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Introduction to ARCH & GARCH models

Recent developments in financial econometrics suggest the use of nonlinear time series structures to model the attitude of investors toward risk and expected return. For example, Bera and Higgins (1993, p.315) remarked that "a major contribution of the ARCH literature is the finding that apparent changes in the volatility of economic time series may be predictable and result from a specific type of nonlinear dependence rather than exogenous structural changes in variables."

Campbell, Lo, and MacKinlay (1997, p.481) argued that "it is both logically inconsistent and statistically inefficient to use volatility measures that are based on the assumption of constant volatility over some period when the resulting series moves through time." In the case of financial data, for example, large and small errors tend to occur in clusters, i.e., large returns are followed by more large returns, and small returns by more small returns. This suggests that returns are serially correlated.

When dealing with nonlinearities, Campbell, Lo, and MacKinlay (1997) make the distinction between:

- **Linear Time Series**: shocks are assumed to be uncorrelated but not necessarily identically independent distributed (iid).
- **Nonlinear Time Series**: shocks are assumed to be iid, but there is a nonlinear function relating the observed time series $\{X_t\}_{t=0}^{\infty}$ and the underlying shocks, $\{\varepsilon_t\}_{t=0}^{\infty}$.

They suggest the following structure to describe a nonlinear process:

$$
\begin{aligned}
X_t &= g(\varepsilon_{t-1}, \varepsilon_{t-2}, ...) + \varepsilon_t h(\varepsilon_{t-1}, \varepsilon_{t-2}, ...) \\
E[X_t|\Psi_{t-1}] &= g(\varepsilon_{t-1}, \varepsilon_{t-2}, ...) \\
Var[X_t|\Psi_{t-1}] &= E[\{(X_t - E[X_t])|\Psi_{t-1}\}^2] \\
&= E[\{\varepsilon_t h(\varepsilon_{t-1}, \varepsilon_{t-2}, ...)|\Psi_{t-1}\}^2] \\
&= Var[\varepsilon_t h(\varepsilon_{t-1}, \varepsilon_{t-2}, ...)|\Psi_{t-1}] \\
&= \{h(\varepsilon_{t-1}, \varepsilon_{t-2}, ...)\}^2
\end{aligned}
\tag{1}
$$

where the function $g(\cdot)$ corresponds to the conditional mean of X_t, and the function $h(\cdot)$ is the coefficient of proportionality between the innovation in X_t and the shock ε_t.

The general form above leads to a natural division in Nonlinear Time Series literature in two branches:

- **Models Nonlinear in Mean**: $g(\cdot)$ is nonlinear;
- **Models Nonlinear in Variance**: $h(\cdot)^2$ is nonlinear.

According to the authors, most of the time series studies concentrate in one form or another. As examples, they mention

- Nonlinear Moving Average Model: $X_t = \varepsilon_t + \alpha\varepsilon_{t-1}^2$. Here the function $g = \alpha\varepsilon_{t-1}^2$ and the function $h = 1$. Thus, it is nonlinear in mean but linear in variance.
- Engle's (1982) ARCH Model: $X_t = \varepsilon_t\sqrt{\alpha\varepsilon_{t-1}^2}$. The process is nonlinear in variance but linear in mean. The function $g(\cdot) = 0$ and the function $h = \sqrt{\alpha\varepsilon_{t-1}^2}$.

Given such motivations, Engle (1982) proposed the following model to capture serial correlation in volatility:

$$\sigma^2 = \omega + \alpha(L)\eta_t^2 \tag{2}$$

where $\alpha(L)$ is the polynomial lag operator, and $\eta_t|\Psi_{t-1} \sim N(0, \sigma_{t-1}^2)$ is the innovation in the asset return. Bera and Higgins (1993) explained that "the ARCH model characterizes the distribution of the stochastic error ε_t conditional on the realized values of the set of variables $\Psi_{t-1} = \{y_{t-1}, x_{t-1}, y_{t-2}, x_{t-2}, ...\}$.

Computational problems may arise when the polynomial presents a high order. To facilitate such computation, Bollerslev (1986) proposed a Generalized Autorregressive Conditional Heteroskedasticity (GARCH) model,

$$\sigma_t^2 = \omega + \beta(L)\sigma_{t-1}^2 + \alpha(L)\eta_t^2 \tag{3}$$

It is quite obvious the similar structure of Autorregressive Moving Average (ARMA) and GARCH processes: a GARCH (p, q) has a polynomial $\beta(L)$ of order "p" - the autorregressive term, and a polynomial $\alpha(L)$ of order "q" - the moving average term.

Properties and Interpretations of ARCH Models

Following Bera and Higgins (1993), two important concepts should be introduced at this point:

Definition 1 *(Law of Iterated Expectations)*: Let Ω_1 and Ω_2 be two sets of random variables such that $\Omega_1 \subseteq \Omega_2$. Let Y be a scalar random variable. Then, $E[Y|\Omega_1] = E[E[Y|\Omega_2]|\Omega_1]$.

Note *(Conditionality versus Inconditionality)*: If $\Omega_1 = \varnothing$, then $E[E[Y|\Omega_2]] = E[Y]$.

Without loss of generality, let a ARCH (1) process be represented by

$$u_t = \varepsilon_t\sqrt{\alpha_0 + \alpha_1 u_{t-1}^2} \tag{4}$$

where $\{\varepsilon_t\}_{t=0}^{\infty}$ is a white noise stochastic process. Johnston and DiNardo (1997) briefly mention the following properties of ARCH models:

- u_t have mean zero.
  Proof:

$$\begin{array}{lcl} u_t & = & \varepsilon_t\sqrt{\alpha_0 + \alpha_1 u_{t-1}^2} \\ E_{t-1}[u_t] & = & \underbrace{E_{t-1}[\varepsilon_t]}\sqrt{\alpha_0 + \alpha_1 u_{t-1}^2} \\ & = & 0 \\ E_{t-2}E_{t-1}[u_t] & = & 0 \\ & (...) & \\ E[u_t] & = & 0 \end{array} \tag{5}$$

- u_t have conditional variance given by $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2$.
  Proof:

$$\begin{aligned} u_t^2 &= \varepsilon_t^2[\alpha_0 + \alpha_1 u_{t-1}^2] \\ E_{t-1}[u_t^2] &= \sigma_\varepsilon^2[\alpha_0 + \alpha_1 u_{t-1}^2] \\ &= 1[\alpha_0 + \alpha_1 u_{t-1}^2] \\ &= \sigma_t^2 \end{aligned} \quad (6)$$

- u_t have unconditional variance given by $\sigma^2 = \frac{\alpha_0}{1-\alpha_1}$.
  Proof:

$$\begin{aligned} E_{t-2}E_{t-1}[u_t^2] &= E_{t-2}[\alpha_0 + \alpha_1 u_{t-1}^2] \\ &= \alpha_0 + \alpha_1 E_{t-2}[u_{t-1}^2] \\ &= \alpha_0 + \alpha_0\alpha_1 + \alpha_1^2 u_{t-2}^2 \\ \\ E_{t-3}E_{t-2}E_{t-1}[u_t^2] &= E_{t-3}[\alpha_0 + \alpha_0\alpha_1 + \alpha_1^2 u_{t-2}^2] \\ &= \alpha_0 + \alpha_0\alpha_1 + \alpha_1^2 E_{t-3}[u_{t-2}^2] \\ &= \alpha_0 + \alpha_0\alpha_1 + \alpha_0\alpha_1^2 + \alpha_1^3 u_{t-3} \\ \\ (...) \\ \\ E_0E_1E_2(...)E_{t-2}E_{t-1}[u_t^2] &= \alpha_0(1 + \alpha_1 + \alpha_1^2 + ... + \alpha_1^{t-1}) + \alpha_1^t u_0^2 \\ &= \tfrac{\alpha_0}{1-\alpha_1} \\ &= \sigma^2 \end{aligned} \quad (7)$$

  Therefore, unconditionally the process is **Homoskedastic**.

- u_t have zero-autocovariances.
  Proof:

$$E_{t-1}[u_t u_{t-1}] \;=\; u_{t-1}E_{t-1}[u_t] \;=\; 0 \quad (8)$$

Regarding kurtosis, Bera and Higgins (1993) show that the process has a heavier tail than the Normal distribution, given that

$$\frac{E[\varepsilon_t^4]}{\sigma_\varepsilon^4} = 3(\frac{1-\alpha_1^2}{1-3\alpha_1^2}) > 3 \quad (9)$$

Heavy tails are a common aspect of financial data, and hence the ARCH models are so popular in this field. Besides that, Bera and Higgins (1993) mention the following reasons for the ARCH success:

- ARCH models are simple and easy to handle
- ARCH models take care of clustered errors
- ARCH models take care of nonlinearities
- ARCH models take care of changes in the econometrician's ability to forecast

In fact, the last aspect was pointed by Engle (1982) as a "random coefficients" problem: the power of forecast changes from one period to another.

In the history of ARCH literature, interesting interpretations of process can be found. E.g.:

- Lamoureux and Lastrapes(1990). They mention that the conditional heteroskedasticity may be caused by a time dependence in the rate of information arrival to the market. They use the daily trading volume of stock markets as a proxy for such information arrival, and confirm its significance.
- Mizrach (1990). He associates ARCH models with the errors of the economic agents' learning processes. In this case, contemporaneous errors in expectations are linked with past errors in the same expectations, which is somewhat related with the old-fashioned "adaptable expectations hypothesis" in macroeconomics.
- Stock (1998). His interpretation may be summarized by the argument that "any economic variable, in general, evolves an on 'operational' time scale, while in practice it is measured on a 'calendar' time scale. And this inappropriate use of a calendar time scale may lead to volatility clustering since relative to the calendar time, the variable may evolve more quickly or slowly" (Bera and Higgins, 1990, p. 329; Diebold, 1986].

Estimating and Testing ARCH Models

Johnston and DiNardo (1997) suggest a very simple test for the presence of ARCH problems. The basic menu (step-by-step) is:

- Regress y on x by OLS and obtain the residuals $\{\varepsilon_t\}$.

- Compute the OLS regression $\varepsilon_t^2 = \hat{\alpha_0} + \hat{\alpha_1}\varepsilon_{t-1}^2 + ... + \hat{\alpha_p}\varepsilon_{t-p}^2 + error$.

- Test the joint significance of $\hat{\alpha_1}, ..., \hat{\alpha_p}$.

In case that any of the coefficients are significant, a straight-forward method of estimation (correction) is provided by Greene (1997). It consists in a four-step FGLS:

- Regress y on x using least squares to obtain $\hat{\beta}$ and ε vectors.

- Regress ε_t^2 on a constant and ε_{t-1}^2 to obtain the estimates of α_0 and α_1, using the whole sample (T). Denote $[\hat{\alpha_0}, \hat{\alpha_1}] = \overrightarrow{\alpha}$.

- Compute $f_t = \hat{\alpha_0} + \hat{\alpha_1}\varepsilon_{t-1}^2$. Then compute the **asymptotically efficient** estimate $\hat{\alpha}$, $\tilde{\alpha} = \overrightarrow{\alpha} + d_\alpha$, where d_α is the least squares coefficient vector in the regression

$$[(\frac{\varepsilon_t^2}{f_t}) - 1] = \hat{z}_0(\frac{1}{f_t}) + \hat{z}_1(\frac{\varepsilon_{t-1}^2}{f_t}) + error \tag{10}$$

  The asymptotic covariance matrix for $\tilde{\alpha}$ is $2(\hat{z}'\hat{z})^{-1}$, where $\hat{z}$ is the regressor vector in this regression.

- Recompute f_t using $\tilde{\alpha}$; then compute

$$\begin{aligned} r_t &= [\tfrac{1}{f_t} + 2(\tfrac{\tilde{\alpha_1}\varepsilon_t}{f_{t+1}})^2]^{1/2} \\ s_t &= \tfrac{1}{f_t} - \tfrac{\tilde{\alpha_1}}{f_{t+1}}[\tfrac{\varepsilon_{t+1}^2}{f_{t+1}} - 1] \end{aligned} \tag{11}$$

  Compute the estimate $\tilde{\beta} = \overrightarrow{\beta} + d_\beta$, where d_β is the least squares coefficient vector in the regression

$$[\frac{\varepsilon_t s_t}{r_t}] = \hat{w}x_t r_t + error \tag{12}$$

The **asymptotic** covariance matrix for $\tilde{\beta}$ is given by $(\hat{w}'\hat{w})^{-1}$, where $\hat{w}$ is the regressor vector on the equation above.

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