On the Agenda

1. Formalizing the Game
2. Systems of Beliefs and Sequential Rationality
3. Weak Perfect Bayesian Equilibrium
4. Exercises
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Formalizing the Game

Let me fix some Notation:

- set of players: \( I = \{1, 2, \ldots, N\} \)
- set of actions: \( \forall i \in I, a_i \in A_i \), where each player \( i \) has a set of actions \( A_i \).
- strategies for each player: \( \forall i \in I, s_i \in S_i \), where each player \( i \) has a set of pure strategies \( S_i \) available to him. A strategy is a complete contingent plan for playing the game, which specifies a feasible action of a player’s information sets in the game.
- profile of pure strategies: \( s = (s_1, s_2, \ldots, s_N) \in \prod_{i=1}^{N} S_i = S \).
  Note: let \( s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \in S_{-i} \), we will denote \( s = (s_i, s_{-i}) \in (S_i, S_{-i}) = S \).
- Payoff function: \( u_i : \prod_{i=1}^{N} S_i \rightarrow \mathbb{R} \), denoted by \( u_i(s_i, s_{-i}) \)
- A mixed strategy for player \( i \) is a function \( \sigma_i : S_i \rightarrow [0, 1] \), which assigns a probability \( \sigma_i(s_i) \geq 0 \) to each pure strategy \( s_i \in S_i \), satisfying \( \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \).
- Payoff function over a profile of mixed strategies:

\[
u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \left[ \prod_{j \neq i} \sigma_j(s_j) \right] \sigma_i(s_i) u_i(s_i, s_{-i})
\]
Formalizing the Game

An extensive form game is defined by a tuple $\Gamma = \{I, \chi, p, A, \alpha, H, h, i, \rho, u\}$

1. A finite set of $I$ players: $I = \{1, 2, \cdots, N\}$
2. A finite set of nodes: $\chi$
3. A function $p : \chi \rightarrow \chi \cup \{\emptyset\}$ specifying a unique immediate predecessor of each node $x$ such that $p(x)$ is the empty-set for exactly one node, called the root node, $x_0$. The immediate successors of node $x$ are defined as $s(x) = y \in \chi : p(y) = x$.
4. A set of actions, $A$, and a function $\alpha : \chi \backslash \{x_0\} \rightarrow A$ that specifies for each node $x \neq x_0$, the action which leads to $x$ from $p(x)$.
5. A collection of information sets, $H$, that forms a partition of $\chi$, and a function $h : \chi \rightarrow H$ that assigns each decision node into an information set.
6. A function $i : H \rightarrow I$ assigning the player to move at all the decision nodes in any information set.
7. For each $H$ a probability distribution function $\rho(H)$. This dictates nature’s moves at each of its information sets.
8. $u = (u_1, \cdots, u_N)$, a vector of utility functions for each $i$. 
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A limitation of the preceding analysis is subgame perfection is powerless in dynamic games where there are no proper subgames.
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\[
\begin{array}{c|cc|c}
E \setminus I & F & A & \\
\hline
\text{Out} & 0,2 & 0,2 & \\
\text{In}_1 & -1,-1 & 3,0 & \\
\text{In}_2 & -1,-1 & 2,1 & \\
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We need a theory of "reasonable" choices by players at all nodes, and not just at those nodes that are parts of proper subgames.

One way to approach this problem in the above example is to ask: could Fight be optimal for Firm I when it must actually act for any belief that it holds about whether Firm E played In₁ or In₂? Clearly, no.

Regardless of what Firm I thinks about the likelihood of In₁ versus In₂, it is optimal for it to play Accommodate.

This motivates a formal development of beliefs in extensive form games.

Definition

A system of beliefs is a mapping $\mu : \chi \rightarrow [0, 1]$ such that, for all $h \in H$, $\sum_{x \in H} \mu(x) = 1$.

In words, a system of beliefs, $\mu$, specifies the relative probabilities of being at each node of an information set, for every information set in the game.
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Let $E[u_i|H, \mu, \sigma_i, \sigma_{-i}]$ denote player i’s expected utility starting at her information set $H$ if her beliefs regarding the relative probabilities of being at any node, $x \in H$ is given by $\mu(x)$, and she follows strategy $\sigma_i$ while the others play the profile of strategies $\sigma_{-i}$.

**Definition**

A strategy profile, $\sigma$, is sequentially rational at information set $H$, given a system of beliefs $\mu$, if

$$E\left[u_i(H) | H, \mu, \sigma_i(H), \sigma_{-i}(H)\right] \geq E\left[u_i(H) | H, \mu, \tilde{\sigma}_i(H), \sigma_{-i}(H)\right]$$

for all $\tilde{\sigma}_i(H) \in \Delta(S_i(H))$.

A strategy profile is sequentially rational given a system of beliefs if it is sequentially rational at all information sets given that system of beliefs.

In words, a strategy profile is sequentially rational given a system of beliefs if there is no information set such that once it is reached, the actor would strictly prefer to deviate from his prescribed play, given his beliefs about the relative probabilities of nodes in the information set and opponents’ strategies.
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Weak Perfect Bayesian Equilibrium

With these concepts in hand, we now define a Weak Perfect Bayesian Equilibrium (WPBE). The idea is straightforward:
- strategies must be sequentially rational, and beliefs must be derived from strategies whenever possible via Bayes rule.

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

**Definition**

A profile of strategies, \( \sigma \), and a system of beliefs, \( \mu \), is a Weak Perfect Bayesian Equilibrium (WPBE), \((\sigma, \mu)\), if:

1. \( \sigma \) is sequentially rational given \( \mu \)
2. \( \mu \) is derived from \( \sigma \) through Bayes rule whenever possible. That is, for any information set \( H \) such that \( P(H|\sigma) > 0 \), and any \( x \in H \),

\[ \mu(x) = \frac{P(x|\sigma)}{P(H|\sigma)} \]
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\[ \mu(x) = \frac{P(x|\sigma)}{P(H|\sigma)} \]
Keep in mind that strictly speaking, a WPBE is a strategy profile-beliefs pair.

However, we will sometimes be casual and refer to just a strategy profile as a WPBE.

This implicitly means that there is at least one system of beliefs such the pair forms a WPBE.

The "weak" in WPBE is because absolutely no restrictions are being placed on beliefs at information sets that do not occur with positive probability in equilibrium.

To be more precise, no consistency restriction is being placed; we do require that they be well-defined in the sense that beliefs are probability distributions.

As we will see, in many games, there are natural consistency restrictions one would want to impose on out of equilibrium information sets as well.
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As we will see, in many games, there are natural consistency restrictions one would want to impose on out of equilibrium information sets as well.
Consider the following two person game played by an Entrant E and an Incumbent I (where the first of the two payoffs given belongs to the Entrant (E) and the second to the Incumbent (I)).

a. Derive the Subgame Perfect Nash Equilibrium (SPNE) in this game.

b. Derive both pure and mixed Weak Perfect Bayesian Nash Equilibria in this game.
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\[
\begin{array}{c|cc}
E \backslash I & L & R \\
\hline
\text{Out} & \frac{3}{2}, 2 & \frac{3}{2}, 2 \\
\text{In}_1 & 1, 0 & 0, 1 \\
\text{In}_2 & 1, 2 & 2, 1 \\
\end{array}
\]

- In pure strategies the NE is E: Out and I: L
- No NE in mix strategies (why?)
a. Derive the Subgame Perfect Nash Equilibrium (SPNE) in this game.

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- No NE in mix strategies (why?)
b. Derive both pure and mixed Weak Perfect Bayesian Nash Equilibria in this game.

We first solve for the value of $p$ for which I chooses L over R. L produces an expected payoff of $2(1 - p)$ and R produces an expected payoff of 1. Therefore, L is at least as good as R for I if

$$2 - 2p \geq 1 \iff p \leq \frac{1}{2}$$
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- If I chooses L, then E chooses Out. This is our first equilibrium:

  \[ E : \text{Out} \]
  \[ I : L, \ p \leq \frac{1}{2} \]
b. Derive both pure and mixed Weak Perfect Bayesian Nash Equilibria in this game.

We next try to construct an equilibrium in which I chooses R. This results in E choosing IN2, which requires $p = 0$.

We need $p \geq 1/2$ to support I’s choice of R over L. So there is no pure strategy WPBNE in which I chooses R.
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Finally, we consider the use of a mixed strategy by I. Let $\sigma$ denote the probability that I chooses L. E’s expected payoff from choosing IN1 is $\sigma$ and his expected payoff from choosing IN2 is $\sigma + 2(1 - \sigma)$, which is strictly greater than $\sigma$ for $\sigma < 1$. It is therefore possible to choose $\sigma$ so that E is indifferent between IN1 and IN2, only by setting $\sigma = 1$, which leads us back to the equilibrium derived above.
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b. Derive both pure and mixed Weak Perfect Bayesian Nash Equilibria in this game.

We next consider the possibility that E chooses Out in a mixed strategy equilibrium.

We need

\[ \frac{3}{2} \geq 2 - \sigma \iff \sigma \geq \frac{1}{2} \]
We therefore have derived one more type of WPBNE:

\[
E : \text{Out} \\
I : \sigma \geq \frac{1}{2}, \ p = \frac{1}{2}
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