Multiple Equilibrium under CES Preferences

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1 Introduction

Since the time of Wald [19] and Arrow, Block, and Hurwicz [2], the standard sufficiency condition used to prove uniqueness of general exchange equilibrium has been that of gross substitutability introduced by Metzler [15] (cf. [13, p. 613]). Other sufficient conditions have been introduced by Arrow and Hahn [3, Ch. 9], and further results have been obtained by Kehoe [9], who introduced production adjustments of the activity-analysis type, showing that with at least four commodities and four agents, multiple equilibrium is possible even with gross substitutability. In the two-commodity two-agent model of pure exchange, however, few results appear to have been obtained for the standard functional forms of utility functions used in econometric research. The particular case of consumption in fixed proportions has been analyzed by Mas-Colell [12]. In the case of preferences represented by CES (constant elasticity of substitution) utility functions, it is known (e.g. [4, p. 725–7]) that gross substitutability implies that the elasticity of substitution satisfies $\sigma \geqq 1$.

In this paper I show using elementary methods that in the simple case of two agents and two commodities, and under an assumption that I call supersymmetry (the agents' preferences and endowments are mirror images of one another), a sufficient condition for uniqueness of general equilibrium is the weaker property $\sigma \geqq 1/2$ (Theorem 1). I also show (Theorem 2) that even under those conditions (and this has been remarked upon in another context in [12, p. 285]), preferences must favor the good that is sold (in international trade, the export good) in order for multiple equilibrium to be possible.

The problem of multiple equilibrium has received considerable attention attention in the theory of international trade ([4, p. 735]), owing to the fact that among three equilibria, an egalitarian distribution of welfare between two countries occurs only in the unstable equilibrium, the stable ones necessarily favoring one country over the other. In recent work by Zhou [21] and Wan and Zhou [20], who postulate mirror-image quasilinear-quadratic utility functions for two trading countries, these

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authors argue that the possibility of multiple equilibrium under free trade can result in both countries preferring a unique optimal-tariff Johnson-Nash equilibrium to the prospect of an uncertain stable free-trade equilibrium, since the former will have a more favorable outcome for each party than the worse of the two stable free-trade solutions. This work has provided the main stimulus for the present paper, which is to explore whether their results can be extended to the case of CES preferences, which have played a prominent role in the so-called computable general equilibrium (CGE) models of international trade (e.g. [14]). In view of this, I adopt the terminology of international trade in which "countries" take the place of "agents", and (fixed) production levels take the place of endowments. It is of course assumed that consumers in each country have identical CES preferences of the specified types.

2 CES preferences and competitive equilibrium

Maximization of country k's utility function $U_k(x_{1k}, x_{2k}) = (\alpha_{1k}x_{1k}^{\rho} + \alpha_{2k}x_{2k}^{\rho})^{1/\rho}$ subject to $p_1x_{1k} + p_2x_{2k} \leqq Y_k$ (where Y_k is country k's disposable national income, and where $\rho = 1-1/\sigma$ and σ is the constant elasticity of substitution), entails setting the ratio of marginal utilities to the price ratio. The marginal utility of commodity i to consumers in country k is

$$\frac{\partial U_k}{\partial x_{ik}} = (\alpha_{1k}x_{1k}^{\rho} + \alpha_{2k}x_2^{\rho})^{1/\rho-1}\,\alpha_{ik}x_{ik}^{\rho-1},$$

hence

$$\frac{\partial U_k/\partial x_{1k}}{\partial U_k/\partial x_{2k}} = \frac{\alpha_{1k}}{\alpha_{2k}}\left(\frac{x_{1k}}{x_{2k}}\right)^{\rho-1} = \frac{p_1}{p_2}.$$

From this last equality we obtain (recalling that $\rho = 1 - 1/\sigma$)

$$\frac{x_{1k}}{x_{2k}} = \left(\frac{\alpha_{2k}}{\alpha_{1k}}\cdot\frac{p_1}{p_2}\right)^{-\sigma}.$$

Now, combining this with equality in the budget constraint we obtain country k's demand function for commodity i as a function of the two prices and disposable income,

$$x_{ik} = h_{ik}(p_1, p_2, Y_k) = \frac{\alpha_{ik}^{\sigma}p_i^{-\sigma}Y_k}{\alpha_{1k}^{\sigma}p_1^{1-\sigma} + \alpha_{2k}^{\sigma}p_2^{1-\sigma}}. \tag{2.1}$$

It will be assumed that country k's supply function $y_{ik} = \hat{y}_{ik}(p_1, p_2)$ is a constant ω_{ik}, and that trade is balanced. Country k's *excess-demand function* for commodity i is given by

$$z_{ik} \equiv x_{ik} - y_{ik} = \hat{z}_{ik}(p_1, p_2; \omega_k, \alpha_k, \sigma) = \frac{\alpha_{ik}^{\sigma}p_k^{-\sigma}(p_1\omega_{1k} + p_2\omega_{2k})}{\alpha_{1k}^{\sigma}p_1^{1-\sigma} + \alpha_{2k}^{\sigma}p_2^{1-\sigma}} - \omega_{ik}. \tag{2.2}$$

It will also be assumed that the parameters are such that, in equilibrium, country k exports commodity k to country $j \neq k$. In the case of country 2's excess demand for commodity 1 (its import good), formula (2.2) reduces to

$$(2.3) \qquad z_{12} = \frac{p_1\omega_{12} + p_2\omega_{22}}{p_1 + (\alpha_{22}/\alpha_{12})^\sigma (p_1)^\sigma} - \omega_{12} = \frac{p_2\omega_{22} - (\alpha_{22}/\alpha_{12})^\sigma p_1^\sigma p_2^{1-\sigma}\omega_{12}}{p_1 + (\alpha_{22}/\alpha_{12})^\sigma p_1^\sigma}.$$

3 World equilibrium and supersymmetry

In the special case $\alpha_{ik} = 1$ for $i, k = 1, 2$, from (2.2) we have for the world excess demand for commodity i,

$$z_{i1} + z_{i2} = \frac{p_1(\omega_{11} + \omega_{12}) + p_2(\omega_{21} + \omega_{22})}{p_i^\sigma (p_1^{1-\sigma} + p_2^{1-\sigma})} - (\omega_{i1} + \omega_{i2}).$$

Setting this equal to zero for world equilibrium, we obtain

$$p_i^\sigma(\omega_{i1} + \omega_{i2}) = \frac{p_1(\omega_{11} + \omega_{12}) + p_2(\omega_{21} + \omega_{22})}{p_1^{1-\sigma} + p_2^{1-\sigma}} \quad \text{for } i = 1, 2.$$

It follows that

$$\frac{p_1}{p_2} = \left(\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}}\right)^{1/\sigma}.$$

Assuming that $\omega_{11} + \omega_{12} = \omega_{21} + \omega_{22}$, the equilibrium price ratio is $p_1/p_2 = 1$. Then (2.2) yields $z_{21} = z_{12} = 1/2$.

Relaxing the requirement of equal consumption coefficients α_{ik}, the following *supersymmetry* conditions (with apologies to the cosmologists for the terminology) also lead to a unit equilibrium price ratio, namely that each matrix

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \quad \text{and} \quad \mathrm{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

(a) is symmetric and (b) has equal diagonal elements. If the material-balance conditions $z_{i1} + z_{i2} = 0$ are to be satisfied at a unit price ratio, it follows from (2.2) that

$$(3.4) \quad \frac{\omega_{11} + \omega_{21}}{(\alpha_{i1})^{-\sigma}[(\alpha_{11})^\sigma + (\alpha_{21})^\sigma]} + \frac{\omega_{12} + \omega_{22}}{(\alpha_{i2})^{-\sigma}[(\alpha_{12})^\sigma + (\alpha_{22})^\sigma]} = \omega_{i1} + \omega_{i2} \quad \text{for } i = 1, 2.$$

From the supersymmetry of Ω, each of the numerators on the left is equal to the expression on the right, so they may be cancelled out. From the supersymmetry of A, whose elements will also be assumed to be positive, we have $\alpha_{11}/\alpha_{21} = \alpha_{22}/\alpha_{12} = \alpha$, say. Then (3.4) reduces to the identity

$$(3.5) \qquad \frac{1}{1 + \alpha^\sigma} + \frac{1}{1 + \alpha^{-\sigma}} = 1,$$

whose validity is easily verified. Thus, *under supersymmetry there will always exist a world equilibrium at a unit price ratio.*

Given our assumption of supersymmetry, the notation in the remainder of this paper will be simplified to $\omega_{ii} = \omega$ for $i = 1, 2$ and $\omega_{ij} = \delta$ for $i \neq j$. To give each country k a comparative advantage in commodity k, I shall assume throughout that $\omega > \delta$. And since the scale of world equilibrium is immaterial to the analytic results, we may assume that $\delta = 1$ in the case of diversification, and of course $\delta = 0$ in the case of specialization. Moreover, since consumer preferences are affected only by the ratios of the α_{ij}s, we shall denote $\alpha_{kk} = \alpha$ and $\alpha_{ij} = 1$ for $i \neq j$.

With this notation, we have

$$\Omega = \begin{bmatrix} \omega & \delta \\ \delta & \omega \end{bmatrix} \quad \text{and} \quad \mathrm{A} = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}, \quad \text{where } \omega > \delta,\ \delta = 1 \text{ or } 0, \text{ and } \alpha > 0,$$

and denoting $r_1 = p_1/p_2$ we may write the equation of world equilibrium as

$$\frac{\omega r_1 + \delta}{r_1 + \alpha^{-\sigma} r_1^{\sigma}} + \frac{\delta r_1 + \omega}{r_1 + \alpha^{\sigma}} = \omega + \delta. \tag{3.6}$$

4 Marshallian offer functions and the stability of the symmetric equilibrium

Country 2's offer function, expressing its export quantity as a function of its import quantity, is defined as follows. In terms of the relative price $r_1 = p_1/p_2$, in the notation of the previous section (and assuming supersymmetry), country 2's excess demand (2.3) for its import good (commodity 1) becomes (dropping the parametric arguments for convenience)

$$z_{12} = \hat{z}_{12}(r_1) = \frac{\omega - \delta \alpha^{\sigma} r_1^{\sigma}}{r_1 + \alpha^{\sigma} r_1^{\sigma}}. \tag{4.7}$$

Country 2's *inverse excess-demand function* $\hat{r}_1(z_{12})$ is then defined as the solution of the equation[1]

$$f_2(z_{12}) = (\delta + z_{12})[\hat{r}_1(z_{12}) + \alpha^{\sigma} \hat{r}_1(z_{12})^{\sigma}] - \delta \hat{r}_1(z_{12}) - \omega = 0. \tag{4.8}$$

This is well defined provided

$$J(r_1) = \frac{d\hat{z}_{12}}{dr_1} = \frac{(1-\sigma)\alpha^{\sigma} r_1^{\sigma} \delta - (1 + \alpha^{\sigma} \sigma r_1^{\sigma-1})\omega}{(r_1 + \alpha^{\sigma} r_1^{\sigma})^2} \neq 0. \tag{4.9}$$

In particular, $J(r_1) < 0$ (the law of demand holds for the excess-demand function (4.7)) if either $\sigma \geqq 1$ (the commodities are weak gross substitutes) or $\delta = 0$ (the countries specialize). In the case of diversification ($\delta = 1$), one must choose the

[1]It may be noted that a zero price ratio $r_1 = 0$ cannot be a solution to this equation, hence the example in Mas-Colell [12, p. 285, Figure 4(b)] of zero prices in stable equilibria under consumption in fixed proportions does not carry over to CES preferences with low elasticity of substitution. On the other hand, as Figures 3 and 6 below illustrate, under multiple equilibrium, stable CES equilibria will typically be characterized by extremely low price ratios.

parameters ω, α, and σ so as to assure that the preference ($\alpha > 1$) for the export good does not counterbalance comparative advantage ($\omega > 1$) so much as to cause each country to import the good in which it has a comparative advantage. From (4.7) this requires $\omega > \alpha^\sigma r_1^\sigma$. Assuming this to be the case, it follows that $J(r_1) < 0$. In the case of the symmetric equilibrium price ratio $r_1 = 1$ this requires $\omega > \alpha^\sigma$. For example, if $\alpha = 1024$ and $\sigma = .1$, $J(1) < 0$ requires $\omega > 2$.

Differentiating (4.8) with respect to z_{12} we obtain

$$\frac{d\hat{r}_1}{dz_{12}} = -\frac{\hat{r}_1(z_{12}) + \alpha^\sigma \hat{r}_1(z_{12})^\sigma}{(\delta + z_{12})[1 + \sigma\alpha^\sigma \hat{r}_1(z_{12})^{\sigma-1}] - \delta}. \tag{4.10}$$

Evaluated at the symmetric equilibrium point, given by (4.7) for $r_1 = 1$, this is

$$\left.\frac{d\hat{r}_1}{dz_{12}}\right|_{z_{12}=\frac{\omega-\delta\alpha^\sigma}{1+\alpha^\sigma}} = -\frac{(1+\alpha^\sigma)^2}{\omega(1+\sigma\alpha^\sigma) + (\sigma-1)\delta\alpha^\sigma}. \tag{4.11}$$

The *Marshallian offer function* for country 2 is now defined by

$$-z_{22} = F_2(z_{12}) \equiv z_{12}\hat{r}_1(z_{12}), \tag{4.12}$$

expressing the quantity of its exports as a function of the quantity of its imports. Its derivative at the symmetric equilibrium point is

$$\gamma_2 \equiv \left.\frac{dF_2}{dz_{12}}\right|_{z_{12}=\frac{\omega-\delta\alpha^\sigma}{1+\alpha^\sigma}} = 1 - \frac{(\omega-\delta\alpha^\sigma)(1+\alpha^\sigma)}{\omega(1+\sigma\alpha^\sigma) + (\sigma-1)\delta\alpha^\sigma}. \tag{4.13}$$

By supersymmetry, exactly the same formula holds for the slope γ_1 of country 1's offer function dF_1/dz_{21} evaluated at the same expression for z_{21}.

The Marshall-Samuelson *non-tâtonnement* dynamic-adjustment process ([10], [17, p. 266]) is

$$\dot{z}_{12} = F_1(z_{21}) - z_{12} \equiv P_1(z_{12}, z_{21}), \quad \dot{z}_{21} = F_2(z_{12}) - z_{21} \equiv P_2(z_{12}, z_{21}). \tag{4.14}$$

From a theorem of Liapunov (cf. [1, pp. 272–3, 279]), the stability of a process such as (4.14) in the neighborhood of a stationary point may be determined under certain circumstances by that of the first-order Taylor approximation to the P_i around that point. In particular, we examine the stability of (4.14) around the symmetric equilibrium values $\bar{z}_{ij} = (\omega - \delta\alpha^\sigma)/(1+\alpha^\sigma) > 0$ for $i \neq j$ given by (4.7) at $r_1 = 1$. Defining $u_i = z_{ij} - \bar{z}_{ij}$ $(i \neq j)$, as well as $\gamma_1 = dF_1/dz_{21}$ and $\gamma_2 = dF_2/dz_{12}$ at this equilibrium point, we obtain the approximating dynamical system

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -1 & \gamma_1 \\ \gamma_2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.15}$$

with characteristic roots λ and distribution coefficients ν satisfying

$$\begin{bmatrix} \lambda+1 & -\gamma_1 \\ -\gamma_2 & \lambda+1 \end{bmatrix} \begin{bmatrix} 1 \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.16}$$

namely $\lambda = -1 \pm \sqrt{\gamma_1\gamma_2}$ and $\nu = \pm\sqrt{\gamma_2/\gamma_1}$ (cf. [1, p. 258], [6, p. 935]). Equilibrium of this system is attained at $(u_1, u_2) = (0, 0)$. The conditions imposed by Liapunov's theorem are that the real parts of the roots λ must not be zero; if one of them is zero, the stability of the equilibrium cannot be determined by this method.

Asymptotic stability follows if both roots have negative real parts, which are -1 if the roots are complex, and real if the γ_k have the same sign. In the latter case, stability is assured if $\gamma_1\gamma_2 < 1$. Geometrically, referring the slopes of both of the offer curves to the z_{12} axis, this sufficient condition for asymptotic stability may be written $|\gamma_2| < 1 < 1/|\gamma_1|$, i.e., country 2's offer curve is less steep (in absolute terms) than country 1's (cf. [11, p. 353]). If the γ_k are both negative (the only case in which multiple equilibrium is possible), the relevant stability condition is $\gamma_2 > -1 > 1/\gamma_1$.

With supersymmetry, $\gamma_1 = \gamma_2 = \gamma$, hence $\lambda = -1 \pm \gamma$ and $\nu = \pm 1$. The system (4.16) is satisfied by each of the two pairs $(\lambda, \nu) = (-\gamma-1, -1)$ and $(\lambda, \nu) = (\gamma-1, 1)$. Accordingly, the solution of (4.15) may be written

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} b_1 e^{-(\gamma+1)t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} b_2 e^{(\gamma-1)t}, \tag{4.17}$$

where the b_i are constant parameters (cf. [6, p. 935]). The symmetric equilibrium is then a stable node if $-1 < \gamma < 1$ and an unstable saddle point if $\gamma < -1$. Stability in the case of "neutral equilibrium" ($\gamma = -1$) cannot be determined by Liapunov's above ("first") method.[2] However, this does not mean that it cannot be determined. In fact, it is determined by Liapunov's "second method" (cf. [7, pp. 192–209]):

Lemma. *Under supersymmetry, the symmetric equilibrium is asymptotically stable for* $-1 \leqq \gamma < 1$.

Proof. By Liapunov's main theorem, an equilibrium at $u = 0$ is stable if, for a positive-definite Liapunov function $V(u) > 0$ for $u \neq 0$ and $V(0) = 0$, its total derivative $\dot V(u) = \sum \partial V/\partial u_i \dot u_i$ is non-positive along any trajectory in a neighborhood U of $u = 0$, and asymptotically stable if $\dot V(u) < 0$ in $U\backslash 0$.

Let us take the simplest case $V(u) = u_1^2 + u_2^2$. Then from (4.15),

$$\tfrac{1}{2}\dot V(u) = -(u_1^2 - 2\gamma u_1 u_2 + u_2^2) \leqq -(u_1 + u_2)^2 < 0 \quad \text{for } -1 \leqq \gamma.$$

The asymptotic stability for $0 \leqq \gamma < 1$ follows from (4.17). □

It follows from this lemma that for competitive equilibrium with CES utility functions under supersymmetry, asymptotic stability is equivalent to uniqueness.

In the following development, it will be convenient to deal with the function

$$\psi(\sigma; \omega, \delta, \alpha) = \frac{(\omega - \delta\alpha^\sigma)(1 + \alpha^\sigma)}{\omega(1 + \sigma\alpha^\sigma) + (\sigma - 1)\delta\alpha^\sigma}. \tag{4.18}$$

Since $\psi = 1 - \gamma$, the sufficient condition $-1 \leqq \gamma < 1$ for asymptotic stability may be written simply as $0 < \psi \leqq 2$.

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[2] The same is true of the other neutral equilibrium given by $\gamma = 1$, but this case has been ruled out by the above assumption $\omega > \delta\alpha^\sigma$ (cf. (4.9) and (4.13)).

Theorem 1. *A sufficient condition for asymptotic stability (hence uniqueness) of the symmetric equilibrium, under the hypothesis of supersymmetry, is that* $\sigma \geqq 1/2$.

Proof. We are to show that $\sigma \geqq 1/2$ implies $\psi \leqq 2$ (i.e., $\gamma \geqq -1$), or conversely, that $\psi > 2$ ($\gamma < -1$) implies $\sigma < 1/2$.

Let us first consider the case of specialization ($\delta = 0$). Then

$$\psi = (1 + \alpha^{\sigma})/(1 + \sigma\alpha^{\sigma}) > 2 \tag{4.19}$$

if and only if $(1 - 2\sigma)\alpha^{\sigma} > 1$, which implies $\sigma < 1/2$.

Next we consider case of diversification ($\delta > 0$); without loss of generality we take $\delta = 1$. First we observe that if $\sigma \geqq 1/2$, then since $\omega > 1$ by hypothesis, the denominator of (4.18) is positive—in fact > 1:

$$\omega(1 + \sigma\alpha^{\sigma}) + (\sigma - 1)\alpha^{\sigma} > 1 + 2(\sigma - 1/2)\alpha^{\sigma} \geqq 1.$$

Now to show that $\sigma \geqq 1/2$ implies $\psi \leqq 2$, suppose by way of contradiction that $\sigma \geqq 1/2$ as well as $\psi > 2$; then by cross-multiplication in the formula (4.18) we obtain

$$\omega + \alpha^{2\sigma} + 2(\omega + 1)(\sigma - 1/2)\alpha^{\sigma} < 0,$$

which is impossible. □

Theorem 2. *Multiple or neutral equilibrium ($\psi \geqq 2$) is possible only if the trading parties have a relative preference for their export goods ($\alpha > 1$).*

Proof. We prove that $\alpha \leqq 1$ implies $\psi < 2$. From (4.18) we may write

$$\psi - 2 = -\frac{(1 - \alpha^{\sigma})(\omega - \delta\alpha^{\sigma}) + 2\sigma\alpha^{\sigma}(\omega + \delta)}{\omega - \delta\alpha^{\sigma} + \sigma\alpha^{\sigma}(\omega + \delta)}.$$

If $\alpha \leqq 1$, this is clearly negative. □

5 On the probability of multiple equilibrium

Theorem 1 provides a *necessary* condition ($\sigma < 1/2$) for *nonuniqueness* of competitive equilibrium under supersymmetry; but this condition is far from being sufficient. In fact it turns out that multiple equilibrium is quite rare even when $\sigma < 1/2$. For the case of specialization ($\delta = 0$), Table 1 provides a tabulation of consecutive values of σ in the first column, and of

$$\inf_{\psi \geqq 2} \alpha = \frac{1}{(1 - 2\sigma)^{1/\sigma}} \tag{5.1}$$

(see (4.19)) in the second, referred to as the minimum associated value of α, above which multiple (triple) equilibrium occurs. The third column of Table 1 provides the share of the exportable in consumer expenditures at the unit price ratio, given by (2.1), namely $1/(1 + \alpha^{-\sigma})$ (see (3.5)).

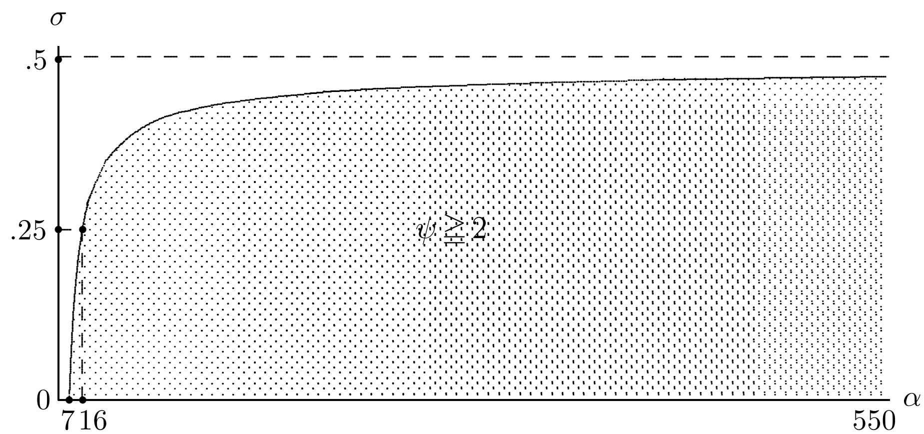

Figure 1

σ as a function of α for $\delta = 0$, when $\psi = 2$

TABLE 1 ($\delta = 0$)

σ	minimum associated α	export share
0+	7.38905609895	.5
.01	7.54036607387	.505050505
.05	8.22526333997	.526315789
.1	9.31322574615	.555555556
.15	10.7815135548	.588235294
.2	12.8600823045	.625
.25	16.	.666666667
.3	21.2063876296	.714285714
.35	31.1845349756	.769230769
.4	55.9016994375	.833333333
.45	166.810053719	.909090909
.49	2932.82211971	.980392157
.5	∞.	1

The function $\hat{\sigma}(\alpha)$ inverse to (5.1) is depicted in Figure 1. It starts with $\sigma = 0$ at $\alpha = 7.389$ (rounded down to 7 in the figure) and rises asymptotically to $\sigma = 1/2$ as $\alpha \to \infty$. The interior of the shaded area is the "zone of instability" (or of multiple equilibrium), consisting of those combinations (α, σ) for which $\psi > 2$ ($\gamma < -1$). While it forms most of the area of the extended rectangle, when limited to small values of α such as $\alpha = 16$ as shown in the figure, it forms the smaller part of the extended rectangle.

Table 1 provides cases of so-called neutral equilibrium on the borderline of instability, at which $\psi = 2$. The case $\sigma = .25$ and $\alpha = 16$ is illustrated in Figure 2. The equilibrium occurs where the share of each country's importable in consumer expenditure is 1/3. It is unique (from (5.1) and the Lemma) and stable. Starting

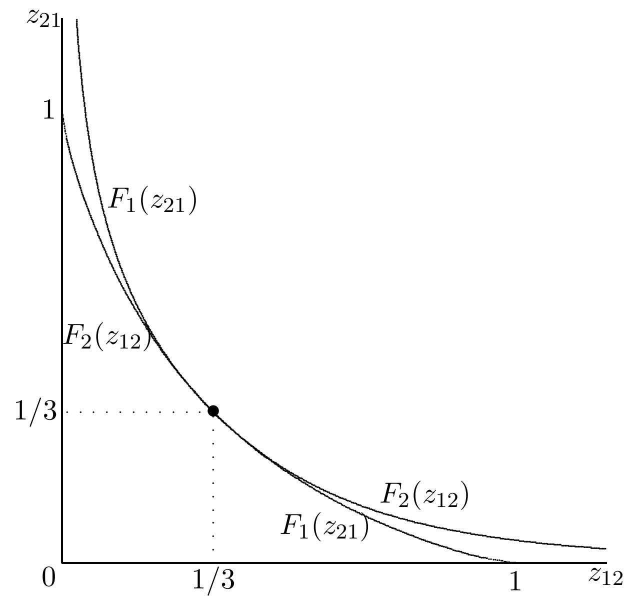

Figure 2

Marshallian offer curves under "neutral" equilibrium: the case $\omega = 1,\ \delta = 0,\ \alpha = 16$, and $\sigma = .25.\ \gamma = -1$.

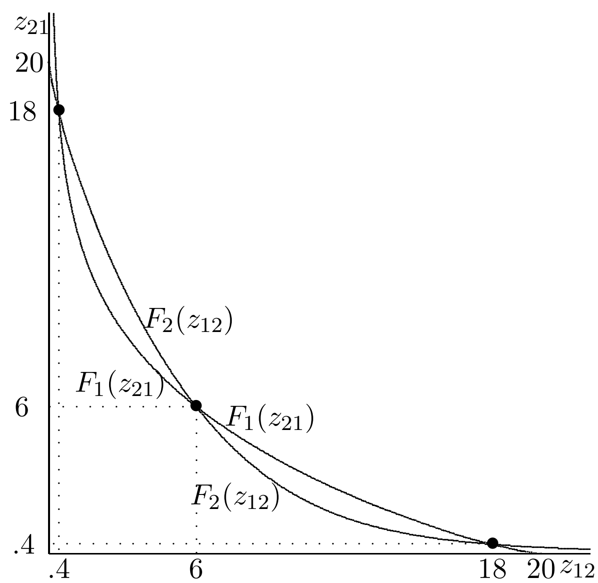

Figure 3

Marshallian offer curves with $\omega = 20,\ \delta = 0,\ \alpha = 70$ and $\sigma = .2.\ \gamma = -1.275$. Equilibrium relative prices .02174, 1, 45.9962, trades 18.0285, 5.9899, .39196.

from such a neutral equilibrium, as depicted by the dashed lines in Figure 1, in order to find an unstable equilibrium one must either increase α above 16, or decrease σ below .25, or both. Table 1 (and Figure 1) also show that under specialization ($\delta = 0$), multiple equilibrium is never possible unless $\alpha > 7.389$.[3] Figure 3 illustrates a case in which α is increased to 70 and σ is reduced to .2. Note that when $\delta = 0$ the level of ω affects only the scale.

The case of diversification is more complex. Table 2 provides, for given $\omega > 1$ and $\delta = 1$, the maximum value of σ for which a solution exists to the equation $\psi = 2$, and the corresponding critical value of α at which this maximum is attained. This may be obtained, for given ω, by solving the equation $\psi = 2$ in (4.18) for σ, given successive values of α, and determining that value of α at which σ reaches a maximum; alternatively, by solving, given ω, for the associated value of α at successive values of σ, until no solution is found. An illustration is given in Figure 4 for the case $\omega = 20$ and $\delta = 1$, where σ reaches a maximum of approximately .287 at $\alpha = 184.62$. The figure also shows the minimum value 9.12 of α in the domain of this function. This is obtained by tabulating α as a function of σ for successively smaller values of σ. The relationship of Table 2 to Table 1 should be noted: as $\omega \to \infty$ (with $\delta = 1$), diversification approaches specialization, and thus as ω increases, the corresponding minimum value of α in column 2 of Table 2 approaches the limiting value of α in the first row of Table 1; in fact, for $\omega = 100,000$, the minimum α in Table 2 is approximately 7.389, achieved at $\sigma = 10^{-6}$.

It is apparent from Figure 4 that the function $\hat{\sigma}(\alpha)$ is very flat in the neighborhood of the maximum point, so that obtaining accurate results by direct tabulation is difficult. However, the critical α value and the corresponding maximum σ value may be computed accurately by the following simple procedure which underlies the numbers in columns 3 and 4 of Table 2.

Setting $\psi = 2$ in (4.18) we arrive at the quadratic equation

$$\chi(\alpha, \sigma; \omega) \equiv (\alpha^\sigma)^2 - 2(\omega + 1)(1/2 - \sigma)\alpha^\sigma + \omega = 0 \tag{5.2}$$

in α^σ, with roots

$$\alpha^\sigma = \beta \pm \sqrt{\beta^2 - \omega}, \quad \text{where } \beta = (\omega + 1)(1/2 - \sigma).$$

Thus the critical value of α is

$$\alpha_{\text{crit}} = \left((\omega + 1)(1/2 - \sigma) \pm \sqrt{(\omega + 1)^2(1/2 - \sigma)^2 - \omega}\right)^{1/\sigma}. \tag{5.3}$$

Now it turns out that the roots are repeated, hence the discriminant is zero. This is seen as follows.

[3] This contradicts the example in Exercise 17.D.1 of Mas-Colell et al. [13, p. 644], where the authors take $\sigma = .2$ and $\alpha = 2$. For $\sigma = .2$, Table 1 shows that when $\delta = 0$, in order to obtain an example of multiple equilibrium one must take $\alpha > 12.86$.

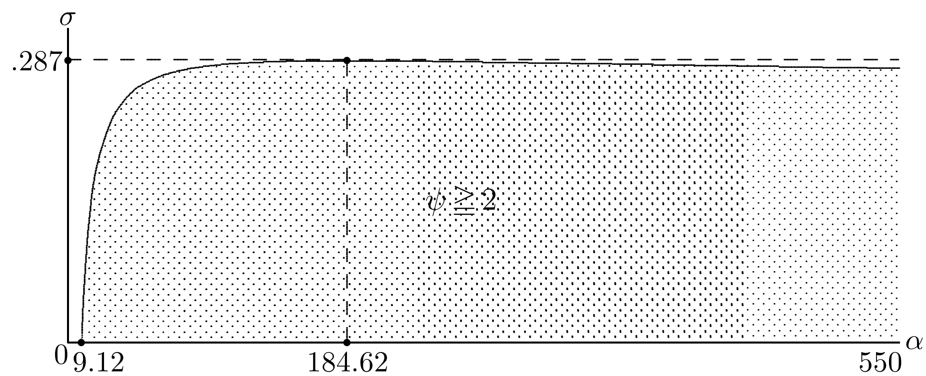

Figure 4

Maximum σ as a function of α for $\omega = 20$ and $\delta = 1$, when $\psi = 2$

TABLE 2 ($\delta = 1$)

ω	minimum α	maximum σ	critical α	export share
3/2	22025.974	.010102051444	519539777.807	.550510257216
2	403.410	.02859547921	183482.300248	.585786437626
3	54.579	.066987298108	3641.51017644	.633974596215
4	28.028	.1	1024.	.666666666667
5	20.085	.12732200375	555.764551652	.690983005625
10	11.524	.212520212712	225.277277594	.759746926646
19	9.226	.282055052823	184.863460212	.813394503137
20	9.120	.287041145	184.61963199	.817256002368
21	9.025	.291701104775	184.66902107	.82087121525
30	8.482	.323315304031	192.45779337	.845612911202
40	8.187	.345742553162	207.414672881	.863472940503
50	8.017	.361351611532	224.312943463	.876100656899
100	7.694	.40099009901	311.764858269	.909090909091
1000	7.419	.468408814584	1593.42217089	.96934656997
10000	7.392	.4900009999	12067.6949665	.990099009901
100000	7.389	.496837753962	107599.933218	.996847648654

Letting the identity (5.2) implicitly define the function $\hat{\sigma}(\alpha;\omega)$, we seek its maximum with respect to α by setting

$$(5.4) \qquad \frac{d\hat{\sigma}}{d\alpha} = -\frac{\partial\chi/\partial\alpha}{\partial\chi/\partial\sigma} = -\frac{\sigma\alpha^{\sigma-1}[\alpha^\sigma - (\omega+1)(1/2-\sigma)]}{[\alpha^\sigma - (\omega+1)(1/2-\sigma)]\alpha^\sigma \log\alpha + (\omega+1)\alpha^\sigma}$$

equal to zero. The maximum of $\hat{\sigma}(\alpha;\omega)$ for given ω is achieved when $d\xi/d\alpha = 0$ in (5.4), at

$$(5.5) \qquad \alpha_{\text{crit}} = [(\omega+1)(1/2-\sigma)]^{1/\sigma},$$

since at this value the denominator of (5.4) is clearly positive. Thus, setting the discriminant in (5.3) equal to zero, we obtain

$$\sigma_{\max} = \frac{1}{2} - \frac{\sqrt{\omega}}{\omega + 1}, \tag{5.6}$$

which is the maximum σ for which multiple equilibrium is possible for given $\omega > 1$ and $\delta = 1$. Then substituting (5.6) into (5.5) we obtain

$$\alpha_{\text{crit}} = \omega^{1/2\sigma_{\max}}. \tag{5.7}$$

These are the two values tabulated in columns 3 and 4 of Table 2.

The lowest critical value 184.62 of α shown in Table 2, at which 81.7% of income is spent on the exportable good, occurs at an endowment ratio of $\omega = 20$ exportables to 1 importable. An interesting case of neutral equilibrium is that of the third row of the table. It is illustrated in Figure 5, where the small arrows partly indicate the movement toward erquilibrium. Finally, Figure 6 displays a case of multiple equilibrium with $\sigma = .1$ and $\alpha = 1024$. The diagram differs markedly from those depicted in Marshall [10, Figs. 4, 8], [11, p. 353, Fig. 20], and subsequent textbooks.

I come now to the question of probabilities. The main question, stemming from Theorem 2, is that of how likely it is that inhabitants of a country will spend a larger proportion of their incomes on the country's export good than on its import good. This question has been approached in the economic literature largely in relation to the "transfer problem", since this hypothesis has been behind the presumption that a country making a unilateral transfer to another will incur a "secondary burden", that is, an additional welfare loss resulting from a deterioration in its terms of trade (its export price relative to its import price). Samuelson [18, p. 290] has read into Pigou [16, pp. 539–40] the interpretation that "Pigou leans heavily on the simple econometric fact that Europeans tend to spend more relatively on European products than Americans do; Americans likewise have a greater avarage propensity to consume American products than do Europeans." Accepting this as a datum, he attributes it to the fact that tariffs and transport costs raise the relative cost of foreign goods, and assumes that in their absence, relative propensities to consume will be the same across countries. Likewise, Jones [8, p. 179]) makes the "basic assumption" that "a country's taste pattern is independent of its endowment (production) pattern." If preferences are identical and homothetic (as both authors appear to assume), then of course world equilibrium is determinable by maximization of a common world utility function subject to world production constraints (cf. [5]), and it must be unique. But homothetic preferences violate Engel's law, and there appears to be little in the way of solid information to back up any presumptions concerning relative expenditure patterns across countries.

One could argue that habits are formed in a state of autarky in which consumers have no choice but to conform to the existing endowment of goods, and that such habits may become culturally ingrained and may persist for a long time. The prevailing literature on habit formation appears to assume that such habits are purely temporary and cannot last very long, but whether this is really so we do not seem

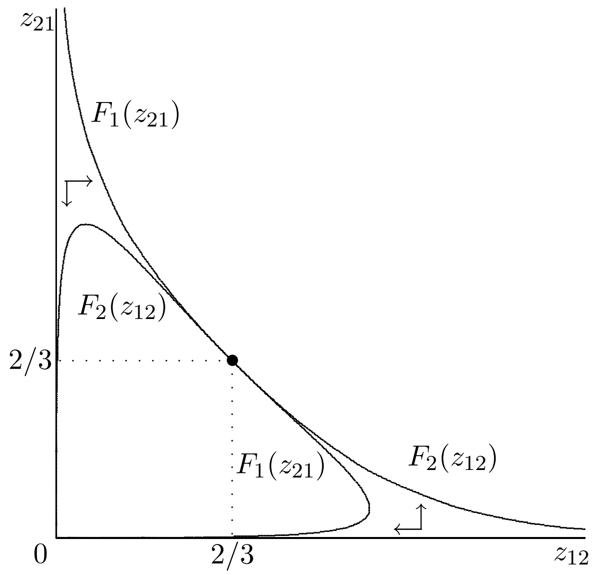

Figure 5

Marshallian offer curves under "neutral equilibrium": The case $\omega = 4,\ \delta = 1,\ \alpha = 1024$, and $\sigma = .1.\ \gamma = -1.$

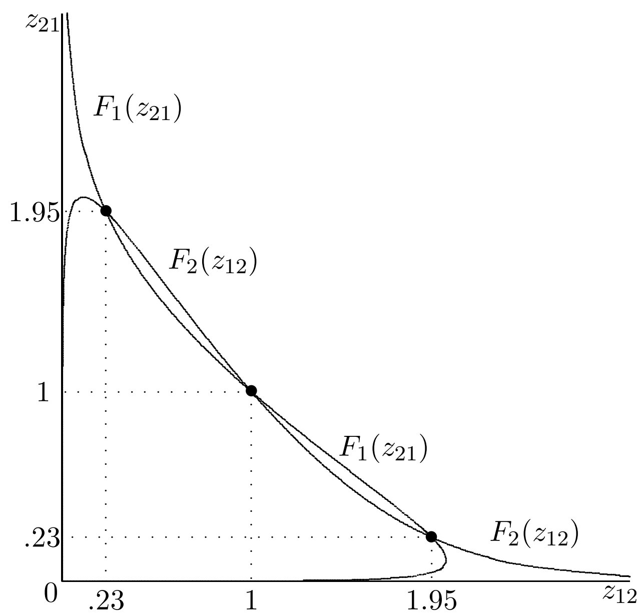

Figure 6

Marshallian offer curves with $\omega = 5,\ \delta = 1,\ \alpha = 1024$, and $\sigma = .1.\ \gamma = -1.14285.$
Equilibrium prices .11809, 1, 8.4679, trades 1.95265, 1, .23059.

to know. In the model considered in this paper, there clearly must be a positive correlation between a country's endowments and its preference patterns in order for multiple equilibrium to result. The parameter α used to measure the relative preference for the export good must in all cases exceed 7.389 for $\sigma > 0$, and a higher number with higher values of σ and lower values of ω in accordance with Tables 1 and 2. However, it is difficult to attach any intuition to specific values of α as an indicator of relative preference for the export good, as opposed to the export share itself, $\varepsilon = 1/(1+\alpha^{-\sigma})$; but given some of the high required values of the latter, we may conclude that under the assumption of CES preferences, multiple equilibrium must be quite rare.

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