EDGEWORTH ON BANKING

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> These speculations are at an immense height of abstraction above the affairs of earth. Appropriate general conceptions, rather than exact propositions, have been the object of the writer. He does not pretend to base any practical recommendations on the theory. There is some sense as well as absurdity in the sentiment which Thomas Peacock in his amusing *Nightmare Abbey*, puts into the mouth of Mr.Flosky, the caricature of Coleridge. The philosopher having been consulted in a practical emergency, replies: "Madam, I should lose my transcendental reputation, if it could be said of Ferdinand Flosky that he was ever known to give any useful information to any one." The theorist must not pretend to wisdom, if he knows so little what he is about as to mistake his abstract formulae for rules immediately applicable to practice.
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> F.Y. Edgeworth (1888)

1. Introduction

Edgeworth's (1888) paper "A Mathematical Theory of Banking" is generally regarded as laying the foundation of modern inventory theory. Although it is often referenced I have been unable to find an explicit link between Edgeworth's formal theory and modern developments. Arrow's (1958) historical survey emphasizes Edgeworth's contribution, but focuses on his observation that reserves should expand like the square-root of deposits without describing the simple model Edgeworth introduces that anticipates the extensive modern literature on what has become known as the "newsvendor problem." The influential paper of Arrow, Harris, and Marschak (1951), which introduced dynamic "sS" rules, also contains a similar static formulation to that of Edgeworth, albeit with an additional complication of a fixed charge for exceeding the chosen inventory level. No mention is made, however, of Edgeworth, nor is any made in Arrow's (2002) more recent memoir. Wagner's (1969) encyclopedic survey of operations research contains a thorough discussion of what he calls the "newsboy problem," but again there is no mention of Edgeworth. My aim in this note is simply to provide a more explicit exegesis of Edgeworth's model in the hope that someone might eventually be inspired to fill this historical gap.

Edgeworth's basic model, p. 120-1, takes the form of what he calls a parable; it is really just a simple game. Players are each allocated 100 tokens that they may invest in one of three assets: an illiquid asset that pays 1% per period, a semi-liquid asset that he calls "at call" that pays 0.4% per period, and a cash reserve. Periodically, a positive random variable, Y is generated. Players are then required to pay the amount Y to the game authority. This can be done at no penalty out of reserves if they are sufficient, or out of holdings of the "at call" asset if reserves are insufficient. However, drawing upon the "at call" asset entails a penalty of 10% of the amount required. If Y exceeds the amount invested in the two liquid assets combined, the player must declare bankruptcy and pay a large penalty.

The mechanism for generating Y is amusingly quaint: 22 decimal digits are selected "at random from the pages of some mathematical or statistical tables," and then summed. Presuming these digits are drawn from the least significant in the tables, i.e. ignoring Benford's Law, they will

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be approximately uniformly distributed and consequently their sum, say X, will be approximately Gaussian with mean 99 and variance, 181.5. Edgeworth computes the modulus of this distribution as 19, which is defined, e.g. Stigler (1990), as the square root of twice the variance. In effect the random variable he then considers is $Y = |X - EX|$.

Edgeworth argues, somewhat cavalierly, that the probability that the half-normal Y exceeds 75 is "one in 100 million" and therefore it is safe to invest 25 of the initial tokens in the illiquid asset. A more pedantic estimate of the risk based on the Gaussian approximation yields one in 259 million, an even more pedantic estimate of the risk based on the discrete uniform convolution is one in 1.561 billion.

How should the remaining 75 be divided? To this question Edgeworth responds very tersely: "the portion held at call should be 56, the reserve 19." Having noted earlier that "it need not be pointed out that the professor of Probability will have a great advantage in this game," it is a matter of natural curiosity to try to determine how he arrived at this split. Although the game as originally described almost begs for a dynamic formulation Edgeworth is at pains to emphasize that it is intended to be a series of one-shot decisions with no reinvested earnings and the original setting restored at the beginning of each play.

We can begin our inquiry by noting that if the portion, say $\hat{Y}$, devoted to the reserve is set too high so $Y < \hat{Y}$, we would have sacrificed $0.004(\hat{Y} - Y)$ in lost interest, while if it is set too low so $Y > \hat{Y}$, we must pay the penalty of $0.1(Y - \hat{Y})$. In such circumstances we know that $\hat{Y}$ should be chosen as the $\tau = 1/1.04$ quantile of the distribution of Y. Now it is easily seen that the quantile function of Y is $Q_Y(\tau) = \sigma\Phi^{-1}((\tau + 1)/2)$, and consequently that $\hat{Y} \approx 27.89$. The split should be 28 in reserve and $75 - 28 = 47$ at call. Something is clearly amiss. Having folded the distribution of Y to obtain a positive random variable, it is tempting to imagine that Edgeworth might have introduced a spurious two. If $Y = 2|X - EX|$, so $Q_Y(\tau) = 2\sigma\Phi^{-1}((\tau + 1)/2)$, then $\hat{Y} \approx 55.77$, which Edgeworth would have rounded to 56. Perfect! But wait! Now Edgeworth seems to have gotten it backwards! Our calculation would imply that 56 should be placed in reserve and 19 at call. We might also want to reconsider allocating 25 to the illiquid asset since the more dispersed Y now has a somewhat higher probability ($\approx 1/827{,}000$) of exceeding the threshold we had previously set at 75. We are left with something of a quandary. Edgeworth's formulation of the problem is very precise and he is obviously aware of the need for an asymmetric asset allocation in response to the asymmetric losses, but the rationale for his computations remains rather murky. Our epigraph, drawn from the concluding paragraph of Edgeworth's article, thus seems well vindicated.

Admittedly, this wouldn't seem to be an auspicious beginning for the theory of inventory policy, however it should be remembered that problem formulation is often more important than an authoritative solution. It is fair to say that the extensive literature now referred to as the "newsvendor problem" reviewed in Choi (2012) flows directly from Edgeworth's formulation. And not coincidently, so does quantile regression.

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