 weeks.

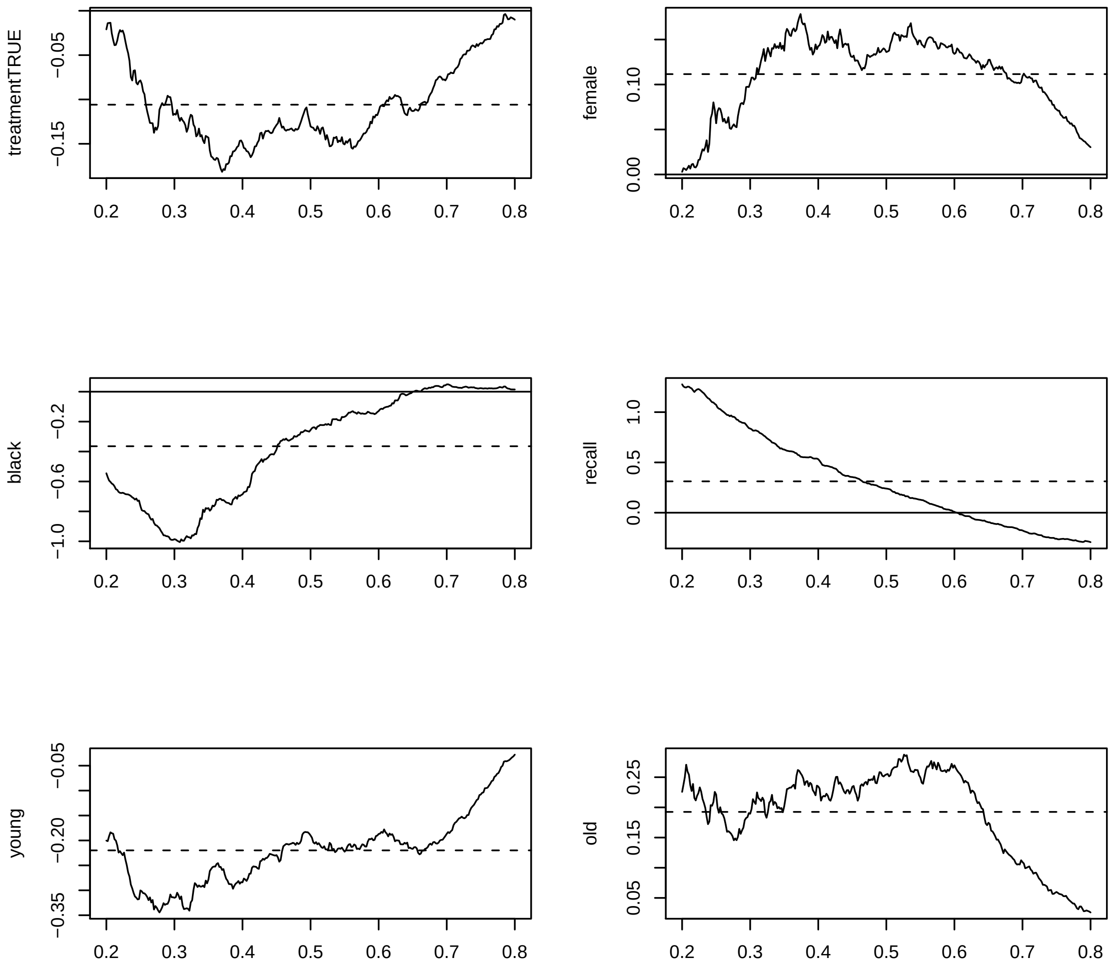
Sample Sizes

Groups	Target n	Collected n	Analysis n
Control	3,000	3,392	3,354
Treatment 1	1,030	1,395	1,385
Treatment 2	2,240	2,456	2,428
Treatment 3	1,740	1,910	1,885
Treatment 4	1,590	1,771	1,745
Treatment 5	1,740	1,860	1,831
Treatment 6	1,780	1,302	1,285
Total	13,120	14,086	13,913

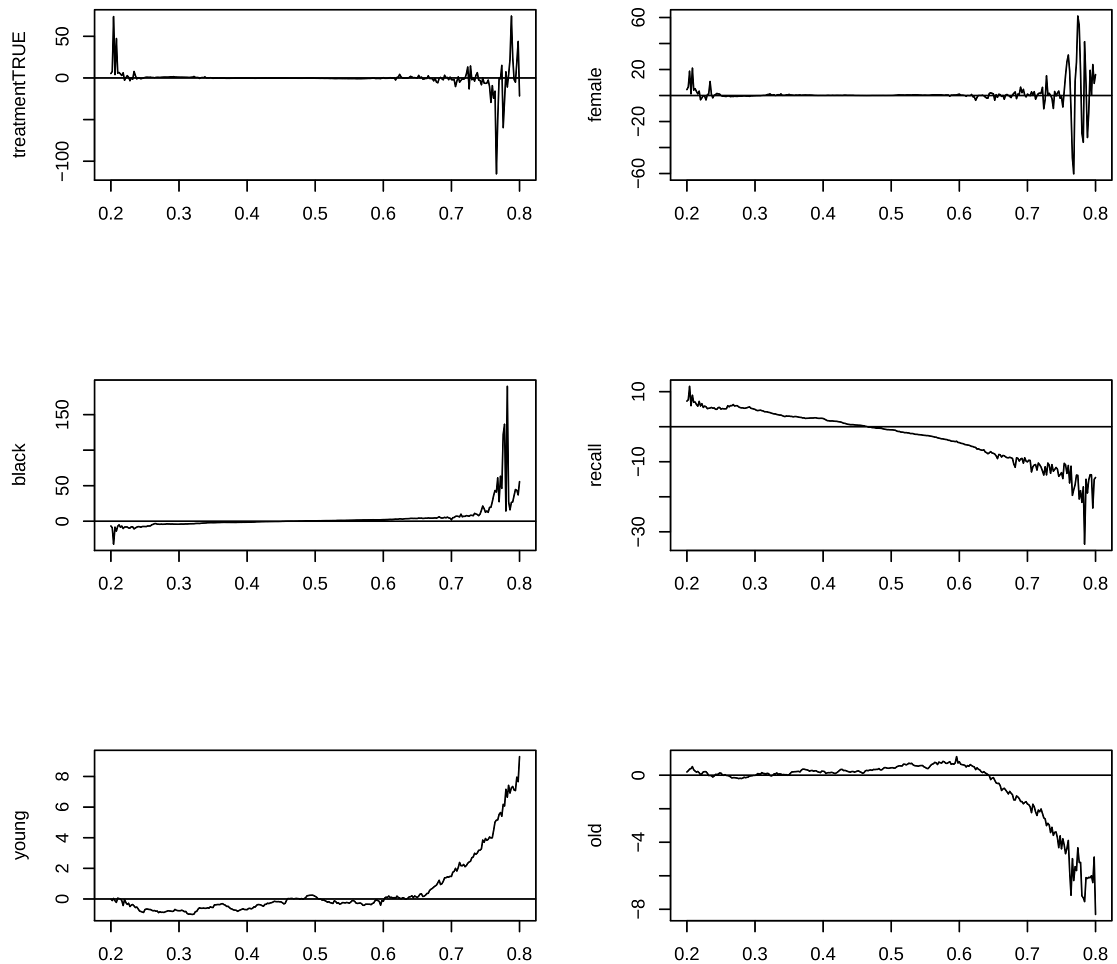
Quantile Regression Process



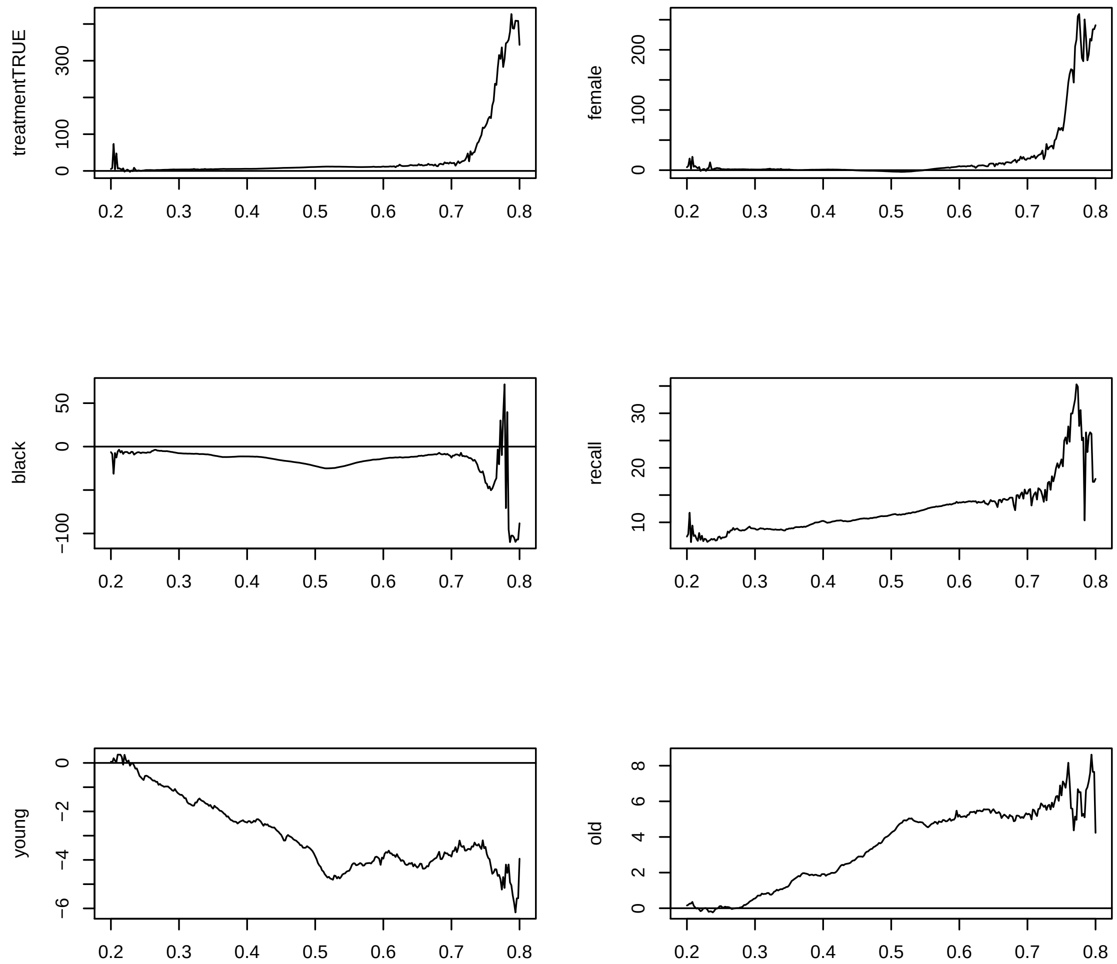
Fitted Quantile Regression Process



Standardized Residual Quantile Regression Process



Khmaladzized Quantile Regression Process



Test Results

Variable	Location Scale Shift	Location Shift
Treatment	5.41	5.48
Female	4.47	4.42
Black	5.77	22.00
Hispanic	2.74	2.00
N-Dependents	2.47	2.83
Recall Effect	4.45	16.84
Young Effect	3.42	3.90
Old Effect	6.81	7.52
Durable Effect	3.07	2.83
Lusd Effect	3.09	3.05
Joint Effect	112.23	449.83

Table 2: Tests of the Location-Scale and Location Shift Hypotheses: Critical values for the univariate tests are 1.92 at .05 and 2.42 at .01. For the joint tests the .01 critical value is 16.0.

Conclusions

- Quantile regression methods complement established survival analysis methods.
- By focusing on local slices of the conditional distribution, they offer a useful deconstruction of conditional mean models.
- They offer a more flexible role for covariate effects allowing them to influence location, scale *and shape* of the response distribution.
- The Khmaladze transformation approach offers a flexible way to handle nuisance parameter problems in semi-parametric inference for quantile regression.