Dynamic Games: Backward Induction and Subgame Perfection

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Jun 22th, 2017
On the Agenda

1. Formalizing the Game
2. Extensive Form Refinements of Nash Equilibrium
3. Backward Induction
4. Subgame Perfect Nash Equilibrium
5. Exercises
Formalizing the Game

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Let me fix some Notation:

- set of players: \( I = \{1, 2, \cdots, N\} \)
- set of actions: \( \forall i \in I, \ a_i \in A_i \), where each player \( i \) has a set of actions \( A_i \).
- strategies for each player: \( \forall i \in I, \ s_i \in S_i \), where each player \( i \) has a set of pure strategies \( S_i \) available to him. A strategy is a complete contingent plan for playing the game, which specifies a feasible action of a player’s information sets in the game.
- profile of pure strategies: \( s = (s_1, s_2, \cdots, s_N) \in \prod_{i=1}^N S_i = S \). Note: let \( s_{-i} = (s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N) \in S_{-i} \), we will denote \( s = (s_i, s_{-i}) \in (S_i, S_{-i}) = S \).
- Payoff function: \( u_i : \prod_{i=1}^N S_i \to \mathbb{R} \), denoted by \( u_i(s_i, s_{-i}) \)
- A mixed strategy for player \( i \) is a function \( \sigma_i : S_i \to [0, 1] \), which assigns a probability \( \sigma_i(s_i) \geq 0 \) to each pure strategy \( s_i \in S_i \), satisfying \( \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \).
- Payoff function over a profile of mixed strategies:

\[
u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \left[ \prod_{j \neq i} \sigma_j(s_j) \right] \sigma_i(s_i) u_i(s_i, s_{-i}) \]
An extensive form game is defined by a tuple $\Gamma_E = \{I, \chi, p, A, \alpha, H, h, i, \rho, u\}$

1. A finite set of $I$ players: $I = \{1, 2, \cdots, N\}$
2. A finite set of nodes: $\chi$
3. A function $p: \chi \rightarrow \chi \cup \{\emptyset\}$ specifying a unique immediate predecessor of each node $x$ such that $p(x)$ is the empty-set for exactly one node, called the root node, $x_0$. The immediate successors of node $x$ are defined as $s(x) = y \in X : p(y) = x$.
4. A set of actions, $A$, and a function $\alpha: \chi \setminus \{x_0\} \rightarrow A$ that specifies for each node $x \neq x_0$, the action which leads to $x$ from $p(x)$.
5. A collection of information sets, $H$, that forms a partition of $\chi$, and a function $h: \chi \rightarrow H$ that assigns each decision node into an information set.
6. A function $i: H \rightarrow I$ assigning the player to move at all the decision nodes in any information set.
7. For each $H$ a probability distribution function $\rho(H)$. This dictates nature’s moves at each of its information sets.
8. $u = (u_1, \cdots, u_N)$, a vector of utility functions for each $i$. 
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We now turn to a study of sequential or dynamic games, i.e. games where players are not moving simultaneously, but rather in a sequence over time.

The underlying theme will be to refine the set of Nash equilibria in these games.

One way to go is to simply write down their normal form, and then proceed as we did when studying simultaneous games.

The problem is that certain Nash equilibria in dynamic games can be very implausible predictions.

**Predation Game:** Firm E (the entrant) can choose whether to enter a market against a single incumbent, Firm I, or exit. If Firm E enters, Firm I can either respond by fighting or accommodating.
Nash Equilibria:

\((\sigma_E, \sigma_I)_1 = ((1, 0), (1, 0))\)

\((\sigma_E, \sigma_I)_2 = ((0, 1), (0, 1))\)

\((\sigma_E, \sigma_I)_3 = ((1, 0), (2/5, 3/5))\) (Why?)
However, (Out, Fight) does not seem like a plausible prediction:

conditional upon Firm E having entered, Firm I is strictly better off accommodating rather than fighting. Hence, if Firm E enters, Firm I should accommodate.

But then, Firm E should foresee this and enter, since it prefers the outcome (In, Accommodate) to what it gets by playing Out.
The problem in the Example is that the "threat" of playing Fight upon entry is not credible.

The outcome (Out, Fight) is Nash Equilibrium because if Firm I would fight upon entry, then Firm E is better off exiting.

However, in the dynamic game, Firm E should not believe such an "empty threat".
The crux of the matter is that the Nash Equilibrium concept places no restrictions on players’ behavior at nodes that are never reached on the equilibrium path.

In this example, given that Firm E is playing Out, any action for Firm I is a best response, since all its actions are at a node that is never reached when Firm E places Out.

Thus, by choosing Fight, that certainly wouldn’t want to play, Firm I can ensure that Firm E’s best response is to play Out, guaranteeing that Firm I in fact won’t have to act.
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The natural way to solve the problem above is to require that a player’s strategy specify *optimal actions at every node of the game tree*.

When contemplating an action, a player takes as given that the relevant node has been reached, and thus should playing something that is optimal here on out (given her opponents’ strategies).

In the previous example Fight is not optimal, conditional on the relevant node being reached.

We can apply this logic to any extensive game in the following way:

Start at the "end" of the game tree, and work "back" up the tree by solving for optimal behavior at each node.

This procedure is known as *backward induction*. In the class of finite games with perfect information (finite number of nodes and singleton information sets), this is a powerful procedure.
Backward Induction

Theorem

(Zermelo). Every finite game of perfect information has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then backward induction results in a unique Nash equilibrium.


Remark: Every finite game of perfect information has a PSNE.

Zermelo’s Theorem says that backward induction can be powerful in various finite games.

For example it implies that even a game as complicated as chess is solvable through backward induction.

In this sense, chess is "solvable" although, no-one knows what the solution is!
Quality Choice:

- Player I is an internet service provider and player II a potential customer. They consider entering into a contract of service provision for a period of time.
- The provider decides between two levels of quality of service: High or Low
- The buyer decides between two actions: to buy or not to buy
- The service provider, player I, makes the first move, choosing High or Low quality of service. Then the customer, player II, is informed about that choice.
- Player II can then decide separately between buy and don't buy in each case.
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**Centipede Game**: Two players, 1 and 2, take turns choosing one of two actions each time, continue or stop.

They both start with $1 in their respective piles, and each time i says continue, $1 is taken away from his pile, and $2 are added to the other player’s pile. The game automatically stops when both players have $100 in their respective piles.

Backward induction implies that a player should say stop whenever it is his turn to move.

In particular, Player 1 should say stop at the very first node, and both players leave with just the $1 they start out with.
Notice the independence of history: a player makes a decision at a node in anticipation of its future consequences and without regard to the sequence of moves that have been made to place him at that node.

Is this what you think would happen if this game were played in an experimental setting?

How would you as a player interpret a choice of C by your opponent?

If you had seen him choose C repeatedly, would your expectations of the future play of the game be determined solely by looking forward?
**Ultimatum Game**: Two players have a continuous dollar to divide.

Player 1 proposes to divide the dollar at $x \in [0, 1]$, where he will keep $x$ and player 2 will receive $1 - x$.

Player 2 can choose to either Accept this division of the dollar, or Reject it, in which case each player receives 0.
Any division \((p, 1 - p)\) of the dollar can be sustained as a Nash equilibrium:

\[
\begin{align*}
1 & : \quad x = p \\
2 & : \quad \begin{cases} 
R & \text{if } 1 - x < 1 - p \\
A & \text{if } 1 - x \geq 1 - p
\end{cases}
\end{align*}
\]

Notice that 2's strategy specifies how he responds to any offer, not just the one that 1 actually makes in equilibrium.

We might interpret this equilibrium as "2 demands at least \(1 - p\), and 1 offers 2 the minimal amount that 2 will accept."

In the case of \(p < 1\), however, player 2's strategy involves a not believable threat to reject a positive amount \(1 - p\) in favor of 0. Not rational behavior\(^1\).

If 2 accepts any offer, however, then we can define an equilibrium:

\[
\begin{align*}
1 & : \quad x = 1 \\
2 & : \quad A
\end{align*}
\]

\(^1\) Rationality requires that player 2 accept any non negative offer. If we assume that 2 accepts only positive offers, there is not best response for 1: he wants to choose \(x < 1\) as large as possible, which is not well-defined.
The interest of the previous game is to behavioral and experimental economics.

Do people have a sense of fairness in how they play games?

If so, should this be modeled as part of economic theory and game theory?

The role of anonymity in allowing players to focus on their narrow self-interests.
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We are now going to define a refinement of Nash equilibrium that captures the notion of backward induction.

Recall that an extensive form game, $\Gamma_E$, specifies a host of objects, including a set of nodes, $\chi$, an immediate predecessor mapping $p(x)$ that induces a successor nodes mapping $s(x)$, and a mapping $h(x)$ from nodes to information sets.

**Definition**

A **subgame** of an extensive form game, $\Gamma_E$, is a subset of the game such that

1. There is a unique node in the subgame, $x^*$, such that $p(x^*)$ is not in the subgame. Moreover, $h(x^*) = \{x^*\}$ and $x^*$ is not a terminal node.
2. A node, $x$, is in the subgame if and only if $x \in \{x^*\} \cup s(x^*)$.
3. If node $x$ is in the subgame, then so is any $\tilde{x} \in h(x)$.
Subgame Perfect Nash Equilibrium

Figure 9.B.5
Three parts of the game in Figure 9.B.4 that are not subgames.
Subgame Perfect Nash Equilibrium

- Notice that any extensive form game as whole is always a subgame (of itself).
- We use the term *proper subgame* to refer to a subgame where $x^* \neq x_0$.
- The key feature of a subgame is that it is a game in its own right, and hence, we can apply the concept of Nash equilibrium to it.
- We say that a strategy profile, $\sigma$, in the game $\Gamma_E$ induces a Nash equilibrium in a particular subgame of $\Gamma_E$ if the moves specified by $\sigma$ for information sets in the subgame constitute a Nash equilibrium when the subgame is considered as a game by itself.

**Definition**

A Nash equilibrium, $\sigma^*$, in the extensive form game, $\Gamma_E$, is a Subgame Perfect Nash Equilibrium (SPNE) if it induces a Nash equilibrium in every subgame of $\Gamma_E$. 
In finite extensive form games with possibly imperfect information, we conduct generalized backward induction as follows:

1. Consider the maximal subgames (that is, subgames that have no further proper subgames) and pick a Nash equilibrium in each maximal subgame (one exists! why?).

2. Replace each maximal subgame with a "terminal" node that has the payoffs of the Nash equilibrium we picked in the subgame. After replacement, call this a "reduced" game.

3. Iterate the process on a successive sequence of "reduced" games until the whole tree has been replaced with a single terminal node.
Subgame Perfect Nash Equilibrium

- Let's consider the example of Predation with Niches:
  Firm E, (the Entrant) first chooses to enter or not. If it enters, then the two firms simultaneously choose a niche of the market to compete in: a (A) or b (B). Niche b is the "larger" niche.
To find the pure strategy SPNE equilibria of this game, we employ generalized backward induction as follows.

Notice that there is only one proper subgame here.

There are two PSNE in the subgame: (a,B) and (b,A).

If we replace the subgame with a terminal node corresponding to (a,B) payoffs, then it follows that Firm E prefers to play Out at its first move.

If we replace the subgame with a terminal node corresponding to (b,A) payoffs, then it follows that Firm E prefers to play In at its first move.

Therefore, the two pure strategy SPNE of this game are ((Out,a),(B)) and ((In,b),(A)).
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Consider the following games:

a. Apply backward induction. State the rationality/knowledge assumptions necessary for each step in this process.

b. Write the game in normal form.

c. Find all the rationalizable strategies in this game using the normal form of the game. State the rationality/knowledge assumptions necessary for each elimination.

c. Find all the Nash equilibria in this game.
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Consider the following two person game played by an Entrant E and an Incumbent I (where the first of the two payoffs given belongs to the Entrant and the second to the Incumbent):

Find the pure strategy SPNE equilibria of this game