CONGESTION GAMES AND POTENTIALS RECONSIDERED

MARK VOORNEVELD*, PETER BORM, FREEK VAN MEGEN and STEF TIJS
Department of Econometrics and CentER, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg, The Netherlands

GIOVANNI FACCHINI
Department of Economics, Stanford University, Stanford, CA 94305, USA

In congestion games, players use facilities from a common pool. The benefit that a player derives from using a facility depends, possibly among other things, on the number of users of this facility. The paper gives an easy alternative proof of the isomorphism between exact potential games and the set of congestion games introduced by Rosenthal (1973). It clarifies the relations between existing models on congestion games, and studies a class of congestion games where the sets of Nash equilibria, strong Nash equilibria and potential-maximising strategies coincide. Particular emphasis is on the computation of potential-maximising strategies.

1. Introduction

In recent years, there has been a growing interest in the study of specific classes of non-cooperative games for which there exist pure strategy Nash equilibria. This paper deals with games derived from congestion models. In a congestion model, players use several facilities — also called machines or (primary) factors — from a common pool. The costs or benefits that a player derives from the use of a facility are, possibly among other factors, determined by the number of users of that same facility.

Congestion models can, for instance, be used to model the foraging behaviour of a population of bees in a field of flowers. In deciding which flower to visit, each insect will take into account the quantity of nectar available and the number of bees already on the flower, because, as is intuitively clear, the more crowded the source of nectar, the less food is available per capita. In economics, this kind of problems is studied in the literature on local public goods, where it is common to speak about “anonymous crowding” [cf. Wooders (1989)] to describe the negative externality arising from the presence of more than one user of the same facility. Another example is the problem faced by a set of unemployed workers who have to decide where to emigrate to get a job. The attraction of different countries depends

*Corresponding author. E-mail: m.voorneveld@kub.nl. This author’s research is financially supported by the Netherlands Mathematics Research Foundation, part of the Netherlands Organization for Scientific Research (NWO). Comments of the two referees are gratefully acknowledged.
on the conditions of the local labor market and, on the other hand, a crowding-out effect reduces the appeal of emigrating.

Rosenthal (1973) constitutes one of the pioneering papers on congestion games. In his model, each player chooses a subset of facilities. The benefit associated with each facility is a function only of the number of players using it. The payoff to a player is the sum of the benefits associated with each facility in his strategy choice, given the choices of the other players. Monderer and Shapley (1996) define exact potential games, games where information concerning the Nash equilibria can be incorporated in a potential function, a single real-valued function on the strategy space. Strategy profiles maximising the potential are Nash equilibria of the potential game. By constructing a potential function for Rosenthal’s congestion games, the existence of pure-strategy equilibria can be established. Monderer and Shapley (1996) not only prove that every such congestion game is an exact potential game, but also establish that every exact potential game is isomorphic to a congestion game.

Konishi, Le Breton and Weber (1997), Milchtaich (1996), and Quint and Shubik (1994) considered different classes of congestion games which in general do not admit a potential function, but were still able to prove the existence of pure Nash equilibria. Konishi, Le Breton and Weber (1997), considering the same model as Milchtaich, have even shown the existence of a strong Nash equilibrium.

This paper has several goals. First, after reviewing some results on exact potential games in Sec. 2, we provide a new, simple proof of the isomorphism between exact potential games and Rosenthal’s congestion games in Sec. 3.

Secondly, the relations and differences between the congestion models of Konishi et al. (1997), Milchtaich (1996), and Quint and Shubik (1994) are clarified in Sec. 4.

Thirdly, in Secs. 5 and 6 we return to potential games and focus on a class of congestion games that combines features from the congestion models mentioned above. This class is shown to have interesting properties. In particular, it is shown that for each game in this class, the set of strong Nash equilibria is non-empty and coincides with the set of Nash equilibria and the set of potential-maximising strategies.

In Sec. 5, we analyse the geometric properties of this class of games, showing that it can be represented by a finitely generated cone. The aim of this section is twofold. First, it provides an easy way to compute potential-maximising strategies, and second, it facilitates the proof that the sets of strong Nash equilibria, Nash equilibria, and potential-maximising strategies are equal.

Implications of relaxing some of the assumptions underlying the congestion effect are discussed in Sec. 7.

2. Exact Potential Games

This section defines exact potential games and surveys some results that are used in the remainder of the paper. A (strategic) game is a tuple $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$, where $N$ is a non-empty, finite set of players, each player $i \in N$ has a non-empty,
finite set $X_i$ of pure strategies and a payoff function $u_i : \prod_{j \in N} X_j \rightarrow \mathbb{R}$ specifying for each strategy profile $x = (x_j)_{j \in N} \in \prod_{j \in N} X_j$ player $i$'s payoff $u_i(x) \in \mathbb{R}$. Mixed strategies are not considered in this paper. Conventional game-theoretic notation is used: $X = \prod_{j \in N} X_j$ denotes the set of strategy profiles. Let $i \in N$. $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ denotes the strategy profiles of $i$'s opponents. Let $S \subseteq N$. $X_S = \prod_{j \in S} X_j$ denotes the set of strategy profiles of players in $S$. With a slight abuse of notation strategy profiles $x = (x_j)_{j \in N}$ will be denoted by $(x_i, x_{-i})$ or $(x_S, x_{NS})$ if the strategy choice of player $i$ or of the set $S$ of players needs stressing.


**Definition 2.1.** A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an exact potential game if there exists a function $P : X \rightarrow \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $x_i, y_i \in X_i$:

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

The function $P$ is called an (exact) potential (function) for $G$.

In other words, a strategic game is an exact potential game if there exists a real-valued function on the strategy space which exactly measures the difference in the payoff that accrues to a player if he unilaterally deviates.

**Example 2.1.** The Prisoner’s Dilemma game of Fig. 1(a) is an exact potential game with an exact potential function given by $P(c, c) = 5$, $P(c, d) = 4$, $P(d, d) = 3$. The game in Fig. 1(b) is an exact potential game with an exact potential function given by $P(T, L) = 0$, $P(T, R) = 1$, $P(B, L) = 2$, $P(B, R) = 3$.

If $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has an exact potential $P$, the definition of an exact potential game immediately implies that the Nash equilibria of $\langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$, the game obtained by replacing each payoff function by the potential $P$, and the game $G$ coincide.

**Proposition 2.1.** [Monderer and Shapley (1996), Corollary 2.2] Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be an exact potential game. Then $G$ has at least one (pure-strategy) Nash equilibrium.

**Proof.** Let $P$ be an exact potential for $G$. Since $X$ is finite, $\arg\max_{x \in X} P(x)$ is a non-empty set. Clearly, all elements in this set are pure-strategy Nash equilibria.

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$d$</th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>1.1</td>
<td>4.0</td>
<td>0.2</td>
<td>2.3</td>
</tr>
<tr>
<td>$d$</td>
<td>0.4</td>
<td>3.3</td>
<td>2.5</td>
<td>4.6</td>
</tr>
</tbody>
</table>

(a) (b)

Fig. 1. Two exact potential games.
Facchini et al. (1997) provide a characterisation of exact potential games by splitting them up into coordination games and dummy games.

**Definition 2.2.** A game \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is a

- coordination game if there exists a function \( u : X \to \mathbb{R} \) such that \( u_i = u \) for all \( i \in N \);
- dummy game if for all \( i \in N \) and all \( x_{-i} \in X_{-i} \) there exists a \( k \in \mathbb{R} \) such that \( u_i(x_i, x_{-i}) = k \) for all \( x_i \in X_i \), or equivalently, if for all \( i \in N \), all \( x_{-i} \in X_{-i} \), and all \( x_i, y_i \in X_i \): \( u_i(x_i, x_{-i}) = u_i(y_i, x_{-i}) \).

In a coordination game, players pursue the same goal, reflected by the identical payoff functions. In a dummy game, a player’s payoff does not depend on his own strategy. Coordination games and dummy games are exact potential games.

**Theorem 2.1.** Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a strategic game. \( G \) is an exact potential game if and only if there exist functions \( (c_i)_{i \in N} \) and \( (d_i)_{i \in N} \) such that

- \( u_i = c_i + d_i \) for all \( i \in N \),
- \( \langle N, (X_i)_{i \in N}, (c_i)_{i \in N} \rangle \) is a coordination game, and
- \( \langle N, (X_i)_{i \in N}, (d_i)_{i \in N} \rangle \) is a dummy game.

**Proof.** The “if”-part is obvious: the payoff function of the coordination game is an exact potential function of \( G \). To prove the “only if”-part, let \( P \) be an exact potential for \( G \). For all \( i \in N \), \( u_i = P + (u_i - P) \). Clearly, \( \langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle \) is a coordination game. Let \( i \in N \), \( x_{-i} \in X_{-i} \) and \( x_i, y_i \in X_i \). Then \( u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}) \) implies \( u_i(x_i, x_{-i}) - P(x_i, x_{-i}) = u_i(y_i, x_{-i}) - P(y_i, x_{-i}) \). So \( \langle N, (X_i)_{i \in N}, (u_i - P)_{i \in N} \rangle \) is a dummy game.

The difference between two exact potential functions of a game is a constant function [see also Monderer and Shapley (1996), Lemma 2.7].

**Proposition 2.2.** Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a game with exact potential functions \( P \) and \( Q \). Then \( P - Q \) is a constant function.

**Proof.** Let \( i \in N \). By Theorem 2.1, \( u_i - Q \) and \( u_i - P \) do not depend on the strategy choice of player \( i \). Hence \( (P - Q) = (u_i - Q) - (u_i - P) \) does not depend on the strategy choice of player \( i \). This holds for every player \( i \in N \): \( P - Q \) is a constant function.

Proposition 2.2 implies that the set of strategy profiles maximising a potential function of an exact potential game does not depend on the particular potential function that is chosen. Potential-maximising strategies were used in the proof of Proposition 2.1 to show that exact potential games have pure-strategy Nash
equilibria. The potential maximiser, formally defined for an exact potential game \( G = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) as

\[ PM(G) = \{ x \in X | x = \arg \max_{y \in X} P(y) \text{ for some potential function } P \text{ of } G \} \]

can therefore, as shown by Monderer and Shapley (1996), act as an equilibrium refinement tool. Peleg, Potters and Tijs (1996) provide an axiomatic approach to potential-maximising strategies.

3. Rosenthal’s Congestion Model

In a congestion model, players use several facilities — also called machines or (primary) factors — from a common pool. The costs or benefits that a player derives from the use of a facility are, possibly among other factors, determined by the number of users of a facility. In this section we focus on the congestion model of Rosenthal (1973). In this model, each player chooses a subset of facilities. The benefit (possibly negative) associated with each facility is a function only of the number of players using it. The payoff to a player is the sum of the benefits associated with each facility in his strategy choice, given the choices of the other players. By constructing an exact potential function for such congestion games, the existence of pure-strategy Nash equilibria can be established. Moreover, Monderer and Shapley (1996) showed that every exact potential game is isomorphic to a congestion game. Their proof is rather complex. In this section, we present a different proof which is shorter and in our opinion more intuitive. In fact, we use the decomposition of exact potential games into dummy games and coordination games stated in Theorem 2.1 to decompose the problem into two sub-problems. It is shown that each coordination game and each dummy game is isomorphic to a congestion game as defined by Rosenthal.

A congestion model is a tuple \( (N, F, (X_i)_{i \in N}, (w_f)_{f \in F}) \), where

- \( N = \{1, \ldots, n\} \) is a non-empty, finite set of players;
- \( F \) is a non-empty, finite set of facilities;
- For each player \( i \in N \), his collection of pure strategies \( X_i \) is a non-empty, finite family of subsets of \( F \);
- For each facility \( f \in F \), \( w_f : \{1, \ldots, n\} \to \mathbb{R} \) is the benefit function of facility \( f \), with \( w_f(r), r \in \{1, \ldots, n\} \), the benefits to each of the users of facility \( f \) if there is a total of \( r \) users.

This gives rise to a congestion game \( G = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) where \( N \) and \( (X_i)_{i \in N} \) are as above and for \( i \in N \), \( u_i : X \to \mathbb{R} \) is defined thus: for each \( x = (x_1, \ldots, x_n) \in X \), and each \( f \in F \), let \( n_f(x) = |\{ i \in N : f \in x_i \}| \) be the number of users of facility \( f \) if the players choose \( x \). Then \( u_i(x) = \sum_{f \in x_i} w_f(n_f(x)) \). This definition implies that each player derives benefit from the facilities he uses, with benefits depending only on the number of users of the facility. Notice that benefit functions can achieve negative values, representing costs of using a facility.
The main result from Rosenthal’s paper, formulated in terms of exact potentials, is given in the next proposition. Its proof is straightforward and therefore omitted.

**Proposition 3.1.** Let \( \langle N, F, (X_i)_{i \in N}, (w_f)_{f \in F} \rangle \) be a congestion model and \( G \) its congestion game. Then \( G \) is an exact potential game. A potential function is given by \( P : X \rightarrow \mathbb{R} \) defined for all \( x = (x_i)_{i \in N} \in X \) as

\[
P(x) = \sum_{f \in \cup_{i \in N} X_i} \sum_{\ell=1}^{n_f(x)} w_f(\ell).
\]

Since \( X \) is finite, the game has a Nash equilibrium in pure strategies.

Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) and \( H = \langle N, (Y_i)_{i \in N}, (v_i)_{i \in N} \rangle \) be two strategic games with identical player set \( N \). \( G \) and \( H \) are isomorphic if for all \( i \in N \) there exists a bijection \( \varphi_i : X_i \rightarrow Y_i \) such that

\[
u_i(x_1, \ldots, x_n) = v_i(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \quad \text{for all} \quad (x_1, \ldots, x_n) \in X.
\]

A congestion game where the facilities have non-zero benefits only if all players use it as part of their strategy choice is clearly a coordination game. Also, each coordination game can be expressed in this form, as shown in the proof of the next theorem.

**Theorem 3.1.** Each coordination game is isomorphic to a congestion game.

**Proof.** Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be an \( n \)-player coordination game in which each player has payoff function \( u \). Introduce for each \( x \in X \) a different facility \( f(x) \). Define the congestion model \( \langle N, F, (Y_i)_{i \in N}, (w_f)_{f \in F} \rangle \) with \( F = \cup_{x \in X} \{ f(x) \} \), for each \( i \in N \): \( Y_i = \{ g_i(x_i) | x_i \in X_i \} \) where \( g_i(x_i) = \cup_{x_{-i} \in X_{-i}} \{ f(x_i, x_{-i}) \} \), and for each \( f(x) \in F \):

\[
w_{f(x)}(r) = \begin{cases} u(x) & \text{if } r = n \\ 0 & \text{otherwise.} \end{cases}
\]

For each \( x \in X : \cap_{i \in N} g_i(x_i) = \{ f(x) \} \), so the game corresponding to this congestion model is isomorphic to \( G \) (where the isomorphisms map \( x_i \) to \( g_i(x_i) \)).

The proof is illustrated with a simple example.

**Example 3.1.** Consider the coordination game in Fig. 2(a). For each strategy profile we introduce a facility as in Fig. 2(b). These are the facilities that we want to be used by both players if they play the corresponding strategy profile. To do this, give each player in a certain row (column) all facilities mentioned in this row (column). For instance, the second strategy of the row player will correspond with choosing facility set \( \{ C, D \} \). Now indeed, if both players play their second strategy, facility \( D \) is used by both players and all other facilities have one or zero users. Defining the benefits of \( D \) in case of two simultaneous users to be 3 and in case of
less users zero, we obtain the payoff (3,3) in the lower right-hand corner of Fig. 2(c). Similar reasoning applies to the other cells.

Consider a congestion game in which the benefits for a facility are non-zero only if it is used by a single player. If for each player, given the strategy choices of the other players, it holds that his benefits arise from using one and the same facility, irrespective of his own strategy choice, we have a dummy game. Also, as shown in the next theorem, each dummy game is isomorphic to a congestion game with this property.

**Theorem 3.2.** Each dummy game is isomorphic to a congestion game. 

**Proof.** Let \( G = \langle N, (X_{i})_{i \in N}, (u_{i})_{i \in N} \rangle \) be a dummy game. Introduce for each \( i \in N \) and each \( x_{-i} \in X_{-i} \) a different facility \( f(x_{-i}) \). Define the congestion model \( \langle N, F, (Y_{j})_{j \in F}, (w_{f})_{f \in F} \rangle \) with \( F = \bigcup_{i \in N} \cup_{x_{-i} \in X_{-i}} \{ f(x_{-i}) \} \), for each \( i \in N \); \( Y_{i} = \{ h_{i}(x_{i}) | x_{i} \in X_{i} \} \) where

\[
    h_{i}(x_{i}) = \{ f(x_{-i}) | x_{-i} \in X_{-i} \}
\]

\[
\cup \{ f(y_{-j}) | j \in N \setminus \{ i \} \}
\]

and for each \( f(x_{-i}) \in F \):

\[
w_{f(x_{-i})}(r) = \begin{cases} u_{i}(x_{i}, x_{-i}) & \text{if } r = 1 \text{ (with } x_{i} \in X_{i} \text{ arbitrary) } \\ 0 & \text{otherwise} \end{cases}
\]

For each \( i \in N \), \( \bar{x}_{-i} \in X_{-i} \) and \( \bar{x}_{i} \in X_{i} \): \( i \) is the unique user of \( f(\bar{x}_{-i}) \) in \( (h_{j}(\bar{x}_{j}))_{j \in N} \) and all other facilities in \( h_{i}(\bar{x}_{i}) \) have more than one user. Why? Let \( i \in N \), \( \bar{x}_{-i} \in X_{-i} \), and \( \bar{x}_{i} \in X_{i} \). Then \( f(\bar{x}_{-i}) \in h_{i}(\bar{x}_{i}) \) and for each \( j \in N \setminus \{ i \} \): \( f(\bar{x}_{-i}) \notin h_{j}(\bar{x}_{j}) \), so \( i \) is indeed the unique user of \( f(\bar{x}_{-i}) \) in \( (h_{j}(\bar{x}_{j}))_{j \in N} \). Let \( f \in h_{i}(\bar{x}_{i}), f \neq f(\bar{x}_{-i}) \).

- If \( f = f(y_{-i}) \) for some \( y_{-i} \in X_{-i} \), then \( y_{-i} \neq \bar{x}_{-i} \) implies that \( y_{j} \neq \bar{x}_{j} \) for some \( j \in N \setminus \{ i \} \), so \( f = f(y_{-i}) \in h_{j}(\bar{x}_{j}) \).
- If \( f = f(y_{-j}) \) for some \( j \in N \setminus \{ i \} \) and \( y_{-j} \in X_{-j} \) with \( y_{i} \neq \bar{x}_{i} \), then \( f = f(y_{-j}) \in h_{j}(\bar{x}_{j}) \).

In both cases \( f \) has more than one user. So the game corresponding to this congestion model is isomorphic to \( G \) (where the isomorphisms map \( x_{i} \) to \( h_{i}(x_{i}) \)). \( \square \)

Once again, this argumentation is illustrated by an example.
Example 3.2. Consider the dummy game in Fig. 3(a). Introduce a different facility for each profile of opponent strategies as in Fig. 3(b). Facility \( \alpha \), for instance, is associated with the strategy profile in which player 2, the only opponent of player 1, choosing the left column. Include a facility \( f(x_{-i}) \) in each strategy of each player, except for the strategies of players \( j \in N \setminus \{i\} \) specified by the profile \( x_{-i} \). For instance, facility \( \alpha \) was introduced for the first column of player 2; then \( \alpha \) is part of every strategy, except for the first column of player 2. This yields the strategies as in Fig. 3(c). Define benefits for multiple users equal to zero. No matter what player 1 does, if his opponent chooses his second strategy, the benefits to player 1 can be attributed to facility \( \beta \). Assign benefit 1 to a single user of this facility. Similar reasoning for the other payoffs yields the isomorphic congestion game in Fig. 3(c).

In the previous two theorems it was shown that coordination and dummy games are isomorphic to congestion games. Using the decomposition of Theorem 2.1 we obtain that every exact potential game is isomorphic to a congestion game.

**Theorem 3.3.** Every exact potential game is isomorphic to a congestion game.

**Proof.** Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be an exact potential game. Split it into a coordination game and a dummy game as in Theorem 2.1 and take their isomorphic congestion games as in Theorems 3.1 and 3.2. Without loss of generality, take their facility sets disjoint. Construct a congestion game isomorphic to \( G \) by taking the union of the two facility sets, benefit functions as in Theorems 3.1 and 3.2, and strategy sets \( Y_i = \{g_i(x_i) \cup h_i(x_i) | x_i \in X_i \} \).

Example 3.3. The exact potential game in Fig. 1(b) is the sum of the coordination game from Example 3.1 and the dummy game from Example 3.2. Combining the two isomorphic congestion games from these examples yields a congestion game isomorphic to the exact potential game. See Fig. 4.
4. Congestion Games

The games introduced by Konishi, Le Breton and Weber (1997), Milchtaich (1996), and Quint and Shubik (1994) are similar, in the sense that the utility functions of the players are characterised by a congestion effect. The various classes of games we discuss are identified by means of different sets of properties concerning the structure of the strategic interaction. In particular, Konishi et al. (1997) impose the following assumptions (P1)–(P4) on a game $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$.

(P1) There exists a finite set $F$ such that $X_i = F$ for all players $i \in N$.

The set $F$ is called the “facility set” and a strategy for player $i$ is choosing an element of $F$.

(P2) For each strategy profile $x \in X$ and all players $i, j \in N$: if $x_i \neq x_j$ and $x'_j \in X_j$ is such that $x_i \neq x'_j$, then $u_i(x_j, x_{-j}) = u_i(x'_j, x_{-j})$.

Konishi et al. (1997) call this assumption independence of irrelevant choices: for each player $i \in N$ and each strategy profile $x$ the utility of $i$ will not be altered if the set of players that choose the same facility as player $i$ is not modified.

Let $x \in X, f \in F$. Denote as before by $n_f(x)$ the number of users of facility $f$ in the strategy profile $x$. Then the third property can be stated as follows:

(P3) For each player $i \in N$ and all strategy profiles $x, y \in X$ with $x_i = y_i$: if $n_f(x) = n_f(y)$ for all $f \in F$, then $u_i(x) = u_i(y)$.

This anonymity condition reflects the idea that the payoff of player $i$ depends on the number of players choosing the facilities, rather than on their identity. The fourth assumption, called partial rivalry, states that each player $i$ would not regret that other players, choosing the same facility, would select another one. Formally:

(P4) For each player $i \in N$, each strategy profile $x \in X$, each player $j \neq i$ such that $x_j = x_i$ and each $x'_j \neq x_i$: $u_i(x_j, x_{-j}) \leq u_i(x'_j, x_{-j})$.

Although Milchtaich (1996) introduces his model in a slightly different way, the resulting class of games is the same. More specifically Milchtaich (1996) introduces the conditions (P1), (P4) and the following assumption:

(P2') For each player $i \in N$ and all strategy profiles $x, y$ with $x_i = y_i = f$: if $n_f(x) = n_f(y)$, then $u_i(x) = u_i(y)$.

In other words the utility of player $i$ depends only on the number of users of the facility that $i$ has chosen. Assuming (P1), it is straightforward to prove that (P2') implies both (P2) and (P3). The converse implication is also true.

Lemma 4.1. Any game $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ satisfying (P1), (P2) and (P3) satisfies (P2').
Theorem 4.2. For each game satisfying (P1), (P2) and (P3), the set of strong Nash equilibria is non-empty.

Finally, we mention the model introduced by Quint and Shubik (1994), where the assumption that all players have the same set of facilities [as stated by (P1)] is relaxed.

(P1') There exists a finite set $F$ such that $X_i \subseteq F$ for all players $i \in N$.

Assuming that (P1') holds, it is still easy to see that (P2') implies (P2) and (P3). But the analogon of Lemma 4.1 does not hold.

Example 4.1. Take $N = \{1, 2, 3\}$, $F = \{a, b, c\}$ and strategy sets $X_1 = \{a, b\}$, $X_2 = \{a\}$, $X_3 = \{a, c\}$. This game satisfies (P1'). Assumption (P3) imposes no additional requirements and (P2) requires that $u_1(b, a, a) = u_1(b, a, c)$ and $u_3(a, a, c) = u_3(b, a, c)$. This does not imply $u_2(a, a, c) = u_2(b, a, a)$, which is required by (P2').
Quint and Shubik (1994) show:

**Theorem 4.3.** All strategic games satisfying (P1'), (P2') and (P4) possess a pure Nash equilibrium.

Games in the classes considered so far do not necessarily admit a potential function. Consider now the following cross-symmetry condition, which states that the payoffs on a certain facility are player-independent, provided that the number of users is the same.

(P5) For all strategy profiles $x, y \in X$ and all players $i, j \in N$: if $x_i = y_j = f$ and $n_f(x) = n_f(y)$, then $u_i(x) = u_j(y)$.

Notice that (P5) together with (P1) implies (P2'), and thus (P2) and (P3). Moreover, (P1) and (P5) guarantee the existence of a potential.

**Theorem 4.4.** Each game satisfying (P1) and (P5) is an exact potential game.

**Proof.** Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfy (P1) and (P5). For any $f \in F$ and $x, y \in X$ such that $n_f(x) = n_f(y)$, we have by (P5): if there are $i, j \in N$ such that $x_i = y_j = f$, then $u_i(x) = u_j(y)$. This shows that for all $f \in F$ there exists a benefit function $w_f : \{1, \ldots, n\} \to \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. This makes the game $G$ a congestion game as defined in Sec. 3.\(^a\) The result now follows from Proposition 3.1.

**Remark 4.1.** The theorem still holds if (P1') is substituted for (P1). It also follows from Proposition 3.1, that the benefit functions $(w_f)_{f \in F}$ give rise to a potential $P : x \mapsto \sum_{f \in \cup_{i \in N} (x_i)} \sum_{\ell = 1}^{n_f(x)} w_f(\ell)$.

The remainder of this paper focuses on games that admit an exact potential and have strong Nash equilibria. Therefore, attention is restricted to the class $\mathcal{C}$ of congestion games satisfying not only (P1) and (P5), but also (P4). So

$$\mathcal{C} = \{ G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle | G \text{ satisfies (P1), (P4) and (P5)} \}.$$  \hspace{1cm} (1)

**5. On the Structure of the Class $\mathcal{C}$**

We analyse the structure of the class $\mathcal{C}$ defined in (1). Let $n \in \mathbb{N}$ be a number of players and $F$ a finite set of facilities. Denote by $\mathcal{C}(F, n)$ the set of all games in $\mathcal{C}$ with $n$ players and facility set $F$. Identifying each game $G \in \mathcal{C}(F, n)$ with a set of vectors $(w_f)_{f \in F}$ like in the proof of Theorem 4.4, it is shown that $\mathcal{C}(F, n)$ is a finitely generated cone in $(\mathbb{R}^n)_F$. The vector notation of the games simplifies the proofs of the theorems on strong equilibria and the potential maximiser presented in Secs. 6 and 7.

\(^a\)Where we identify choosing a facility $f \in F$ with choosing facility set $\{f\} \subseteq F$. 

---

**Congestion Games and Potentials Reconsidered**

May 11, 2000 9:44 WSPC/151-IGTR 0010
Let $G \in \mathcal{C}(F,n)$. Recall from Theorem 4.4 that for every $f \in F$ there exists a function $w_f : \{1, \ldots, n\} \rightarrow \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. From (P4), we have for each $f \in F$ and $t \in \{1, \ldots, n-1\}$ that $w_f(t) \geq w_f(t + 1)$. For convenience and without loss of generality (by adding a positive constant to all functions $w_f$ if necessary) we assume that $w_f(t) = 0$ for all $f \in F$ and $t \in \{1, \ldots, n-1\}$. This means that the game $G \in \mathcal{C}(F,n)$ is described by $jFj$ vectors of the form $(w_f(1), \ldots, w_f(n))$, $f \in F$, each in the set $V = \{v = (v_1, \ldots, v_n) \in \mathbb{R}_+^n | v_t \geq v_{t+1} \text{ for all } t \in \{1, \ldots, n-1\}\}$.

**Proposition 5.1.** The set $V$ is a finitely generated cone in $\mathbb{R}_+^n$. The extreme directions of $V$ are the vectors $b^1, b^2, \ldots, b^n$ with $b^i = (1,1,1,1,0,\ldots,0)$. Furthermore, $\dim(V) = n$.

**Proof.** The vectors 

$$b^1 = (1,0,0,\ldots,0), b^2 = (1,1,1,1,0,\ldots,0), \ldots, b^n = (1,1,1,1,\ldots,1)$$

are elements of $V$ and each vector $v \in V$ can be uniquely written as a non-negative combination of $b^1, b^2, \ldots, b^n$. To show this, let $v \in V$ and define

$$B_n = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{bmatrix}.$$ 

So $B_n$ is the $n \times n$ matrix whose $i$th row is $b^i$. Since $\det(B_n) = 1$, the equation $\alpha B_n = v$ has exactly one solution. Clearly, $\alpha$ is non-negative because of the decreasingness property of $v$. The set $V$ is therefore the cone $C(B_n)$ where $C(B_n) := \{\alpha B_n \mid \alpha \in \mathbb{R}_+^n\}$.

The extreme directions of the cone $C(B_n)$ are the vectors $b^i$, $i \in \{1, \ldots, n\}$. This cone has furthermore the property that its dimension is the number of extreme directions. In other words we have that $\dim(C(B_n)) = \text{rank}(B_n) = n$. 

This proves:

**Corollary 5.1.** The class of games $\mathcal{C}(F,n)$ can be identified with a cone in $(\mathbb{R}_+^n)^F$ and $\dim(\mathcal{C}(F,n)) = |F| \times n$.

In the next example we consider an extreme game of $\mathcal{C}(F,n)$, i.e. a game with facility set $F$ such that $w_f$ is an extreme direction in the cone $V$ for each $f \in F$.

**Example 5.1.** Let $G$ be a game in $\mathcal{C}(\{f,g\},4)$ such that $w_f = (1,0,0,0)$ and $w_g = (1,1,0,0)$. Nash equilibria are either those strategy profiles in which one of
the players chooses $f$ and the other three $g$, or those in which both facilities are chosen by two players. These situations will be depicted

\[
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]

for the first case and

\[
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]

for the second one, where the numbers in the square boxes indicate the payoff received by each player choosing this facility. Notice furthermore that the players are interchangeable as suggested by the cross-symmetry condition (P5). One easily checks that all Nash equilibria are strong.

6. Strong Nash Equilibria and the Potential Maximiser

In this section it is shown that on the class $C$, the set of Nash equilibria, strong Nash equilibria, and potential maximisers coincide:

Theorem 6.1. On the class $C$ of games, $SNE = NE = PM$.

A proof of this result is given in parts. Recall that for any strategic game $G$, $SNE(G) \subseteq NE(G)$ and that for any exact potential game $G$, $PM(G) \subseteq NE(G)$. It therefore suffices to prove the following propositions.

Proposition 6.1. For each game $G \in C$, $NE(G) \subseteq PM(G)$.

Proposition 6.2. For each game $G \in C$, $NE(G) \subseteq SNE(G)$.

The proofs are based on the structure of the class $C$ described in the previous section. We assume $n \in \mathbb{N}$ and a finite facility set $F$ to be fixed. Each game $G \in \mathcal{C}(F, n)$ is given by a collection of vectors

\[
((w_f(1), \ldots, w_f(n)))_{f \in F},
\]

\[
(w_f(1), \ldots, w_f(n)) \in \{v \in \mathbb{R}^n_+ | v_t \geq v_{t+1} \text{ for all } t \in \{1, \ldots, n-1\}\}.
\]

To compute the potential of Remark 4.1, it is necessary to add the utilities of the used facilities up to the number of users. This means that in each vector $w_f$ all the first $n_f(x)$ numbers are added.

As a consequence it is clear that a potential maximising profile is found by $n$ times consecutively choosing the facilities with highest remaining numbers, from left on, in the set of vectors $\{(w_f(1), \ldots, w_f(n))\}_{f \in F}$. This is illustrated in the following example.
Example 6.1. Let $G \in \mathcal{C}([f, g], 4)$ such that
\[
    w_f = (4, 3, 2, 1)
    \quad w_g = (5, 2, 1, 0).
\]
In the first step we take the first cell in $w_g$, in the second step the first cell in $w_f$, in the third step the second cell of $w_f$, and finally, in the fourth step either the third cell of $w_f$ or the second cell of $w_g$. Consequently, the potential maximising strategy combinations are those $x \in F^N$ with $n_f(x) = 3$, $n_g(x) = 1$ and those with $n_f(x) = 2$, $n_g(x) = 2$. Notice that for these $x$, $P(x) = 14$ and that all Nash equilibria are potential maximising.

Based on a switching argument the next lemma shows the similarities in utilities for different Nash equilibria.

Lemma 6.1. Let $G \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f \in F}$ and let $x$ and $y$ be Nash equilibria of $G$. For all $f, g \in F$ such that $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$, and for all $l \in \{n_f(x) + 1, \ldots, n_f(y)\}$ and $m \in \{n_g(y) + 1, \ldots, n_g(x)\}$ it holds that
\[
    w_f(l) = w_f(n_f(y)) = w_f(n_g(x)) = w_g(m).
\]

Proof. Let $f, g \in F$ and $l, m$ be as described in the lemma. Both $x$ and $y$ are Nash equilibria, so $w_f(n_f(y)) = w_g(n_g(y) + 1) = w_g(m)$.

This lemma is used in the proof of Proposition 6.1.

Proof. [Proposition 6.1] Let $G \in \mathcal{C}(F, n)$ be determined by the set of vectors $((w_f(1), \ldots, w_f(n)))_{f \in F}$. It suffices to show that $P(x) = P(y)$ if $x$ is a Nash equilibrium and $y$ a potential maximising strategy combination. Let $x \in NE(G)$ and $y \in PM(G)$. Facilities $f \in F$ such that $n_f(x) = n_f(y)$ add as much to $P(x)$ as to $P(y)$. Furthermore, by Lemma 6.1, if $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$ for certain $f, g \in F$, then there exists a $w \in \mathbb{R}$ such that $w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m)$ for all $l \in \{n_f(x) + 1, \ldots, n_f(y)\}$ and $m \in \{n_g(y) + 1, \ldots, n_g(x)\}$. By this argument, the total contribution of the facilities in the set $\{f \in F| n_f(x) \neq n_f(y)\}$ to the potentials $P(x)$ and $P(y)$ is the same:

\[
    P(x) - P(y) = \sum_{\{f \in F| n_f(x) > n_f(y)\}} \sum_{k = n_f(y) + 1}^{n_f(x)} w_f(k) - \sum_{\{f \in F| n_f(x) < n_f(y)\}} \sum_{k = n_f(x) + 1}^{n_f(y)} w_f(k)
\]
\[
    = \sum_{\{f \in F| n_f(x) > n_f(y)\}} [n_f(x) - n_f(y)] w - \sum_{\{f \in F| n_f(x) < n_f(y)\}} [n_f(y) - n_f(x)] w
\]
\[ = w \left[ \sum_{\{f \in F | n_f(x) > n_f(y)\}} (n_f(x) - n_f(y)) - \sum_{\{f \in F | n_f(x) < n_f(y)\}} (n_f(y) - n_f(x)) \right] \]
\[ = 0, \]
completing the proof.

Remains to prove Proposition 6.2.

**Proof. [Proposition 6.2].** Let \( G \in \mathcal{C}(F,n) \) be given by \((w_f(1), \ldots, w_f(n))\)_{f \in F} and let \( x \in NE(G) \). Suppose \( S \subseteq N \) can strictly improve the payoff for all its members by switching to a strategy combination \( y_S \in F^S \). Call the resulting strategy combination \( y = (y_S, x_{N\setminus S}) \). If \( n_f(y) > n_f(x) \) for some \( f \in F \), a player \( i \in S \) exists such that \( y_i = f \) and \( x_i = g, g \neq f \). This implies \( w_f(n_f(x) + 1) \geq w_f(n_f(y)) > w_g(n_g(x)) \), which contradicts the fact that \( x \) is a Nash equilibrium. So \( n_f(x) = n_f(y) \) for all \( f \in F \). Therefore every player in \( S \) chooses a new facility already chosen by a member of \( S \) and obtains a higher utility. Among the utilities assigned to members of \( S \) there is a maximum, since \( S \) is finite. Any player in \( S \) rewarded with this maximum cannot get more in the new configuration. Hence a contradiction arises. Every Nash equilibrium is strong.

In the last part of this section we consider strictly strong Nash equilibria. Recall that given a game \( (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \), a strategy profile \( x \in X \) is a strongly Nash equilibrium if for all coalitions \( S \subseteq N \) and strategy combinations \( y_S \in X_S, u_i(y_S, x_{N\setminus S}) = u_i(x) \) for all \( i \in S \) or \( u_i(y_S, x_{N\setminus S}) < u_i(x) \) for at least one \( i \in S \). The following example illustrates that the properties of \( \mathcal{C} \) do not guarantee the existence of strictly strong Nash equilibria.

**Example 6.2.** Consider the game \( G \in \mathcal{C}\{f, g\}, 3 \) with \( w_f, w_g \) given by

\[
\begin{align*}
w_f &= \begin{pmatrix} 4 & 2 & 0 \end{pmatrix}, \\
w_g &= \begin{pmatrix} 3 & 2 & 1 \end{pmatrix},
\end{align*}
\]

where the squared numbers depict a strong Nash equilibrium payoff. If the two players choosing \( f \) agree that one of them switches to \( g \) and the other one sticks to \( f \), the utility will still be 2 for the switching one but increases from 2 to 4 for the remaining player. A similar argument holds for the other type of strong Nash equilibria given by

\[
\begin{align*}
w_f &= \begin{pmatrix} 4 & 2 & 0 \end{pmatrix}, \\
w_g &= \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}
\end{align*}
\]

Since these are the only two types of strong Nash equilibria, and neither of them is strictly strong, strictly strong Nash equilibria do not exist.
7. Extensions of the Model

The class $\mathcal{C}$ is defined by properties (P1), (P4) and (P5). It is obvious that relaxation of those properties will have consequences on the result presented in Sec. 6.

First of all, the classes of congestion games of Quint and Shubik (1994), Milchtaich (1996) and Konishi et al. (1997) without (P5) do not necessarily admit an exact potential.

Secondly, consider the class $\mathcal{CP}$ of strategic games which satisfy the properties (P1) and (P5). Each $n$-player game $G$ in $\mathcal{CP}$ is a potential game and can be represented by a collection of arbitrary vectors $((w_f(1), \ldots, w_f(n)))_{f \in F} \in (\mathbb{R}^n)^F$. It is obvious that not every game $G \in \mathcal{CP}$ has a strong Nash equilibrium. For instance, the Prisoner’s Dilemma in Example 2.1 is an element of $\mathcal{CP}$ with $F = \{c,d\}$, $w_c = (4,1)$ and $w_d = (0,3)$, but does not have a strong Nash equilibrium. But even the existence of a strong Nash equilibrium for a game $G \in \mathcal{CP}$ does not guarantee that each Nash equilibrium is strong too, nor that a strong equilibrium is a potential maximiser. The next example gives a game $G \in \mathcal{CP}$ such that $\emptyset \neq SNE(G) \subset NE(G)$ and $SNE(G) \cap PM(G) = \emptyset$.

**Example 7.1.** Let $G \in \mathcal{CP}(\{f,g\}, 3)$ with

$$w_f = (4,0,5)$$

$$w_g = (4,2,0).$$

The unique strong Nash equilibrium in which all three players chooses facility $f$ is indicated. By Theorem 4.4, the potential can be computed as in Remark 4.1. The maximal potential arises at the non-strong equilibria which are given by

$$w_f = (4,0,5)$$

$$w_g = (4,2,0).$$

Finally, consider the class of strategic games $\mathcal{C}'$ satisfying (P1'), (P4) and (P5). Similarly to Proposition 6.2 one can show:

**Theorem 7.1.** For every game $G \in \mathcal{C}'$, $NE(G) = SNE(G)$.

This result coincides with that of Holzman and Law-Yone (1997, Theorem 2.1). In the class $\mathcal{C}'$, however, the set of potential maximising strategy combinations need not coincide with the set of Nash equilibria, as can be seen in the following example.

**Example 7.2.** Consider the game $G \in \mathcal{C}'(\{f,g,h\}, 5)$ in which three players have strategy set $\{f,h\}$ and two $\{g,h\}$. The benefit vectors are

$$w_f = (4,2,1,-,-)$$

$$w_g = (3,2,-,-,-)$$

$$w_h = (2,1,1,0,0).$$
where the squared numbers depict a Nash equilibrium payoff. It represents strategy combinations in which the three players with strategy set \{f, h\} all play \(f\). Consider now the equilibrium in which two of those three play \(f\) and the other plays \(h\).

\[
\begin{align*}
w_f &= (4, 2, 1, -), \quad w_g = (3, 2, -, -), \quad w_h = (2, 1, 0, 0)
\end{align*}
\]

The potential can be computed as in Remark 4.1. For the first type of equilibrium in this example, the potential value equals \(4 + 2 + 1 + 3 + 2 = 12\), which is less than \(4 + 2 + 3 + 2 + 2 = 13\), the potential value associated to the second type of equilibrium.

References


