My Theory of Lemmas: Advice for Thesis Students

At the outset of his textbook *Probability & Measure* Patrick Billingsley quotes Edward Davenant (1634 - 1679)

> "I would have a man knockt on the head that should write anything in Mathematiques that had been written of before."

By this standard very little should be written – exceptions granted for Ramanujan's Notebooks – but if we interpret the injunction strictly, it could be more pithily reformulated as: Don't reprove the proven. Once in a long while you may have a better, clearer, simpler proof of an old result that really justifies a new exposition, but this is rare in econometrics. Usually, arguments follow well trodden prior footsteps.

My impression is that there is quite a lot of this going on, particularly in econometrics. Partly this is the fault of prior authors who don't provide modular arguments. I like to think of mathematical arguments like computer programs; in some brave new world of the future proofs could be checked by computer, both for correctness and for originality. This would save a lot of time for thesis advisors, and journal editors.

The crux of the problem is that very similar results can be formulated and proven under very minor perturbations in conditions. When this is allowed, then the usual system of credit for mathematical innovation begins to break down, and clarity is also sacrificed.

The objectives in formulating propositions and their proofs is much the same as the objectives in writing good software:

- *Keep things Modular* : Just as you shouldn't write one big subroutine with many pieces that could be helpful in other circumstances, you shouldn't formulate theorems that are specialized to one particular instance and then prove them in an extended series of steps that are logically distinct and might be useful in other contexts. A rule of thumb is that a good subroutine shouldn't exceed 20 or 30 lines, the size of a typical computer screen window; this could be extended to proofs. Develop the argument in digestible chunks. If parts of the argument can be formulated as independent propositions requiring only a subset of the required conditions for the main result, by all means treat them independently. They can be used and reused, and like good subroutines they are easier to debug.

- *Keep the Role of the Regularity Conditions Clear:* This is intimately connected to the modularity objective. Often conditions are needed at only one stage of a long and complicated argument and it is crucial to a clear exposition that this is clear to the reader. Conditions are like inputs to a subroutine, when they aren't needed to reach the conclusions – the output of the subroutine – then they just confuse the exposition when they are carried along.

- *Keep the notation clear* : It is worth investing considerable forethought (and afterthought) to finding a concise, clear set of symbols to express results. As much as possible one should follow successful prior precedent. Modularity here too is your friend; chunks of the argument can often be expressed very efficiently with much simpler notation than that used for the final result. This is the notion of local as opposed to global variables in the computing analogy.

- *Credit where Credit is Due* : Don't be afraid to use phrases like "arguing as for (4.8.3) in Huber (1965) we have..." or "By Lemma 4.8 of Bickel (1975)..." in either case it is clearer and more honest to push the argument forward without repetition. This is the essential message of Davenant.

- *Revise, Revise and Revise:* Experiment with various reformulations of the argument, trying to find simpler more natural lines of development. This is the message of Piet Hein's famous grook:

  The road to wisdom?
  Well it's plain and simple to express.
  Err and err and err again,
  but less and less and less.

  This is also one of the themes of Paul Halmos's classic essay, "How to Write Mathematics" which I'll distribute with this note.