A New Approach to Identifying the Real Effects of Uncertainty Shocks

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Abstract

This paper introduces the use of the sign restrictions methodology to identify uncertainty shocks. We apply our methodology to a class of vector autoregression models with stochastic volatility that allow volatility fluctuations to impact the conditional mean. We combine sign restrictions on the conditional mean and conditional second moment impulse responses to identify financial and macro uncertainty shocks. On U.S. data, we find stronger evidence that financial uncertainty shocks lead to a decline in real activity and an easing of the federal funds rate relative to macro uncertainty shocks.

Key words: Uncertainty, sign restrictions, vector autoregression, volatility-in-mean, Wishart process, multivariate stochastic volatility.

JEL codes: C11, C32, E32

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1 Introduction

What are the real effects of uncertainty on the macroeconomy? This question has been challenging to answer empirically. Since the seminal paper by Bloom (2009), many papers have taken a variety of approaches including using uncertainty proxies (Baker et al., 2013; Scotti, 2013; Bachmann et al., 2013; Carriero et al., 2015), structural macro models (Fernandez-Villaverde et al., 2011; Leduc and Liu, 2012; Born and Pfeifer, 2014; Basu and Bundick, 2015; Fernandez-Villaverde et al., 2015; Schorfheide et al., 2016), and two-step factor estimation (Jurado et al., 2015; Ludvigson et al., 2015). Two important issues that come to the fore are that uncertainty is not observed and that different sources of uncertainty may affect agents' decisions in different ways. One would ideally desire an econometric framework that can both accurately measure uncertainty from the data and identify different sources of uncertainty shocks.

In this paper, we propose a new strategy to identify the real effects of uncertainty shocks that seeks to simultaneously address both empirical issues. We analyze a class vector autoregressive (VAR) models with multivariate stochastic volatility that allow for the volatility-in-mean effect. VAR models with stochastic volatility-in-mean have been considered by a variety of papers, including Mumtaz and Zanetti (2013), Jo (2014), Creal and Wu (2017), Mumtaz and Theodoridis (2016), and Carriero et al. (2017a). These models have been useful in providing a unifying framework to measure uncertainty and its real effects.

We bring to this literature the use of sign restrictions (Faust, 1998; Canova and Nicolo, 2002; Uhlig, 2005) on the mean and volatility responses of observed economic variables to an uncertainty shock of interest. The literature on sign restrictions to identify level structural shocks in VARs is well developed. Important recent theoretical advances include Baumeister and Hamilton (2015); Arias et al. (2014); Giacomini and Kitagawa (2015); Moon et al. (2017); Amir-Ahmadi and Drautzburg (2017). Our paper, however, is the first to consider the use sign restrictions on the volatility equation to identify structural second moment shocks, in contrast to its previous use to identify structural first moment shocks. The uncertainty shocks considered in our paper are different from level shocks because uncertainty shocks impact both the conditional mean of variables as well as conditional second moment objects of interest, including the volatility of the VAR innovations and the forecast error variances of the variables. With this insight in hand, we proceed to use sign restrictions on the responses of both first and second moment responses to identify our structural uncertainty shocks, which is new to the sign restrictions literature. As we show in an example in the paper, sign restrictions on the first moment and second moment responses each provide useful identifying information.

Our approach is also novel when viewed in the context of the volatility-in-mean literature. Previous papers, such as Mumtaz and Zanetti (2013) and Jo (2014), initially identify first moment structural shocks and then put stochastic volatility on those shocks. Our complementary approach views the level equation of the VAR as a flexible way to control for the expected movements of our macro variables of interest in order to more accurately measure conditional volatility, as emphasized by Jurado et al. (2015). The heart of our identification problem comes from considering the correlated innovations in the volatility equation. In this sense, our work has much in common with Creal and Wu (2017) and Carriero et al. (2017a), who both face a similar problem. These latter two papers solve the second moment identification problem by recursive ordering assumptions. Mumtaz and Theodoridis (2016) do not have an identification problem for uncertainty shocks, as they specify a factor structure for volatility with a single volatility factor.

A quite general class of volatility models are compatible with our methodology. Our framework is amenable to many of the volatility processes popular in the literature, including Cholesky-type decompositions (Cogley and Sargent, 2005; Primiceri, 2005), factor volatility, and level shocks impacting volatility (Carriero et al., 2017a). We consider as a benchmark case the conditional autoregressive inverse Wishart (CAIW) model. This volatility process, introduced into the financial econometrics literature by Golosnoy et al. (2012) and in macroeconomics by Karapanagiotidis (2012) and Rondina (2012), models time-varying volatility with the Wishart family of distributions (See Philipov and Glickman, 2006; Gourieroux et al., 2009; Fox and West, 2013, for alternative autoregressive Wishart models.). We find this model attractive because of its flexibility and property that its estimation results do not depend on the ordering of the observable variables in the system. Moreover, the volatility process can be written linearly in terms of the one-step-ahead forecast error variances and covariances, which facilitates the imposition of sign restrictions on the volatility responses.

As an illustration of our empirical framework, we identify the real effects of financial versus macro uncertainty shocks with two examples. Across these two examples, we emphasize the importance of spreads in distinguishing between different uncertainty shocks, which is novel to the literature. Namely, we find that the uncertainty shock that increases spreads more leads to stronger evidence of a decline in industrial production and subsequent reaction of the federal funds rate. Our work, therefore, complements Caldara et al. (2016) in that it emphasizes the centrality of spread behavior and financial markets in understanding uncertainty shocks. The difference is that we use spreads to differentiate two second moment shocks. Our alternative identification assumption also adds to the discussion on identifying financial and macro uncertainty shocks (Carriero et al., 2017a; Ludvigson et al., 2015).

Our first example uses a small four-variable monetary VAR with a CAIW model of volatility and industrial production, the consumer price index, the federal funds rate, and the excess bond premium (EBP) of Gilchrist and Zakrajsek (2012). Financial and macro uncertainty shocks are both assumed to increase the volatilities of all innovations in the economy. The key distinguishing feature is that a financial uncertainty shock is assumed to increase the excess bond premium by more than the macro uncertainty shock. We assume a conditional Haar prior over the rotation matrix and conduct an extensive analysis of the prior implications for the impulse response functions (IRF) and an identified set analysis (Baumeister and Hamilton, 2015; Moon et al., 2017). We find more evidence that a financial uncertainty shock leads to a decline in industrial production relative to a macro uncertainty shock. We also find that a financial uncertainty shock increases the forecast error variance of industrial production and EBP more than a macro uncertainty shock does.

Our second example asks whether we can robustly identify the effects of financial and macro uncertainty shocks in the sense of Giacomini and Kitagawa (2015). Robust sign restrictions account for the implications of all possible priors over the rotation matrices, and hence are immune to the concerns of Baumeister and Hamilton (2015). In considering the robust implications of these two types of uncertainty shocks, we find it necessary to improve our estimation of financial and macro volatility. Therefore, we turn to the model of Carriero et al. (2017a), which is a state-of-the-art volatility factor model. Carriero et al. (2017a) estimate financial and macro volatility factors as the common movements of volatility across a set of 30 macro and financial data series, respectively. The model additionally allows for level shocks to impact volatility as well, thereby controlling for a potentially important source of volatility fluctuations. We impose a similar set of sign restrictions, with a key condition being that a financial uncertainty shock leads to a larger rise in the BAA-10 year Treasury spreads relative to a macro uncertainty shock. The results are largely consistent with those from the small VAR model. There is strong evidence suggesting that industrial production declines following a financial uncertainty shock, but less support for such a decline following a macro uncertainty shock. Moreover, we also find that a financial uncertainty shock leads to a decline in the federal funds rate, while we find little evidence that a macro uncertainty shock does.

The plan of the paper is as follows. In section 2, we present our general empirical framework, discuss the volatility identification problem, and lay out our sign restrictions strategy to identify uncertainty shocks. In section 3, we discuss various volatility processes that are amenable to our framework and compare our work with the extant literature. Section 4 contains a discussion of inference issues, including the identification problems introduced by sign restrictions. Section 5 contains our empirical illustration on the effects of financial and macro uncertainty shocks and Section 6 concludes.

2 Empirical Framework

2.1 Model

We analyze VAR models with stochastic volatility in which the volatility movements may impact the conditional mean.

We begin with a general framework that encompasses many popular models in the literature. Consider the following non-linear VAR for a vector of economic variables $y_t = [y_{1,t}, y_{2,t}, ..., y_{n,t}]'$,

$$y_t = \mu^y + \Phi^y y_{t-1} + B^y g^y(\Sigma_t) + \epsilon_t, \quad \epsilon_t | \mathcal{F}_{t-1}, \Sigma_t \sim_{i.i.d.} \mathcal{N}(0,\ \Sigma_t) \tag{1}$$

$$\Sigma_t = f(h_t; A) \tag{2}$$

$$h_t = \mu^h + \Phi^h h_{t-1} + B^h g^h(Y_{t-1}) + v_t, \quad v_t | \mathcal{F}_{t-1} \sim_{m.d.s.} \mathcal{V}(0, \Omega_t) \tag{3}$$

where Σ_t is a positive definite $n \times n$ matrix and h_t is a $k \times 1$ vector. The function $g^y(\cdot)$ takes an $n \times n$ matrix and returns an $l \times 1$ vector and $g^h(\cdot)$ takes an $n \times 1$ vector and returns an $m \times 1$ vector. The function $f(\cdot; A)$ with parameter matrix A takes an $k \times 1$ vector and returns an $n \times n$ matrix and its inverse mapping $f^{-1}(\Sigma_t, A) = h_t$ is assumed to be well-defined on the support of A. A primary example for this function in the literature is $f(h_t; A) = Adiag(h_t)A'$ where A is a $k \times k$ lower triangular matrix with ones on the diagonal. μ^y and μ^h are constant vectors with dimensions $n \times 1$ and $k \times 1$, respectively. The coefficient matrices Φ^y, B^y, Φ^h, B^h govern the dynamic relationships among the elements in $[y_t', h_t']'$. We can allow for lags on these coefficient matrices as well, which we do in our empirical application.

There are two types of innovation vectors, ϵ_t and v_t. The innovation ϵ_t is a vector of shocks on the y_t equation and is identically and independently distributed (iid) as the multivariate normal distribution with mean zero and variance-covariance matrix Σ_t conditional on the time $t-1$ information set $\mathcal{F}_{t-1} = \{y_{t-1}, \Sigma_{t-1}, y_{t-2}, \Sigma_{t-2}, ...\}$ and Σ_t. Another type of innovation v_t, on the h_t equation, is a vector of martingale difference sequences with respect to $\mathcal{F}_{t-1}$ and its conditional distribution $\mathcal{V}$ has mean zero and variance-covariance matrix Ω_t, whose elements are deterministic transformations of h_{t-1} and parameter ω, $\Omega_t = \Omega(h_{t-1}; \omega)$. We let the conditional distribution of v_t be slightly more general than the conditional distribution of ϵ_t to maintain the positive definiteness of Σ_t almost surely. For many volatility processes, the distribution of v_t is i.i.d. multivariate normal.

Of key interest in our investigation is the $n \times n$ matrix $\Sigma_t = Var(y_t|\mathcal{F}_{t-1}, \Sigma_t)$, which is the conditional variance-covariance of y_t conditional on the past information set $\mathcal{F}_{t-1}$ and Σ_t. This is the one-step-ahead forecast error variance-covariance of the economic variables (y_t) formed by economic agents at time t (e.g. in DSGE models). It is time-varying and intimately related with the uncertainty economic agents have over the observables y_t. Our empirical framework allows for two sources of volatility fluctuations: exogenous shocks to volatility v_t and endogenous responses of volatility to past movements of economic variables $B^h g^h(y_{t-1})$. In our paper, we are specifically focused on measuring the macroeconomic implications of the former.

Many of the papers in the literature, including Mumtaz and Zanetti (2013), Jo (2014), and Carriero et al. (2017a), fall into this general class of models. Creal and Wu (2017) is a consequential related paper that adopts an alternative timing assumption that allows time t volatility Σ_t to be determined after time t level shocks ϵ_t. We leave a discussion of the commonalities and differences between our setup and previous papers for the next section. A salient point, however, is that as long as some nonlinear transformation of Σ_t, $h_t = f^{-1}(\Sigma_t; A)$, admits a linear structure, our idea developed in this section applies.

There are several important practical issues that complicate the analysis. First, econometricians do not observe Σ_t directly. Therefore, we must use equations 1, 2, and 3 to infer volatility from the observable data $y_{1:T}$. The equation with y_t is as in a standard VAR except that 1) the conditional variance of its innovations are stochastic and time-varying and 2) the conditional variance enters the conditional mean equation.

Second, just as in the standard linear VAR for y_t without stochastic volatility and its in-mean effect, there is a shock labeling issue when we have more than one source of volatility fluctuations. Without further assumptions, elements in the vector v_t are correlated contemporaneously, which makes it difficult to interpret impulse response functions and variance decompositions in terms of v_t shocks. An increase in one of the elements in v_t (e.g., the volatility of the innovation to industrial production) is potentially due to different sources (e.g., either uncertainty originating in the real economy or the financial markets). The principal difference with most of the literature, however, is that this shock labeling issue is over second moment as opposed to first moment shocks.

2.2 Identification of uncertainty shocks

In this section, we discuss how we can distinguish various sources of fluctuations in the economic variables' forecast error variances and covariances (Σ_t) using a sign restrictions methodology. In the structural VAR language, we identify contemporaneously uncorrelated shocks that affect Σ_t and study how these isolated shocks impact y_t and Σ_t over time. Our key observation is that the linear VAR framework for h_t (equation 3) naturally allows us to use the identification strategies developed in the structural VAR literature to identify uncertainty shocks. It allows us to put identifying restrictions directly on the uncertainty shocks. In this manner, we can focus on identifying uncertainty shocks alone, which are the objects we are interested in. Our discussion in this section is based on fixed model parameters and volatility processes, $\psi = (\mu^y, \Phi^y, B^y, \mu^h, \Phi^h, B^h, \omega, h_{1:T})$.

Shock labeling problem. We first assume that the time t volatility innovation (v_t) is a linear function of uncorrelated unit-variance shocks (v_t^*), so we have

$$v_t = R_t v_t^* \quad \text{or} \quad R_t^{-1} v_t = v_t^*, \tag{4}$$

where R_t is a $k \times k$ invertible matrix with $R_t R_t' = \Omega_t$. The mean of v_t^* is zero and the conditional variance-covariance matrix is an identity matrix, $E(v_t^* v_t^{*\prime} | \mathcal{F}_{t-1}) = I_k$. We call v_t^* as "structural" uncertainty shocks. Unlike v_t, each element in v_t^* is uncorrelated with each other. Therefore, we can isolate the effect of one element in v_t^* from another, making the analysis of uncertainty shocks more intuitive and interpretable. Unlike first moment shocks, our uncertainty shocks v_t^* have a contemporaneous impact on both the stochastic covariance matrix (Σ_t) as well as the conditional mean of the observed variables (y_t).

Impulse response function. Our empirical framework allows uncertainty shocks to affect both Σ_t and y_t. We therefore consider first moment impulse response functions and second moment impulse response functions.

A *first moment impulse response function* gives the expected change in the conditional means of the observable variables from the jth structural uncertainty shock $v_t^* = e_j$ (e_j is a column vector of length k with a 1 in the jth element and zeros elsewhere),

$$IRF[y_{i,t+s} | v_t^* = e_j; R_t, \psi] = E(y_{i,t+s} | v_t^* = e_j; R_t, \psi) - E(y_{i,t+s} | v_t^* = 0_{k \times 1}; R_t, \psi) \qquad (5)$$

for $s \geq 0$. This captures the s-step-ahead response of the ith variable to the jth uncertainty shock. The expectation is taken with respect to the joint distribution of the future realization of shocks in the system conditional on v_t and $\mathcal{F}_{t-1}$, $[v_{t+1}, v_{t+2}, ..., v_{t+s}]'$ and $[\epsilon_t, \epsilon_{t+1}, ..., \epsilon_{t+s}]'$. Depending on the specification of the volatility process, Ω_t may depend on h_{t-1}, making the IRFs path dependent. We denote the time period in which the uncertainty shock is considered by the time subscripts on v_t^* and R_t.

A *second moment impulse response function* gives the expected change in the conditional variance covariance matrix (or potentially a convenient transformation of the variance covariance matrix) of the innovations to the observable variables (y_t) from the jth uncertainty shock $v_t^* = e_j$. First we define

$$IRF[\Sigma_{ii,t+s}|v_t^* = e_j; R_t, \psi] = E\left(\Sigma_{ii,t+s}|v_t^* = e_j; R_t, \psi\right) - E\left(\Sigma_{ii,t+s}|v_t^* = 0_{k\times 1}; R_t, \psi\right) \tag{6}$$

for $s \geq 0$. This captures the s-step-ahead impact of the jth uncertainty shock realized at time t on the economic agents' forecast error variance about variable $y_{i,t+s}$ formed at time $t+s$. This forecast error variance is at the time before they make a decision, but after they observe Σ_{t+s}.

A more interpretable second moment impulse response function is in terms of the forecast error variance as defined in Jurado et al. (2015):

$$IRF\left[FEV(y_{i,t+s})\middle|v_t^* = e_j; R_t, \psi\right] = Var\left(y_{i,t+s}\middle|v_t^* = e_j; R_t, \psi\right) - Var\left(y_{i,t+s}\middle|v_t^* = 0_{k\times 1}; R_t, \psi\right) \tag{7}$$

for $s \geq 0$. This is the s-step-ahead impact of the jth uncertainty shock realized at time t on the economic agents' s-step-ahead forecast error variance about variable y_{t+s} formed at time t.

It is important to note that these second moment response functions may be different from the impulse response functions for h_t:

$$IRF[h_{i,t+s}|v_t^* = e_j; R_t] = E\left(h_{i,t+s}|v_t^* = e_j; R_t, \psi\right) - E\left(h_{i,t+s}|v_t^* = 0_{k\times 1}; R_t, \psi\right) \tag{8}$$

for $s \geq 0$. In general, imposing restrictions on h_t leads to different implications compared to the case of imposing restrictions on Σ_t.

Putting sign restrictions to restrict the set of admissible Q. The relationship between volatility innovations v_t and structural uncertainty shocks v_t^* is not uniquely pinned down by equation 4. In fact, any R_t with $R_t = \Omega_t^{1/2} Q$ where $Q \in \mathcal{O}(k) = \{Q : QQ' = Q'Q = I_k, Q \text{ is a } k \times k \text{ matrix}\}$ and $\Omega_t^{1/2} = chol(\Omega_t)$ would satisfy the relationship. This implies that we have a range of impulse responses of variables of interest to uncertainty shocks even at a fixed model parameter ψ. Moreover, without further assumptions, we cannot differentiate the effect of one uncertainty shock from another: the range of possible impulse response functions is the same for all uncertainty shocks.

To overcome this problem, we look for restrictions that can be imposed on the signs of the first and second moment IRFs. These economic restrictions ideally would come from economic theory or outside empirical evidence. Imposing economic restrictions on the responses to the uncertainty shocks v_t^* involves conditions on the set of IRFs that we consider.

Intuitively, the idea of applying sign restrictions to identify uncertainty shocks involves choosing constraints on the signs of the first and second moment responses to the shock of interest. Restrictions on the signs of these level impulse response functions we label as *first moment restrictions* while those on the signs of volatility-related impulse response functions we call *second moment restrictions*.

The first moment restrictions can be written as

$$r^{y}_{i,j,s,t} \times IRF[y_{i,t+s}|v_t^* = e_j; R_t, \psi] \geq 0, \tag{9}$$

where $r_{i,j,s,t} = \{-1, 0, 1\}$. For example, if $r_{i,j,s,t} = -1$, then this restriction implies that the s-step-ahead response of the ith variable to the jth uncertainty shock realized at time t is restricted to be negative. If $r_{i,j,s,t} = 0$, then the sign restriction is not imposed on this response. Similarly, we can write second moment restrictions as

$$\begin{aligned} &r^{\Sigma}_{i,j,s,t} \times IRF[\Sigma_{ii,t+s}|v_t^* = e_j; R_t, \psi] \geq 0 \\ &r^{FEV}_{i,j,s,t} \times IRF[FEV(y_{i,t+s})|v_t^* = e_j; R_t, \psi] \geq 0 \\ &r^{h}_{i,j,s,t} \times IRF[h_{i,t+s}|v_t^* = e_j; R_t, \psi] \geq 0. \end{aligned} \tag{10}$$

We define a sign restriction set, $\mathcal{R}$, which collects non-zero $r^{y}_{i,j,s,t}$, $r^{\Sigma}_{i,j,s,t}$, $r^{FEV}_{i,j,s,t}$, and $r^{h}_{i,j,s,t}$ for all i, j, s, and t. This set contains all information about the imposed sign restrictions.

The first and second moment economic restrictions are conditions on the set of impulse response functions following the jth uncertainty shock. These conditions imply restrictions on the admissible set of decompositions $R_t = chol(\Omega_t)Q$. We formally define the admissible set for Q with respect to the sign restriction set $\mathcal{R}$ at parameter ψ:

$$\mathbb{Q}(\psi, \mathcal{R}) = \{Q : Q \in \mathcal{O}(k) \text{ and IRFs with } Q \text{ and } \psi \text{ satisfy all restrictions in } \mathcal{R}\}. \tag{11}$$

This admissible set leads to an identified set of impulse response functions at parameter ψ:

$$\mathbb{IS}(\psi, \mathcal{R}) = \{IRF[.|v_t^* = e_j; R_t = chol(\Omega_t)Q, \psi] : Q \in \mathbb{Q}(\psi, \mathcal{R})\} \tag{12}$$

The restrictions reduce the set of possible impulse response functions because $\mathbb{Q}(\psi, \mathcal{R}) \subseteq \mathcal{O}(k)$. This correspondingly shrinks the identified set of IRFs. That is, by putting enough restrictions, we can narrow down this set to draw a meaningful conclusion about the differential effects of various uncertainty shocks on economic variables.

In section 4, we present how we can make inference over the identified sets. In section B of the appendix, we provide a simulation-based method to compute these IRFs.

2.3 Simple example

Before we close off this section, we present a simplified bivariate example using the popular volatility process discussed by Cogley and Sargent (2005) to guide our discussion.

$$\begin{aligned}
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} &= \begin{bmatrix} B^y_{11} & B^y_{12} \\ B^y_{21} & B^y_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{11,t} \\ \Sigma_{22,t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}, \quad \epsilon_t | \mathcal{F}_{t-1}, \Sigma_t \sim N\left(0, \Sigma_t\right) \\
\Sigma_t &= \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix} \begin{bmatrix} e^{h_{1,t}} & 0 \\ 0 & e^{h_{2,t}} \end{bmatrix} \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} h_{1,t} \\ h_{2,t} \end{bmatrix} &= \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}, \quad v_t | \mathcal{F}_{t-1} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \right)
\end{aligned} \tag{13}$$

Mapping equation 13 into the general framework, we shut down the autoregressive component of y_t ($\Phi^y = 0$) and h_t ($\Phi^h = 0$), and the component that allows level shocks to impact volatility ($B^h = 0$). $\Omega_t = \Omega(\omega)$ where $\omega = [\Omega_{11}, \Omega_{12}, \Omega_{21}, \Omega_{22}]'$ and h_{t-1} does not enter in Ω_t and therefore it is constant over time. We choose $g^y(\Sigma_t) = diag\left(\Sigma_t\right)$, and

$$f(h_t; A) = \begin{bmatrix} e^{h_{1,t}} & Ae^{h_{1,t}} \\ Ae^{h_{1,t}} & A^2 e^{h_{1,t}} + e^{h_{2,t}} \end{bmatrix} \quad \text{and} \quad f^{-1}(\Sigma_t; A) = \begin{bmatrix} \log(\Sigma_{11,t}) \\ \log(\Sigma_{22,t} - A^2 \Sigma_{11,t}) \end{bmatrix}. \tag{14}$$

Finally, we collect all model parameters and volatility series in $\psi = \{B^y, A, \omega, h_{1:T}\}$.

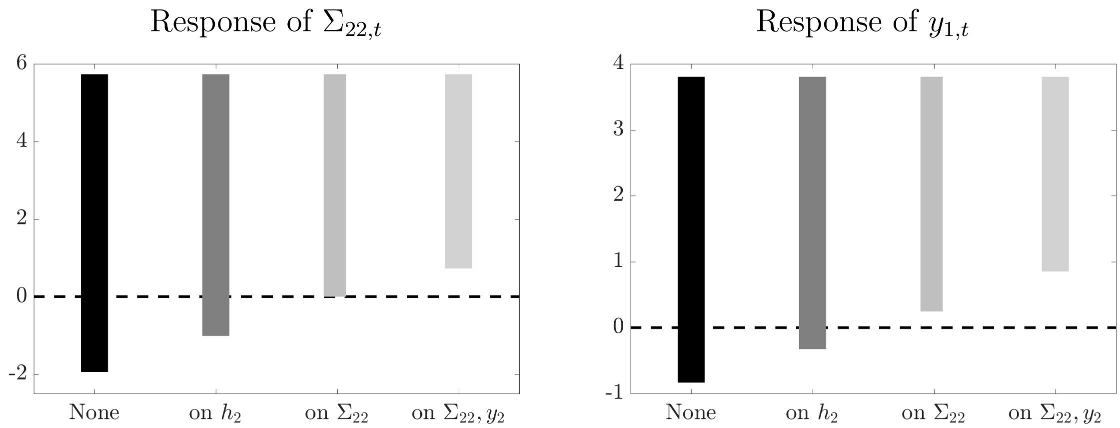
Here, the identification problem can clearly be seen from the volatility equation. Insofar as $\Omega_{12} \neq 0$, the innovations to $h_{1,t}$ and $h_{2,t}$ are correlated and there is a question of how we can decompose these innovations into structural uncertainty shocks.

$$\begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} v^*_{1,t} \\ v^*_{2,t} \end{bmatrix}, \quad \begin{bmatrix} v^*_{1,t} \\ v^*_{2,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \tag{15}$$

We decompose the reduced-form volatility innovations v_t into structural uncertainty shocks v^*_t by equation 15.

Sign restrictions on $v^*_{1,t}$ or $v^*_{2,t}$ then impose directional responses on these first and second

Figure 1 **Identified sets for contemporaneous response of uncertainty shock ($v^*_{1,t}$)**



This figure shows the identified sets for the contemporaneous responses of uncertainty shock $v^*_{1,t}$ on $\Sigma_{22,t}$ (left) and $y_{1,t}$ (right) with various restrictions on the impulse responses. There are four cases. The first case imposes no restriction, the second case assumes that response of $v^*_{1,t}$ on $h_{2,t}$ is positive, the third case assumes that response of $v^*_{1,t}$ on $\Sigma_{22,t}$ is positive, and the fourth case assumes the responses of $v^*_{1,t}$ on $\Sigma_{22,t}$ is positive and $y_{2,t}$ is larger than 1. See Section 2.3 for more details.

moment impulse response functions. They restrict the admissible values of (R_{11}, R_{21}) or (R_{12}, R_{22}). To illustrate this, we assume $B = [1, 1/2; 1/2, 1]$, $\Omega = [1, 1; 1, 2]$, $A = \sqrt{2}$, and compute the range of possible responses of $\Sigma_{22,t}$ and $y_{1,t}$ to the first uncertainty shocks $v^*_{1,t}$ with $E[v^*_t] = 0$, $E[v^*_t v^*_t] = I_2$, and various sign restrictions on v^*_t.

The first vertical bar on the left panel in figure 1 presents all possible responses of $\Sigma_{22,t}$ to $v^*_{1,t}$. The second vertical bar is the responses with an additional restriction that the response of $h_{2,t}$ to $v^*_{1,t}$ is positive. The third vertical bar is the response with a different restriction that the response of $\Sigma_{22,t}$ to $v^*_{1,t}$ is positive. The fourth vertical bar introduces a first moment restriction – that the response of $y_{2,t}$ to $v^*_{1,t}$ is larger than 1 – on top of the restriction in the third bar. The right panel in the figure presents the corresponding set of possible impulse responses of $y_{1,t}$.

We can draw several conclusions from this figure. First, both first moment restrictions (case 4) second moment restrictions (cases 2 and 3) reduce the set of possible IRFs. Second, focusing on the second moment restrictions, conditions on h_t and Σ_t can have differential impacts on other variables (case 2 versus 3). For example, restricting positive response of

$\Sigma_{22,t}$ does not necessarily imply a positive response on $h_{2,t}$. This in turn influences the first moment impulse response functions. A positive response of $\Sigma_{22,t}$ to $v^*_{1,t}$ implies a positive response on $y_{1,t}$. However, just restricting the response of $h_{2,t}$ to be positive may not lead to a positive response of $y_{1,t}$ to the same shock.

3 Discussion

The class of models that we consider in the paper encompasses some popular volatility processes in the literature. Hence, we believe that our proposal in using sign restrictions to identify second moment shocks has quite general applications. In this section, we first give examples of how several different volatility processes could fit into our framework. Then, we compare the sign restrictions identification strategy to others used in the literature.

3.1 Volatility processes

We discuss three popular volatility processes that are amenable to our framework laid out in equations 2 and 3.

Conditional Autoregressive Inverse-Wishart (CAIW) volatility. First, we begin with the inverse Wishart autoregressive volatility process for an $n \times n$ matrix Σ_t.

$$\Sigma_t | \Sigma_{t-1} \sim IW\left((\nu - n - 1)\left(C + \Phi \Sigma_{t-1} \Phi'\right), \nu\right) \tag{16}$$

The time t variance covariance matrix, Σ_t, is assumed to be distributed according to an inverse Wishart distribution conditional upon Σ_{t-1}. The $n \times n$ matrix C is assumed positive definite and influences the long-run mean of the volatility process, the $n \times n$ matrix Φ governs the intertemporal relationship between Σ_t and Σ_{t-1}, and the scalar ν is a degrees of freedom parameter that governs the conditional variability of the time t volatility innovation.

Note that the process can be written in a linear form that relates time t volatility to time $t-1$ volatility and an additive martingale difference sequence:

$$h_t = \overline{C} + \overline{\Phi} h_{t-1} + v_t, \quad E[v_t|\mathcal{F}_{t-1}] = 0 \quad \text{and} \quad E[v_t v_s'|\mathcal{F}_{t-1}] = 0, \quad \forall s \neq t \tag{17}$$

where $h_t = vech(\Sigma_t)$, $\overline{C} = vech(C)$, $\overline{\Phi} = L_n(\Phi \otimes \Phi)D_n$, $vec(x) = D_n vech(x)$, and $vech(x) = L_n vec(x)$. The variance covariance matrix of v_t (Ω_t) is a deterministic function of Σ_{t-1}. Further details regarding this decomposition can be found in section C of the appendix.

A bivariate version of the model leads the following mapping to the general model: $\mu^h = \overline{C}$, $\Phi^h = \overline{\Phi}$, and $B^h = 0$, and

$$\Sigma_t = f(h_t; A) \equiv \begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} \quad \text{and} \quad h_t = f^{-1}(\Sigma_t; A) \equiv vech(\Sigma_t) \tag{18}$$

where A is empty. The choice of $g^y(\cdot)$ is left to the user. In our empirical exercise, we set $g^y(\Sigma_t) = \log(diag(\Sigma_t))$.

The CAIW process has several attractive properties. First, the estimates of Σ_t are invariant to the order of the variables y, as has been noted in Gourieroux et al. (2009). Therefore, simply reshuffling the observables in the VAR has no implications for the volatility estimates. Second, the CAIW model is linear in Σ_t, which makes the imposition of sign restrictions on the one-step ahead forecast error variances straightforward.

Cholesky volatility. Possibly the most popular volatility process for small-scale VAR models with time-varying volatility comes from the work of Cogley and Sargent (2005) and Primiceri (2005). This volatility specification first decomposes the variance covariance matrix into a Cholesky form, and then models the time variation of the diagonal and off-diagonal elements. In the previous section, we have already given a simple example of such a volatility process and how it fits into our general framework. Here we highlight for the reader an important distinction between our specification of the volatility process and the process traditionally used in the literature.

Take a bivariate y_t example to be concrete:

$$\begin{bmatrix} \Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t} & \Sigma_{22,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ h_{3,t} & 1 \end{bmatrix} \begin{bmatrix} e^{h_{1,t}} & 0 \\ 0 & e^{h_{2,t}} \end{bmatrix} \begin{bmatrix} 1 & h_{3,t} \\ 0 & 1 \end{bmatrix} \tag{19}$$

$$h_t = \mu^h + \Phi^h h_{t-1} + v_t, \quad v_t \sim N(0, \Omega)$$

where $h_t = [h_{1,t}, h_{2,t}, h_{3,t}]'$. The relationship between one-step-ahead uncertainty and h_t is given by the following transformation

$$\Sigma_t = f(h_t; A) \equiv \begin{bmatrix} e^{h_{1,t}} & h_{3,t}e^{h_{1,t}} \\ h_{3,t}e^{h_{1,t}} & h_{3,t}^2 e^{h_{1,t}} + e^{h_{2,t}} \end{bmatrix} \quad \text{and} \quad h_t = f^{-1}(\Sigma_t; A) \equiv \begin{bmatrix} \log(\Sigma_{11,t}) \\ \log(\Sigma_{22,t} - \Sigma_{12,t}^2/\Sigma_{11,t}) \\ \Sigma_{12,t}/\Sigma_{11,t} \end{bmatrix}, \tag{20}$$

where A is empty and $B^h = 0$.

Most of the extant literature has specified independent processes for the three second moment shocks (i.e., Φ^h and Ω are restricted to be a diagonal matrix). Independent volatility processes are attractive when viewed through the lens of a Cholesky decomposition of the level VAR economy. The variance processes are the time-varying volatilities of the structural shocks ($e^{h_{i,t}}$), whereas the off diagonal elements are smooth changes in the level structure of the economy ($h_{3,t}$).

For our purposes, however, it is important to allow for general second moment dynamics (i.e., off-diagonal elements in Φ^h and Ω are not restricted to be zero). First, a VAR specification in the volatilities allows for more complex relationships between the different elements in the variance covariance matrix. This may be important to accurately measure volatility. Furthermore, in a more fundamental sense, the volatility identification problem becomes less interesting with independent innovations. Indeed, with a diagonal Ω matrix, the v_t innovations themselves are already independent, so there is no need to orthogonalize them.

Factor volatility. A major concern with which the literature has had to contend is on the accurate measurement of volatility. To this end, exploiting the common movements in volatilities across many data series could improve inference. This insight has pushed Creal and Wu (2017) and Carriero et al. (2017a) to develop models that admit factor structures in volatility. The main advantage of the factor volatility setup is its scalability to much larger dimensions. The factors are then specified to enter into the conditional mean.

As a simple example to illustrate the power of the framework, we consider 3 observable variables y_t.

$$\Sigma_t = \begin{bmatrix} 1 & 0 & 0 \\ A_1 & 1 & 0 \\ A_2 & A_3 & 1 \end{bmatrix} \begin{bmatrix} A_4 e^{h_{1,t}} & 0 & 0 \\ 0 & A_5 e^{h_{2,t}} & 0 \\ 0 & 0 & A_6 e^{\lambda h_{2,t}} \end{bmatrix} \begin{bmatrix} 1 & A_1 & A_2 \\ 0 & 1 & A_3 \\ 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

where Σ_t are then posited to be driven by factors $h_{1,t}$ and $h_{2,t}$ and λ gives the factor loading parameter. The factor dynamics are usually modeled as a VAR.

$$\begin{bmatrix} h_{1,t} \\ h_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi^h_{11} & \Phi^h_{12} \\ \Phi^h_{21} & \Phi^h_{22} \end{bmatrix} \begin{bmatrix} h_{1,t-1} \\ h_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}, \quad \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \right). \tag{22}$$

The relationship between 1-step-ahead uncertainty and h_t is given by the following transformation

$$\Sigma_t = f(h_t; A) \equiv \begin{bmatrix} A_4 e^{h_{1,t}} & A_1 A_4 e^{h_{1,t}} & A_2 A_4 e^{h_{1,t}} \\ A_1 A_4 e^{h_{1,t}} & A_1^2 A_4 e^{h_{1,t}} + A_5 e^{h_{2,t}} & A_1 A_2 A_4 e^{h_{1,t}} + A_3 A_5 e^{h_{2,t}} \\ A_2 A_4 e^{h_{1,t}} & A_1 A_2 A_4 e^{h_{1,t}} + A_3 A_5 e^{h_{2,t}} & A_2^2 A_4 e^{h_{1,t}} + A_3^2 A_5 e^{h_{2,t}} + A_6 e^{\lambda h_{2,t}} \end{bmatrix}, \tag{23}$$

where $A = [A_1, A_2, A_3, A_4, A_5, A_6]'$ with $A_4 > 0$, $A_5 > 0$, $A_6 > 0$, and $B^h = 0$. Unless the researcher considers a one-factor model, the shock labeling problem still appears in the factor volatility equation through Ω, as both structural uncertainty shocks move all factors simultaneously.

Level shocks impacting volatility. For purposes of clarity, our previous discussions of volatility processes neglected the discussion of any feedback from level shocks to the second moments. As our general empirical framework makes clear, however, the volatility sign restrictions identification strategy can accommodate this effect through the term $B^h g^h(y_{t-1})$ in equation 3. A bivariate example in equation 13 allowing lags of level shocks to impact volatility is

$$\begin{aligned}
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} &= \begin{bmatrix} B^y_{11} & B^y_{12} \\ B^y_{21} & B^y_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{11,t} \\ \Sigma_{22,t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}, \quad \epsilon_t | \mathcal{F}_{t-1}, \Sigma_t \sim N(0, \Sigma_t) \\
\Sigma_t &= \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix} \begin{bmatrix} e^{h_{1,t}} & 0 \\ 0 & e^{h_{2,t}} \end{bmatrix} \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} h_{1,t} \\ h_{2,t} \end{bmatrix} &= \begin{bmatrix} B^h_{11} & B^h_{12} \\ B^h_{21} & B^h_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}, \quad v_t | \mathcal{F}_{t-1} \sim N(0, \Omega)
\end{aligned} \tag{24}$$

We can continue to apply our approach to decompose v_t into uncorrelated unit variance shocks.

An important point to be made, which is especially relevant when allowing for feedback effects, is the timing of the realization of v_t, or the time t volatility innovation, and ϵ_t, or the time t level innovation. Implicitly, we are supposing that the time t volatility is determined before the time t level shock. Therefore, current volatility innovations v_t can impact the current level variables, but current level shocks ϵ_t can only impact the volatilities with a lag. This timing assumption is in accordance with standard dynamic stochastic general equilibrium models (Fernandez-Villaverde et al., 2011, 2015; Basu and Bundick, 2015; Born and Pfeifer, 2014) and several papers in the volatility-in-mean literature (Mumtaz and

Zanetti, 2013; Jo, 2014; Carriero et al., 2017a), but it is not the only timing assumption possible. Namely, with additional identification assumptions, it is possible to also allow for the time t level innovations ϵ_t to impact current volatility. This important point is nicely made in Creal and Wu (2017), and their framework allows for this possibility. Additionally, a recent paper by Carriero et al. (2017b) use heteroskedasticity-based identification techniques to identify the effects of volatility and level shocks.

3.2 Other identification strategies for uncertainty shocks

We now discuss how the sign restrictions identification strategy fits in with the literature on identifying the real effects of uncertainty shocks using a VAR model with volatility-in-mean. Most of the extant literature uses zero restrictions on the impact matrix, either on the first or second moment shocks. We discuss both types of identification strategies and how our work differs. A common thread that differentiates our work is the use of sign restrictions.

Zero restrictions on the level equation. Mumtaz and Zanetti (2013) and Jo (2014), two significant early papers in the literature, use the Cogley and Sargent (2005)/Primiceri (2005) decomposition of volatility with exogenous, independent second moment volatility processes. The authors rely on the Cholesky structure of the economy to identify a level structural innovation, and then interpret the volatility fluctuations in that level structural innovation as coming from the corresponding structural uncertainty shock.

In contrast, we view the level VAR equation more as a way to "cleanse" the volatilities of predictable conditional mean movements, which allows us to better measure the conditional volatility of the innovations, as emphasized by Jurado et al. (2015). We then propose a flexible volatility model, and focus our identification efforts directly on the second moment shocks. The disadvantage of this strategy is that we cannot tightly link a level structural shock to our uncertainty structural shock. The upside, however, is that our identification strategy can be applied to a wider class of VAR models with volatility-in-mean (such as a factor volatility model) without identifying level structural shocks.

Zero restrictions on the volatility equation. There are papers that, like us, put identifying restrictions on the second moments to identify uncertainty shocks. The current papers in the literature use recursive ordering assumptions to do so.

Creal and Wu (2017) is the first paper in the VAR with volatility-in-mean literature to propose a factor volatility model in an internally consistent way. Their model has bond yield factors, macro variables, and time-varying volatility factors that can be written in a VAR system. The paper assumes a recursive ordering to identify the VAR. Their model allows volatility to be ordered after level variables contemporaneously, which is a potentially empirically relevant specification that our framework rules out. In this paper, as our main focus is on the identification of different types of volatility shocks, we restrict ourselves to models where volatility is ordered first, with the acknowledgment that alternative orderings are possible.

Carriero et al. (2017a) is the first paper to use the VAR with volatility-in-mean to extract common volatility factors from a large set of macro and financial data in the spirit of Jurado et al. (2015). The Carriero et al. (2017a) paper faces a particularly related identification problem in that they, like us, are focused only on identifying second moment shocks. They pursue a Cholesky identification assumption in the volatilities whereas we use sign restrictions. Our use of sign restrictions allows us to consider more than one structure of the economy. This comes at a cost of potentially weaker conclusions and more involved inference. We see value in both identification strategies and view our approaches as complementary.

4 Inference

In this section, we take parameter uncertainty stemming from the reduced-form parameters $\psi = \{\mu^y, \Phi^y, B^y, \mu^h, \Phi^h, A, \omega, h_{1:T}\}$ and the rotation matrix Q into account and discuss how to make inference about the impulse response functions conditional on observed data. To do so, we write the impulse response functions as a function of v_t^*, Q, and ψ,

$$IRF[y_{i,t+s}; v_t^* = e_j, Q, \psi] = E[y_{i,t+s}|v_t^* = e_j; R_t, \psi] - E[y_{i,t+s}|v_t^* = 0_{k\times 1}; R_t, \psi] \tag{25}$$

where $R_t = chol(\Omega(h_{t-1}; \omega))Q$ and $Q \in \mathcal{O}(k)$.

In this paper, we take a Bayesian approach, and therefore we begin our econometric analysis by placing prior distributions over the unknown objects: ψ and Q. More specifically, we consider a class of joint prior distributions for ψ and Q of the following form:

$$p(\psi, Q) = \frac{1\{\psi \in \mathbb{P}(\mathcal{R})\}p(\psi)}{\int 1\{\psi \in \mathbb{P}(\mathcal{R})\}p(\psi)d\psi} p(Q|\psi). \tag{26}$$

The initial proper prior distribution for ψ, $p(\psi)$, is truncated to the region $\mathbb{P}(\mathcal{R})$, which is the set of ψ in the support of $p(\psi)$ with non-empty $\mathbb{Q}(\psi, \mathcal{R})$. That is, we put positive prior probability only on the set of ψ's that induce at least one restriction-consistent IRF. The second term, $p(Q|\psi)$, is a proper prior density of Q defined at every value of $\psi \in \mathbb{P}(\mathcal{R})$, and its support is a subset of $\mathbb{Q}(\psi, \mathcal{R})$. The joint posterior density of these two unknowns is proportional to the product of the likelihood function $p(Y|\psi)$ and the joint prior density

$$P(\psi, Q|Y) \propto p(Y|\psi)1\{\psi \in \mathbb{P}(\mathcal{R})\}p(\psi)p(Q|\psi), \tag{27}$$

from which we can construct a posterior distribution of $IRF[y_{i,t+s}; v_t^* = e_j, Q, \psi]$.

Because Q does not enter the likelihood function, the likelihood function is flat conditional on ψ over all possible IRFs that satisfy the sign restrictions, leaving both Q and the IRFs set-identified. It is worth mentioning that once we place a proper prior distribution on Q conditional on ψ, the posterior distributions of Q as well as the IRFs are proper, from which we can derive point and interval estimates. Note that the joint distribution of Q and ψ are not independent, and therefore the data are informative about Q (e.g., Poirier, 1998).

A further important point is that the prior for Q injects information into the posterior distributions of the objects of interest, including IRFs. Given that the IRFs depend both on ψ and Q in a nonlinear fashion in our case, it is not always immediate to understand how much prior information we are injecting into the IRFs. With this complication in mind, we take two approaches in our empirical exercises. The first approach (fully Bayesian approach) imposes a conditionally flat prior on Q conditional on ψ (uniform Haar prior) popularized by Uhlig (2005). In our second approach (robust Bayesian approach), we work with multiple priors for Q rather than a single prior. In the next paragraphs, we explain them briefly. Computational details to approximate these quantities can be found in section A of the appendix.

Fully Bayesian approach. Following Uhlig (2005), we impose a flat prior (Haar prior) on the admissible set of Q and we define our prior density function for Q as

$$p(Q|\psi) = \frac{1}{vol(\mathbb{Q}(\psi, \mathcal{R}))} 1\{Q \in \mathbb{Q}(\psi, \mathcal{R})\}, \quad \text{where } vol(\mathbb{Q}(\psi, \mathcal{R})) = \int_{Q \in \mathcal{O}(k)} 1\{Q \in \mathbb{Q}(\psi, \mathcal{R})\}\, dQ. \tag{28}$$

This prior implies that conditional on ψ, we believe that any equal-sized subsets in $\mathbb{Q}(\psi, \mathcal{R})$ have an equal chance. Equipped with a marginal prior distribution over ψ, the posterior distribution given in equation 27 is proper, and so is the posterior distribution of the IRFs. We report a measure of central tendency using the pointwise median and pointwise equal-tailed $\alpha\%$ credible regions for the IRFs in the empirical section.

As Baumeister and Hamilton (2015) discussed, a flat prior on $\mathbb{Q}(\psi, \mathcal{R})$ may introduce unwanted information on the impulse response functions. We are very much sympathetic to their point, and agree that *"any prior beliefs should be acknowledged and defended openly and their role in influencing posterior conclusions clearly identified."* Hence, we carefully analyze whether the prior puts such unwanted information or not in our application and present them in the appendix.

We investigate the prior distribution in two dimensions. First, we investigate the implication of the joint prior distribution of ψ and Q with the sign restrictions. The sign restrictions impose a restriction on the prior support of ψ as well as Q $(\mathbb{P}(\mathcal{R}) \times \cup_{\psi \in \mathbb{P}(\mathcal{R})} \mathbb{Q}(\psi, Q))$. As we start with some parametric prior distribution for ψ and truncate its support according to the sign restrictions, it is important to understand how this truncation affects the marginal prior distribution of ψ and the IRFs, both in terms of magnitude and sign.

Another important aspect of our prior specification is the conditional prior distribution of Q on $\mathbb{Q}(\psi, \mathcal{R})$. Although the data are informative about Q marginally via the sign restrictions and ψ, the data do not tell us about the IRFs conditional on ψ. In addition, a flat prior on Q does not imply a flat prior on the IRFs because the IRFs are some function of Q. To understand the role of the flat prior on Q, we evaluate the amount of information injected by the conditional prior by comparing the identified sets to the credible sets assuming a flat prior at the posterior mean values of ψ (Moon et al., 2017; Baumeister and Hamilton, 2015).

Robust Bayesian approach. To avoid unwanted influences from the prior distribution of Q, we also compute and present IRFs based on the robust Bayesian approach of Giacomini and Kitagawa (2015). We start with all proper prior distributions for Q on $\mathbb{Q}(\psi, \mathcal{R})$ conditional on ψ whose marginal prior density is fixed at a single density as in the first approach. Then, for each joint prior distribution of (ψ, Q), we have a corresponding posterior distribution of (ψ, Q) by applying Bayes rule (equation 27), which leads to a set of posterior distributions for IRFs.

Equipped with this set of posterior distributions, we first report the posterior mean bounds, a range of the maximum and minimum mean responses that are ever possible with some prior Q on $\mathbb{Q}(\psi, \mathcal{R})$. Second, we report the α%-robustified credible set, which is the highest posterior density interval response that covers a corresponding impulse response at least α% with any posterior distribution that is updated from some prior Q on $\mathbb{Q}(\psi, \mathcal{R})$. These two objects are robust to the choice of prior for Q in the sense that it takes all possible prior distributions into account. That is, no matter what prior one has on the set $\mathbb{Q}(\psi, \mathcal{R})$, the posterior mean of the IRFs will lie within the posterior mean bounds and the posterior distribution will have over $\alpha\%$ of its mass within the robust credible set. This set is universal to anyone with the same posterior of the reduced-form parameters and assumed sign restrictions, regardless of their prior choice for Q on $Q(\psi, R)$.

5 Empirical Application: Financial and macro uncertainty shocks

Motivation. Our empirical application investigates the differential impact of financial and macro uncertainty shocks via two examples. In the first example, we consider a flexible volatility model with a small-scale VAR. We use sign restrictions combined with the conditional Haar prior on the rotation matrix to identify the two uncertainty shocks. The appendix contains an analysis of the potentially unwanted prior information on our impulse response functions. The second example asks whether we can robustly identify the effects of these two uncertainty shocks. We employ the methodologies discussed in Giacomini and Kitagawa (2015), Moon et al. (2017), and Amir-Ahmadi and Drautzburg (2017) to conduct inference on the IRFs that are robust to the choice of the prior distribution for the rotation matrix. We leverage as much as data as possible to measure volatility by using the model of Carriero et al. (2017a), which is a state-of-the-art factor volatility model that uses the common movements in the volatilities of many financial and macro data to extract financial and macro volatility.

5.1 Example 1: Small-scale VAR

We first consider the identification problem of financial and macro uncertainty shocks in a small-scale monetary VAR. Our model contains four variables: industrial production, CPI, the federal funds rate, and the excess bond premium (EBP). We use monthly data on log industrial production in the manufacturing sector, log consumer price index, the federal funds rate, and the excess bond premium from $1973M1 - 2012M12$. We obtained the macroeconomic data from the Federal Reserve Bank of St. Louis FRED and the excess bond premium data from Simon Gilchrist's website.

The small number of variables in the model allows us to specify a flexible volatility process. We choose the conditional autoregressive inverse Wishart model of volatility. We estimate a model with 12 lags in the VAR, a contemporaneous volatility-in-mean effect, and 1 lag in the volatilities. Although the volatility model has many second moment shocks, we are specifically interested in identifying two of them, which we call a financial and macro uncertainty shock. We impose the following sign restrictions to identify the two uncertainty shocks (without loss of generality, call the financial uncertainty shock the first shock and the macro uncertainty shock the second shock):

Assumption $\boldsymbol{\mathcal{A}_{uf}}$ (Financial uncertainty shock). The uncertainty shock satisfies $\mathcal{A}_{uf,1}$, $\mathcal{A}_{uf,2}$ and $\mathcal{A}_{ufm}$:

$$\mathcal{A}_{uf,1} : IRF\left[\Sigma_{ii,t+s} \middle| v_t^* = e_1; R_t\right] > 0 \text{ for } s = 0 \text{ and } i = 1, 2, 3, 4.$$

$$\mathcal{A}_{uf,2} : IRF\left[EBP_{t+s} \middle| v_t^* = e_1; R_t\right] > 0 \text{ for } s = 0.$$

$$\mathcal{A}_{ufm} : IRF\left[EBP_{t+s} \middle| v_t^* = e_1; R_t\right] > IRF\left[EBP_{t+s} \middle| v_t^* = e_2; R_t\right] \text{ for } s = 0.$$

Assumption $\boldsymbol{\mathcal{A}_{um}}$ (Macro uncertainty shock). The uncertainty shock satisfies $\mathcal{A}_{um,1}$ and $\mathcal{A}_{ufm}$:

$$\mathcal{A}_{um,1} : IRF\left[\Sigma_{ii,t+s} \middle| v_t^* = e_2; R_t\right] > 0 \text{ for } s = 0 \text{ and } i = 1, 2, 3, 4.$$

$$\mathcal{A}_{ufm} : IRF\left[EBP_{t+s} \middle| v_t^* = e_1; R_t\right] > IRF\left[EBP_{t+s} \middle| v_t^* = e_2; R_t\right] \text{ for } s = 0.$$

Assumption $\boldsymbol{\mathcal{A}_{o}}$ (Other second moment shocks). To differentiate financial and macro uncertainty shocks from other second moment shocks, we impose $\mathcal{A}_o$:

$$\mathcal{A}_o : \text{There is at least one } i \text{ that } IRF\left[\Sigma_{ii,t+s} \middle| v^* = e_j; R_t\right] \leq 0 \text{ for each } j \neq 1, 2 \text{ and } s = 0.$$

We search for orthogonalizations R_t that contain exactly one financial uncertainty shock and one macro uncertainty shock. The IRF definitions are in section 2.2, and its computation is in section B of the appendix.

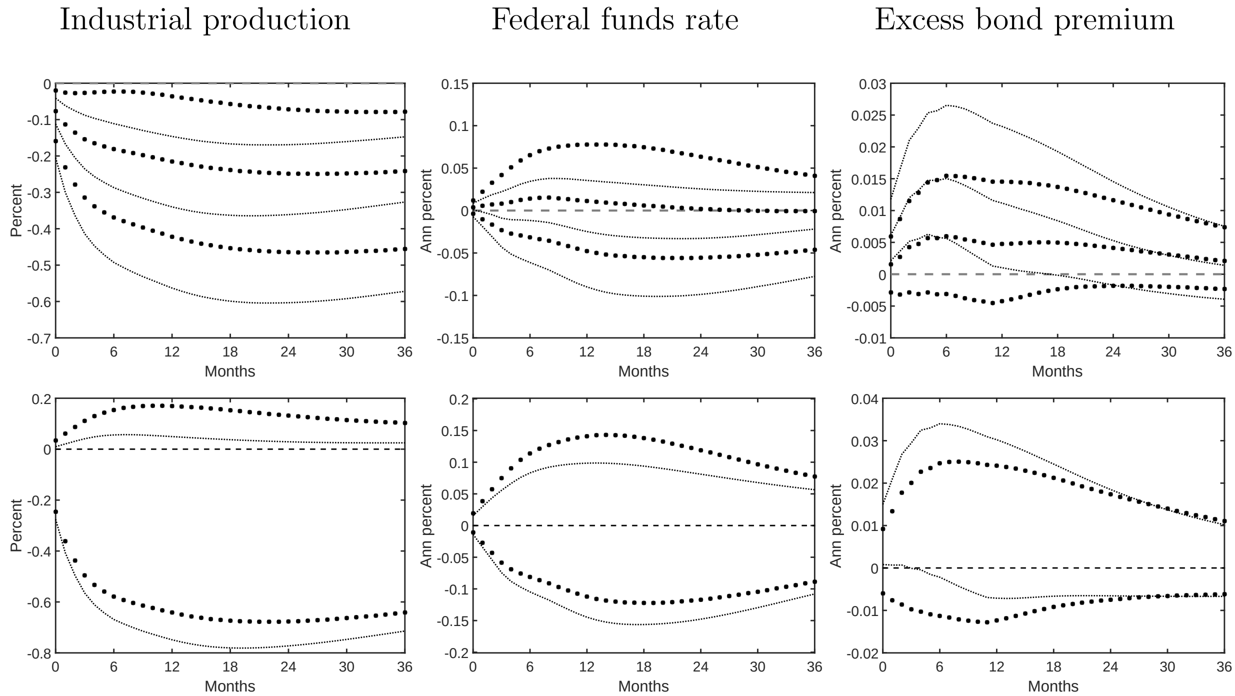
In words, the restrictions impose that both uncertainty shocks increase aggregate uncertainty, so the variances of the innovations on all four data series increase. The key condition that separates the two types of uncertainty shocks is the response of the excess bond premium. A positive financial uncertainty shock must increase the excess bond premium. In addition, a positive financial uncertainty shock is assumed to increase the EBP more than a positive macro uncertainty shock. If one believes that financial uncertainty is a key driver of EBP fluctuations among aggregate second moment shocks, then this assumption naturally follows. Importantly, a macro uncertainty shock's effect on the sign of the EBP response is left unrestricted. The appendix (section D) contains a discussion that provides some model-based and empirical evidence in support of this sign restriction.

Finally, Assumption $\mathcal{A}_o$ is important to our identification strategy. As we are imposing a relative sign restriction between the financial and macro uncertainty shocks' effects on the EBP, it is important for us to specify that the assumptions other than $\mathcal{A}_{ufm}$ narrow down the set to two possible candidate shocks. One implication is that another second moment shock that does not increase aggregate uncertainty may lead to an increase in the EBP by even more than the financial uncertainty shock.

Posterior results. We take $4,000,000$ draws total from the posterior distribution across 20 chains of $200,000$ draws each. The specific prior distribution choice and posterior sampler for the CAIW-in-VAR model is in section C in appendix. These are constructed using $1,000$ parameter draws that are evenly spaced across the final $180,000$ draws for each chain from the posterior our model. We take $1,000$ Q draws per parameter draw. In computing the IRFs, we integrate out the path dependency using the invariant distribution of Σ_t (See section B.2 in the appendix for more details).

The top row of figure 2 shows the effects of financial (broken line) and macro (dark dots) uncertainty shocks on the macroeconomy. At the posterior median impulse response function,

Figure 2 **Financial uncertainty shock (broken line) and macro uncertainty shock (dark dots) on level variables (Haar - top row and Prior robust - bottom row)**



This figure shows impulse response function estimates on the level variables to a 1 standard deviation financial uncertainty shock (broken line) and a 1 standard deviation macro uncertainty shock (dark dots) for the small VAR model. The reduced-form parameters are drawn from their posterior distributions. In the top row, we present pointwise 70% credible sets and pointwise posterior median estimates assuming a Haar prior over the rotation matrices. In the bottom row, we present the posterior mean bounds (an estimator of the identified set) assuming the prior robust framework of Giacomini and Kitagawa (2015). We only keep the impulse response functions that satisfy Assumptions $\mathcal{A}_{uf}$, $\mathcal{A}_{um}$, and $\mathcal{A}_{o}$. To conserve space, we have suppressed the response of the CPI. Those responses can be found in the appendix (section C).

a financial uncertainty shock leads to a decline in industrial production of around -0.12% on impact. The effects peak almost two years after the initial shock, with a maximal decline of -0.36% at the posterior median (70% credible set between -0.60% and -0.17%). The effect on industrial production is significant and long-lived. By 3 years out, the 70% credible set of the IRF is between -0.57% and -0.15%. There is mild evidence of a downward shift in the federal funds rate in response to a financial uncertainty shock, although the credible sets contain 0%. On impact, the posterior median response of EBP rises by 0.006% and peaks at 0.015% 6 months after the shock. Increases of up to 0.027%, however, are within the 70% credible set. The EBP response has a distinct hump-shaped pattern which is not found in the prior and the 70% credible set does not contain 0% for up to 17 months after the uncertainty shock.

Comparing the credible sets, there is evidence that a macro uncertainty shock has a weaker response on industrial production. The posterior median IRF declines by -0.08% on impact. Relative to a financial uncertainty shock, a macro uncertainty shock produces a more gradual decline in industrial production. After 3 years, the response of the posterior median IRF reaches -0.24%. The 70% credible sets of the IRF hovers between -0.46% and -0.08%. Unlike the financial uncertainty shock, there is no evidence of a downward shift in the set of possible federal funds rate responses. There is also little evidence of an increase in the EBP. Appendix C presents additional results on the impact of the conditional Haar prior.

The bottom row in figure 2 shows the prior robust posterior mean bounds of the responses. These bounds are constructed based on all possible priors for Q rather than relying on a single prior, and can be interpreted as an estimator for the identified set. The main qualitative results on the relative impacts continue to hold.

5.2 Example 2: Can we robustly identify the impact of financial and macro uncertainty shocks?

We now turn to the question of robust identification of financial and macro uncertainty shocks. As our results in Example 1 suggest, in weakening the identifying power of the sign restrictions, we find it necessary to improve the estimation of financial and macro volatility. To do so, we use the model and estimates in Carriero et al. (2017a). The Carriero et al. (2017a) paper presents a leading empirical model in measuring the impact of uncertainty shocks on the economy. The parameter estimates of the model from the published version are kindly provided by the authors in the replication dataverse in *The Review of Economics and Statistics*.

A detailed account of the model can be found in the appendix (section C). We sketch out a few important points of the model specification. The model is estimated on 30 data series ($1959M9$ - $2014M17$) which are divided into two categories – financial and macro – for purposes of volatility estimation. The paper considers two volatility factors, a financial volatility factor (f) meant to capture common second moment movements in the financial variables and macro volatility factor (m), which does the same for macro variables. These two volatility factors can impact the conditional mean. They are modeled as a bivariate VAR(2), creating an identification problem, as structural financial and macro uncertainty shocks may increase both volatility factors contemporaneously. The paper also specifies independent idiosyncratic volatility processes that do not impact the conditional mean. An additional attractive feature of the model is that it allows for level shocks to impact the volatility factors with a lag, thereby controlling for a potentially important channel in determining uncertainty fluctuations.

Although the empirical model contains many different financial and macro variables, we are concerned with four of them in our analysis: industrial production, PCE price level, federal funds rate, and the BAA-10 year Treasury spreads ($Spread$). We assume a similar sign restriction as in the previous example to identify the two uncertainty shocks that uses the response of financial spreads to tease out the effects of financial and macro uncertainty shocks.

Assumption $\mathcal{A}^2_{uf}$ (Financial uncertainty shock). The uncertainty shock satisfies $\mathcal{A}^2_{uf,1}$ and $\mathcal{A}^2_{u,2}$:

$$\mathcal{A}^2_{uf,1} : IRF\left[log(f_{t+s})\middle|v_t^* = e_1; R\right] > 0 \text{ for } s = 0.$$

$$\mathcal{A}^2_{ufm} : IRF\left(Spread_{t+s}\middle|v_t^* = e_1; R\right) > IRF\left(Spread_{t+h}\middle|v_t^* = e_2; R\right) \text{ for } s = 0.$$

Assumption $\mathcal{A}^2_{um}$ (Macro uncertainty shock). The uncertainty shock satisfies $\mathcal{A}^2_{um,1}$ and $\mathcal{A}^2_{ufm}$:

$$\mathcal{A}^2_{um,1} : IRF\left(log(m_{t+s})\middle|v_t^* = e_2; R\right) > 0 \text{ for } s = 0.$$

$$\mathcal{A}^2_{ufm} : IRF\left(Spread_{t+s}\middle|v_t^* = e_1; R\right) > IRF_{t+h}\left(Spread_{t+s}\middle|v_t^* = e_2; R\right) \text{ for } s = 0.$$

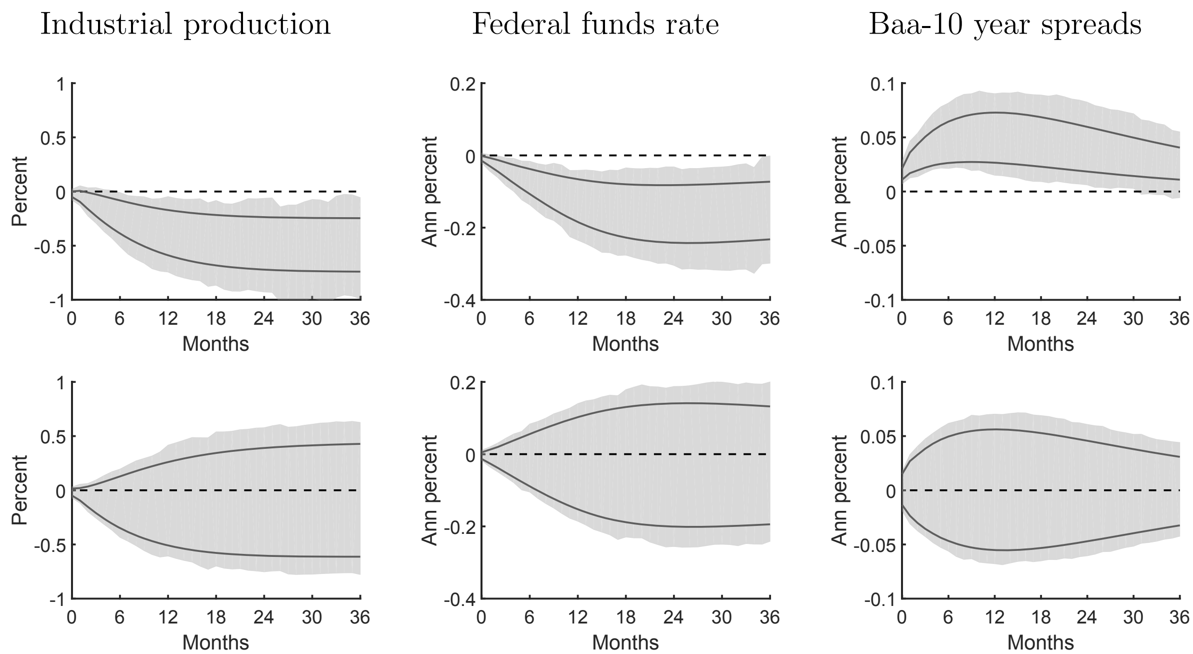
In words, the financial uncertainty shock is assumed to increase the financial volatility factor contemporaneously while the macro uncertainty shock leads to an increase in the macro volatility factor contemporaneously. As the assumed factor structure already specifies a division between financial and macro volatility, we believe it reasonable to only impose that a financial uncertainty shock increases financial volatility and a macro uncertainty shock increases macro uncertainty. We also assume that the financial uncertainty shock leads to a larger increase in the BAA-10 year Treasury spreads compared to the macro uncertainty shock.

Posterior results Figure 3 shows the effects of financial (top row) and macro (bottom row) uncertainty shocks on the economy. We use 500 parameter draws selected evenly from the final $5,000$ parameter draws of Carriero et al. (2017a) to construct our prior robust sign restrictions. For each parameter draw, we draw $10,000$ rotation matrices from the Haar distribution to construct the maximum and minimum of the set.

There is evidence at the 70% prior robust credible set level that industrial production decreases following a financial uncertainty shock. As expected, this credible region is quite wide with such a loose sign restriction. The posterior mean bounds range from -0.74% to -0.25% (70% set from -0.99% to -0.05%) by 3 years out. Similar to the results using the small model, we find evidence that financial uncertainty shocks lead to long-lived declines in industrial production.

Interestingly, the large volatility model identifies a decline in the federal funds rate following a financial uncertainty shock at the 70% credible set level. The posterior mean bounds of the decline is between -0.23% and -0.07% after 3 years, but prior robust credible set suggests that the decline could be as large as 30 basis points 3 years after the financial uncertainty shock. Therefore, the larger volatility model reinforces the weak evidence from the small model of a decline in the federal funds rate after the financial uncertainty shock. This result makes sense in light of the recent evidence provided by Caldara and Herbst (2016), who find evidence of monetary policy responding to changing credit conditions. Therefore,

Figure 3 **Financial uncertainty shock (top row) and macro uncertainty shock (bottom row) on level variables (prior robust)**



This figure shows the pointwise 70% credible set prior robust impulse response functions on the level variables to a 1 standard deviation financial uncertainty shock (top row) and a 1 standard deviation macro uncertainty shock (bottom row) for the Carriero et al. (2017a) model. The lines are the posterior mean bounds. We only keep the impulse response functions that satisfy Assumptions $\mathcal{A}^2_{uf}$ and $\mathcal{A}^2_{um}$. The reduced-form parameters are drawn from their posterior distributions. To conserve space, we have suppressed the response of the PCE price level. Those responses can be found in the appendix (section C).

an interpretation of the behavior of this impulse response function is that the Fed reacts to increases in financial uncertainty by loosening policy because of a rise in spreads. A portion of the easing is also in response to the decline in real activity.

In contrast, there is little evidence that a macro uncertainty shock has any effect on industrial production, the Federal funds rate, and BAA - 10 year Treasury spreads. Many of the same qualitative conclusions from the small VAR model continue to hold. For example, there is stronger evidence that a financial uncertainty shock leads to a decline in industrial production and a decline in the federal funds rates when compared to a macro uncertainty shock.

6 Conclusion and Future direction

We present the sign restrictions methodology as a new approach to identifying the real effects of uncertainty shocks. The methodology is applicable to a wide range of VAR models with volatility-in-mean. We apply these sign restrictions on the conditional first and second moment responses following uncertainty shocks. We investigate the real effects of financial and macro uncertainty shocks, and find more evidence supporting that financial uncertainty shocks lead to a decline in the macroeconomy and an easing of monetary policy. Future work could include extending the framework to allow for a more general relationship between contemporaneous first and second moment shocks.

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