Inference for Quantile Regression

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Inference for Quantile Regression

- Asymptotics of the Sample Quantiles
- QR Asymptotics in iid Error Models
- QR Asymptotics in Heteroscedastic Error Models
- Classical Rank Tests and the Quantile Regression Dual
- Inference on the Quantile Regression Process

Asymptotics for the Sample Quantiles

Minimizing $\sum_{i=1}^n \rho_\tau(y_i - \xi)$ consider

$$g_n(\xi) = -\sum_{i=1}^n \psi_\tau(y_i - \xi) = \sum_{i=1}^n (I(y_i < \xi) - \tau).$$

By convexity of the objective function,

$$\{\hat{\xi}_\tau > \xi\} \Leftrightarrow \{g_n(\xi) < 0\}$$

and the DeMoivre-Laplace CLT yields, expanding F,

$$\sqrt{n}(\hat{\xi}_\tau - \xi) \rightsquigarrow \mathcal{N}(0, \omega^2(\tau, F))$$

where $\omega^2(\tau, F) = \tau(1-\tau)/f^2(F^{-1}(\tau))$. The Bahadur-Kiefer representation theory provides further refinement of this result.

Finite Sample Theory for Quantile Regression

Let $h \in \mathcal{H}$ index the $\binom{n}{p}$ p-element subsets of $\{1, 2, \ldots, n\}$ and $X(h), y(h)$ denote corresponding submatrices and vectors of X and y.

Lemma: $\hat{\beta} = b(h) \equiv X(h)^{-1} y(h)$ is the τth regression quantile iff $\xi_h \in \mathcal{C}$ where

$$\xi_h = \sum_{i \notin h} \psi_\tau(y_i - x_i \hat{\beta}) x_i' X(h)^{-1},$$

$\mathcal{C} = [\tau - 1, \tau]^p$, and $\psi_\tau(u) = \tau - I(u < 0)$.

Theorem: (KB, 1978) In the linear model with iid errors, $\{u_i\} \sim F, f$, the density of $\hat{\beta}(\tau)$ is given by

$$g(b) = \sum_{h \in \mathcal{H}} \prod_{i \in h} f(x_i'(b - \beta(\tau)) + F^{-1}(\tau)) \cdot P(\xi_h(b) \in C) |\mathsf{det}(X(h))|$$

Asymptotic Theory of Quantile Regression I

In the classical linear model,

$$y_i = x_i\beta + u_i$$

with u_i iid from dfF, with density $f(u) > 0$ on its support $\{u|0 < F(u) < 1\}$, the joint distribution of $\sqrt{n}(\hat{\beta}_n(\tau_i) - \beta(\tau_i))_{i=1}^m$ is asymptotically normal with mean 0 and covariance matrix $\Omega \otimes D^{-1}$. Here $\beta(\tau) = \beta + F_u^{-1}(\tau)e_1, e_1 = (1, 0, \ldots, 0)'$, $x_{1i} \equiv 1, n^{-1}\sum x_i x_i' \to D$, a positive definite matrix, and

$$\Omega = ((\tau_i \wedge \tau_j - \tau_i\tau_j)/(f(F^{-1}(\tau_i))f(F^{-1}(\tau_j)))_{i,j=1}^m.$$

Asymptotic Theory of Quantile Regression II

When the response is conditionally independent over i, but not identically distributed, the asymptotic covariance matrix of $\zeta(\tau) = \sqrt{n}(\hat{\beta}(\tau) - \beta(\tau))$ is somewhat more complicated. Let $\xi_i(\tau) = x_i\beta(\tau)$, $f_i(\cdot)$ denote the corresponding conditional density, and define,

$$\begin{aligned} J_n(\tau_1, \tau_2) &= (\tau_1 \wedge \tau_2 - \tau_1\tau_2)n^{-1}\sum_{i=1}^{n} x_i x_i', \\ H_n(\tau) &= n^{-1}\sum x_i x_i' f_i(\xi_i(\tau)). \end{aligned}$$

Under mild regularity conditions on the $\{f_i\}$'s and $\{x_i\}$'s, we have joint asymptotic normality for $(\zeta(\tau_i), \ldots, \zeta(\tau_m))$ with covariance matrix

$$V_n = (H_n(\tau_i)^{-1} J_n(\tau_i, \tau_j) H_n(\tau_j)^{-1})_{i,j=1}^{m}.$$

Rank Based Inference for Quantile Regression

- Ranks play a fundamental $dual$ role in QR inference.
- Classical rank tests for the p-sample problem extended to regression
- Rank tests play the role of Rao (score) tests for QR.

Two Sample Location-Shift Model

$$X_1, \ldots, X_n \sim F(x) \qquad \text{"Controls"}$$
$$Y_1, \ldots, Y_m \sim F(x-\theta) \qquad \text{"Treatments"}$$

Hypothesis:

$$H_0: \quad \theta = 0$$
$$H_1: \quad \theta > 0$$

The Gaussian Model $F = \Phi$

$$T = (\bar{Y}_m - \bar{X}_n)/\sqrt{n^{-1} + m^{-1}}$$

UMP Tests:

$$\text{critical region } \{T > \Phi^{-1}(1-\alpha)\}$$

Wilcoxon-Mann-Whitney Rank Test

Mann-Whitney Form:

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} I(Y_j > X_i)$$

Heuristic: If treatment responses are larger than controls for most pairs (i, j), then H_0 should be rejected.

Wilcoxon Form: Set $(R_1, \ldots, R_{n+m}) = \text{Rank}(Y_1, \ldots, Y_m, X_1, \ldots X_n)$,

$$W = \sum_{j=1}^{m} R_j$$

Proposition: $S = W - m(m+1)/2$ so Wilcoxon and Mann-Whitney tests are equivalent.

Pros and Cons of the Transformation to Ranks

Thought One:

Gain: Null Distribution is independent of F.

Loss: Cardinal information about data.

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Gain: Null Distribution is independent of F.

Loss: Cardinal information about data.

Thought Two:

Gain: Student t-test has quite accurate size provided $\sigma^2(F) < \infty$.

Loss: Student t-test uses cardinal information badly for long-tailed F.

Asymptotic Relative Efficiency of Wilcoxon versus Student t-test

Pitman Alternatives: $H_n : \theta_n = \theta_0/\sqrt{n}$

$(\text{t-test})^2 \rightsquigarrow \chi^2_1(\theta_0^2/\sigma^2(F))$

$(\text{Wilcoxon})^2 \rightsquigarrow \chi^2_1(12\theta_0^2(\int f^2)^2)$

$$\text{ARE(W, t, F)} = 12\sigma^2(\text{F})[\int f^2(x)dx]^2$$

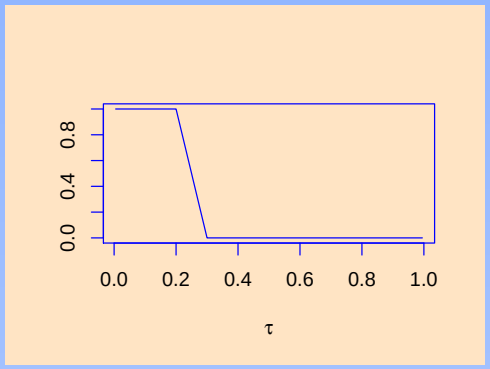
F	N	U	Logistic	DExp	LogN	t_2
ARE	.955	1.0	1.1	1.5	7.35	∞

Theorem (Hodges-Lehmann) For all F, $ARE(W, t, F) \geq .864$.

Hájek 's Rankscore Generating Functions

Let $Y_1, \ldots, Y_n$ be a random sample from an absolutely continuous df F with associated ranks $R_1, \ldots, R_n$, Hájek 's rank generating functions are:

$$\hat{a}_i = \begin{cases} 1 & \text{if } t \leq (R_i - 1)/n \\ R_i - tn & \text{if } (R_i - 1)/n \leq t \leq R_i/n \\ 0 & \text{if } R_i/n \leq t \end{cases}$$



Linear Rank Statistics Asymptotics

Theorem (Hájek (1965)) Let $c_n = (c_{1n}, \ldots, c_{nn})$ be a triangular array of real numbers such that

$$\max_i (c_{in} - \bar{c}_n)^2 / \sum_{i=1}^{n} (c_{in} - \bar{c}_n)^2 \to 0.$$

Then

$$Z_n(t) = (\sum_{i=1}^{n} (c_{in} - \bar{c}_n)^2)^{-1/2} \sum_{j=1}^{n} (c_{jn} - \bar{c}_n) \hat{a}_j(t)$$

converges weakly to a Brownian Bridge.

Some Asymptotic Heuristics

The Hájek functions are approximately indicator functions

$$\hat{a}_i(t) \approx I(Y_i > F^{-1}(t)) = I(F(Y_i) > t)$$

Since $F(Y_i) \sim U[0,1]$, linear rank statistics may be represented as

$$\int_0^1 \hat{a}_i(t) d\varphi(t) \approx \int_0^1 I(F(Y_i) > t) d\varphi(t) = \varphi(F(Y_i)) - \varphi(0)$$

$$\begin{aligned} \int_0^1 Z_n(t) d\varphi(t) &= \sum w_i \int \hat{a}_i(t) d\varphi(t) \\ &= \sum w_i \varphi(F(Y_i)) + o_p(1). \end{aligned}$$

Duality of Ranks and Quantiles

Quantiles may be *defined* as

$$\hat{\xi}(\tau) = argmin \sum \rho_\tau(y_i - \xi)$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$. This can be formulated as a linear program whose dual solution

$$\hat{a}(\tau) = argmax\{y'a | 1_n'a = (1-\tau)n, a \in [0,1]^n\}$$

generates the Hájek rankscore functions.

Reference: Gutenbrunner and Jurečková (1992).

Regression Quantiles and Rank Scores:

$$\hat{\beta}_n(\tau) = \arg\min_{b \in R^p} \sum \rho_\tau(y_i - x_i' b)$$

$$\hat{a}_n(\tau) = \arg\max_{a \in [0,1]^n} \{y'a | X'a = (1-\tau)X'1_n\}$$

$x'\hat{\beta}_n(\tau)$ Estimates $Q_Y(\tau|x)$
Piecewise constant on $[0,1]$.
For $X = 1_n$, $\hat{\beta}_n(\tau) = \hat{F}_n^{-1}(\tau)$.

$\{\hat{a}_i(\tau)\}_{i=1}^n$ Regression rankscore functions
Piecewise linear on $[0,1]$.
For $X = 1_n$, $\hat{a}_i(\tau)$ are the Hajek rank generating functions.

Regression Rank Tests

$$Y = X\,\beta + Z\,\gamma + u$$

$$H_0 : \gamma = 0 \text{ versus } H_n : \gamma = \gamma_0/\sqrt{n}$$

Given the regression rank score process for the restricted model,

$$\hat{a}_n(\tau) = \arg\max \{Y'a \,|\, X'a = (1-\tau)X'1_n\}$$

A test of H_0 is based on the linear rank statistics,

$$\hat{b}_n = \int_0^1 \hat{a}_n(t)\, d\varphi(t)$$

Choice of the score function φ permits test of location, scale or (potentially) other effects.

Regression Rank Tests

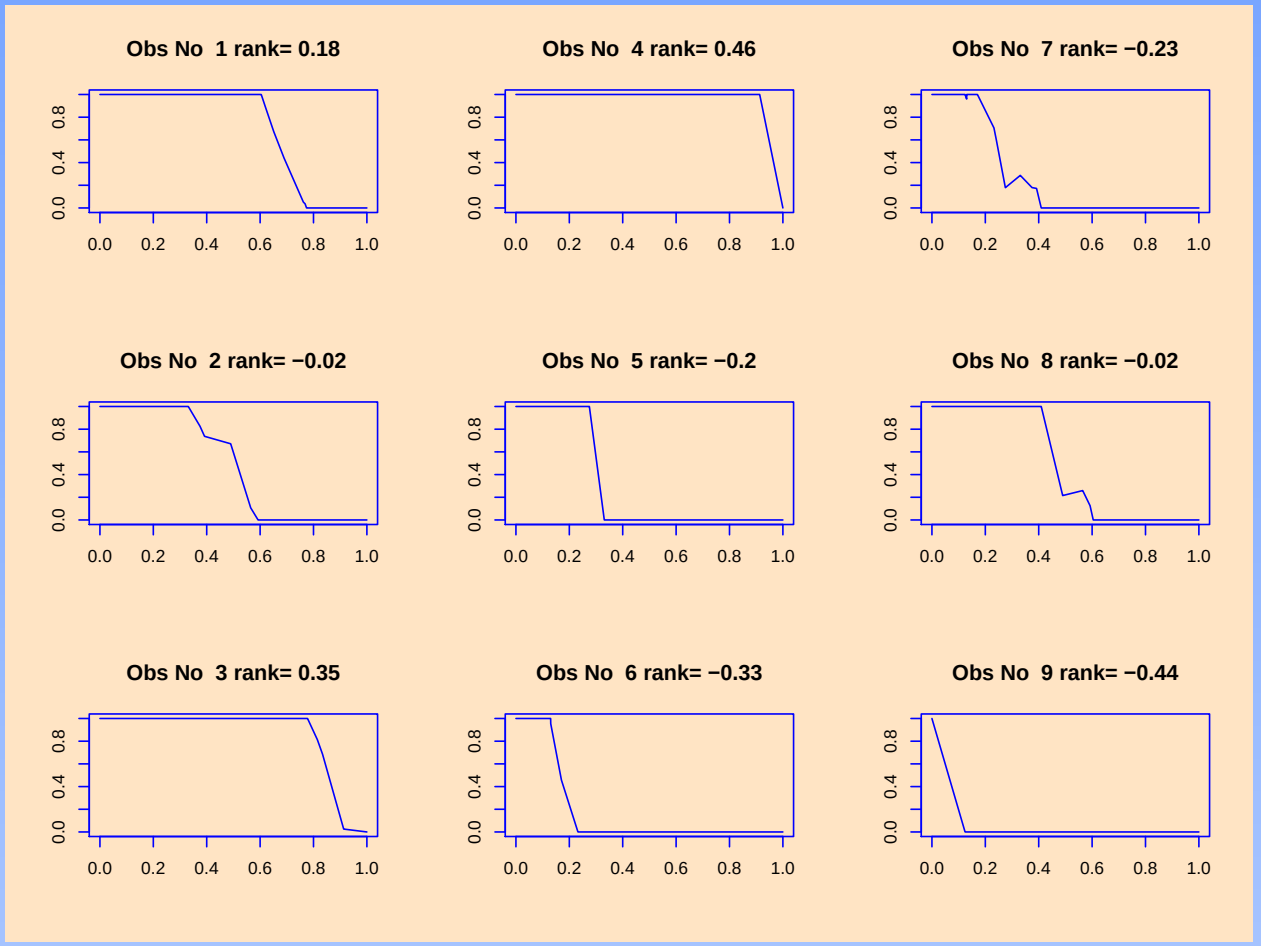
Theorem: (Gutenbrunner, Jurečková , Koenker and Portnoy) Under H_n and regularity conditions, the test statistic $T_n = S_n' Q_n^{-1} S_n$ where $S_n = (Z - \hat{Z})' \hat{b}_n, \quad \hat{Z} = X(X'X)^{-1}X'Z$, $Q_n = n^{-1}(Z - \hat{Z})'(Z - \hat{Z})$

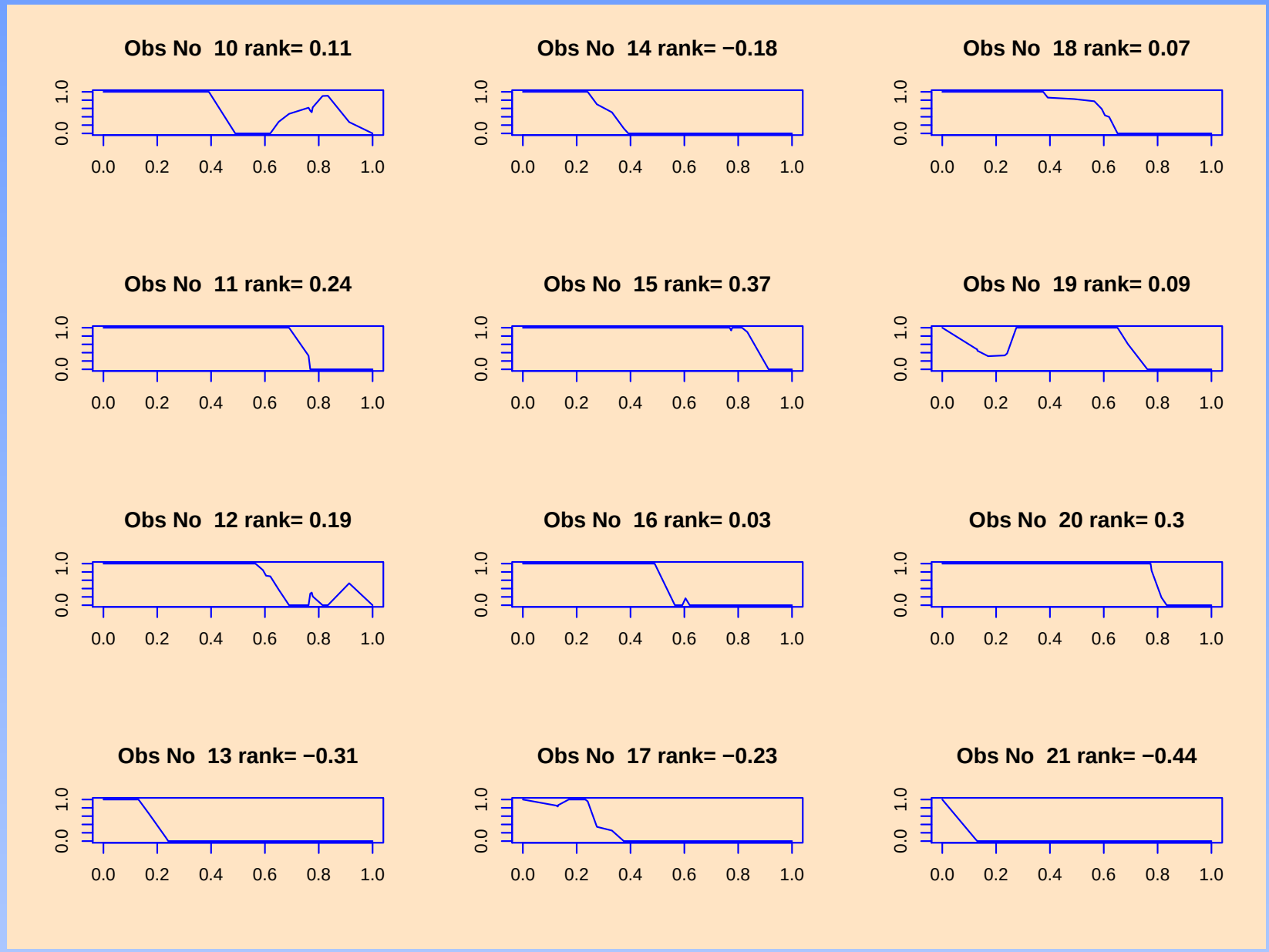
$$T_n \rightsquigarrow \chi_q^2(\eta)$$

where

$$\begin{aligned} \eta^2 &= \omega^2(\varphi, F)\gamma_0' Q \gamma_0 \\ \omega(\varphi, F) &= \int_0^1 f(F^{-1}(t))\, d\varphi(t) \end{aligned}$$

Regression Rankscores for Stackloss Data




Obs No 10 rank= 0.11
Obs No 14 rank= −0.18
Obs No 18 rank= 0.07
Obs No 11 rank= 0.24
Obs No 15 rank= 0.37
Obs No 19 rank= 0.09
Obs No 12 rank= 0.19
Obs No 16 rank= 0.03
Obs No 20 rank= 0.3
Obs No 13 rank= −0.31
Obs No 17 rank= −0.23
Obs No 21 rank= −0.44
0.0
1.0
0.0
0.2
0.4
0.6
0.8
1.0

Inversion of Rank Tests for Confidence Intervals

For the scalar γ case and using the score function

$$\varphi_\tau(t) = \tau - I(t < \tau)$$

$$\hat{b}_{ni} = -\int_0^1 \varphi_\tau(t) d\hat{a}_{ni}(t) = \hat{a}_{ni}(\tau) - (1-\tau)$$

where $\bar{\varphi} = \int_0^1 \varphi_\tau(t)dt = 0$ and $A^2(\varphi_\tau) = \int_0^1 (\varphi_\tau(t) - \bar{\varphi})^2 dt = \tau(1-\tau)$. Thus, a test of the hypothesis $H_0 : \gamma = \xi$ may be based on $\hat{a}_n$ from solving,

$$\max\{(y - x_2\xi)'a | X_1'a = (1-\tau)X_1'1, a \in [0,1]^n\} \tag{1}$$

and the fact that

$$S_n(\xi) = n^{-1/2} x_2' \hat{b}_n(\xi) \rightsquigarrow \mathcal{N}(0, A^2(\varphi_\tau) q_n^2) \tag{2}$$

Inversion of Rank Tests for Confidence Intervals

That is, we may compute

$$T_n(\xi) = S_n(\xi)/(A(\varphi_\tau)q_n)$$

where $q_n^2 = n^{-1}x_2'(I - X_1(X_1'X_1)^{-1}X_1')x_2$. and reject H_0 if $|T_n(\xi)| > \Phi^{-1}(1-\alpha/2)$.

Inverting this test, that is finding the interval of ξ's such that the test fails to reject. This is a quite straightforward parametric linear programming problem and provides a simple and effective way to do inference on individual quantile regression coefficients. Unlike the Wald type inference it delivers asymmetric intervals.

Inference on the Quantile Regression Process

Using the quantile score function, $\varphi_\tau(t) = \tau - I(t < \tau)$ we can consider the quantile rankscore process,

$$T_n(\tau) = S_n(\tau)' Q_n^{-1} S_n(\tau)/(\tau(1-\tau)).$$

where

$$\begin{aligned} S_n &= n^{-1/2}(X_2 - \hat{X}_2)'\hat{b}_n, \\ \hat{X}_2 &= X_1(X_1'X_1)^{-1}X_1'X_2, \\ Q_n &= (X_2 - \hat{X}_2)'(X_2 - \hat{X}_2)/n, \\ \hat{b}_n &= (-\int \varphi(t) d\hat{a}_{in}(t))_{i=1}^n, \end{aligned}$$

Inference on the Quantile Regression Process

Theorem: (Koenker and Machado) Under $H_n : \gamma(\tau) = O(1/\sqrt{n})$ for $\tau \in (0,1)$ the process $T_n(\tau)$ converges to a non-central Bessel process of order $q = \dim(\gamma)$

Related Wald and LR statistics can be viewed as providing a general apparatus for testing goodness of fit for quantile regression models. This approach is closely related to classical p-dimensional goodness of fit tests introduced by Kiefer (1959).

When the null hypotheses under consideration involve unknown nuisance parameters things become more interesting. In Koenker and Xiao (2001) we consider this "Durbin problem" and show that the elegant approach of Khmaladze (1981) yields practical methods.

Some Concluding Comments about Inference

- Asymptotic inference for quantile regression poses some statistical challenges since it involves elements of nonparametric density estimation.
- Classical rank statistics and Hájek 's rankscore process are closely linked via Gutenbrunner and Jurečková 's regression rankscore process.
- Inference on the quantile regression process can be conducted with the aid of Khmaladze's extension of the Doob-Meyer construction.
- Resampling offers many further lines of development for inference in the quantile regression setting.