Quantile Autoregression

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Based (mainly) on joint work with Zhijie Xiao, Boston College.

Outline

Introduction

In classical regression and autoregression models

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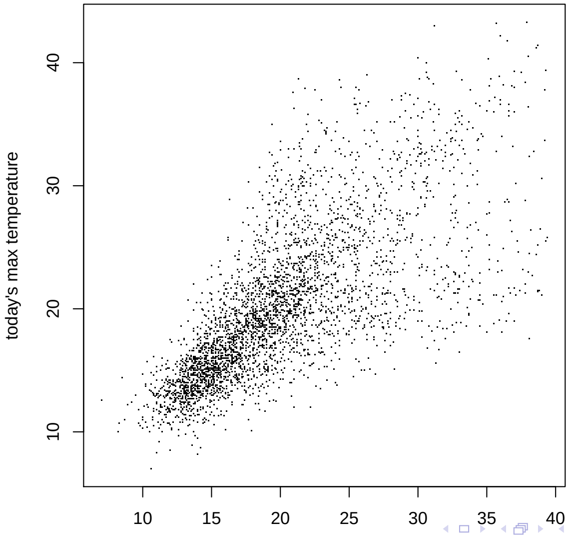
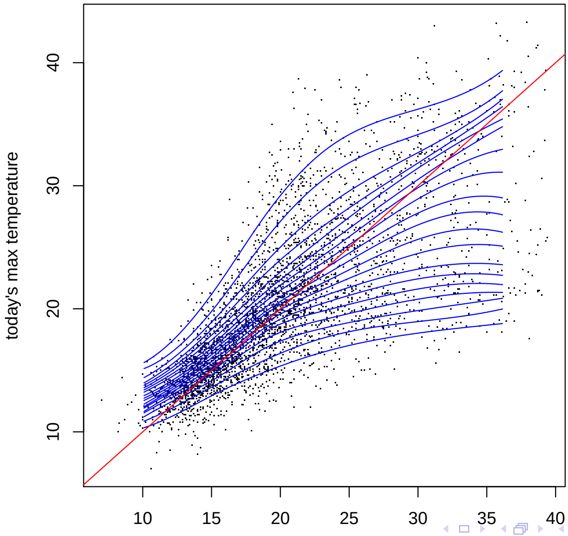
Or perhaps with varying scale as well.

A Motivating Example



Estimated Conditional Quantiles of Daily Temperature



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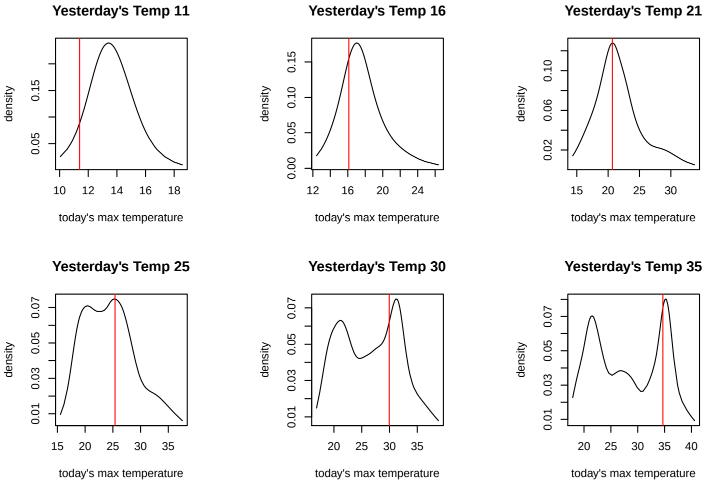

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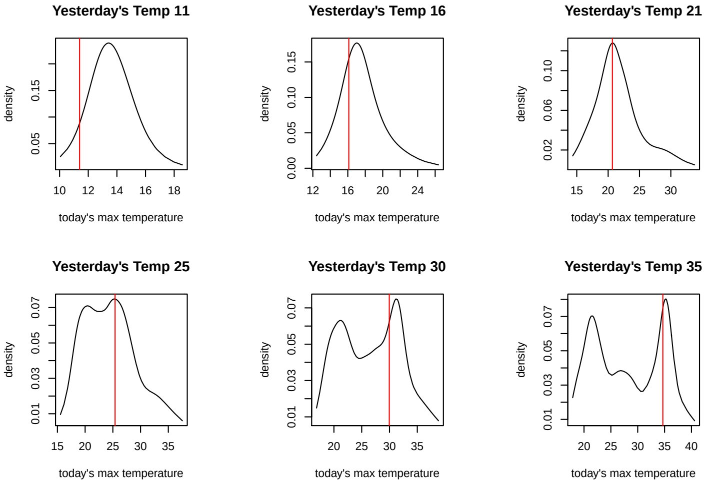

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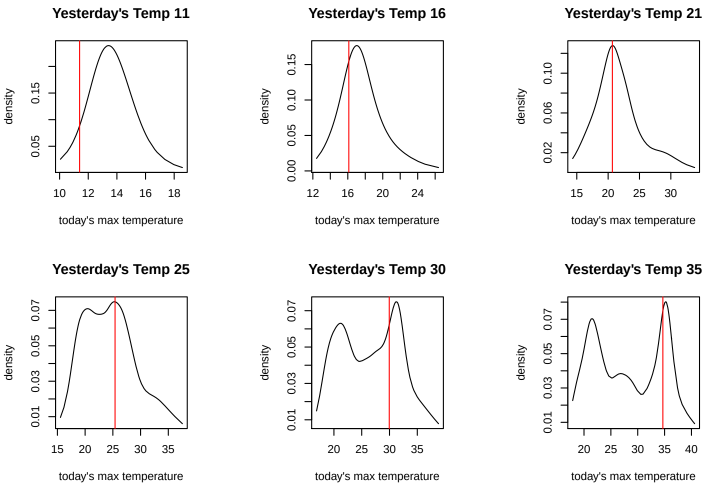

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When today is hot, tomorrow's temperature is bimodal!

Linear AR(1) and QAR(1) Models

The classical linear AR(1) model

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + u_t,$$

with iid errors, $u_t : t = 1, \cdots, T$, implies

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and conditional quantile functions are all parallel:

$$Q_{y_t}(\tau | \mathcal{F}_{t-1}) = \alpha_0(\tau) + \alpha_1 y_{t-1}$$

with $\alpha_0(\tau) = F_u^{-1}(\tau)$ just the quantile function of the u_t's.

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But isn't this rather boring? What if we let α_1 depend on τ too?

A Random Coefficient Interpretation

If the conditional quantiles of the response satisfy:

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \alpha_0(\tau) + \alpha_1(\tau)y_{t-1}$$

then we can generate responses from the model by replacing τ by uniform random variables:

$$y_t = \alpha_0(u_t) + \alpha_1(u_t)y_{t-1} \quad u_t \sim \text{iid } U[0,1].$$

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This is a very special form of random coefficient autoregressive (RCAR) model with comonotonic coefficients.

On Comonotonicity

Definition: Two random variables $X, Y : \Omega \to \mathbb{R}$ are comonotonic if there exists a third random variable $Z : \Omega \to \mathbb{R}$ and increasing functions f and g such that $X = f(Z)$ and $Y = g(Z)$.

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- If X and Y are comonotonic they have rank correlation one.
- From our point of view the crucial property of comonotonic random variables is the behavior of quantile functions of their sums, X, Y comonotonic implies:

$$F_{X+Y}^{-1}(\tau) = F_X^{-1}(\tau) + F_Y^{-1}(\tau)$$

- X and Y are driven by the same random (uniform) variable.

The QAR(p) Model

Consider a p-th order QAR process,

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \alpha_0(\tau) + \alpha_1(\tau)y_{t-1} + \ldots + \alpha_p(\tau)y_{t-p}$$

Equivalently, we have random coefficient model,

$$\begin{aligned} y_t &= \alpha_0(u_t) + \alpha_1(u_t)y_{t-1} + \cdots + \alpha_p(u_t)y_{t-p} \\ &\equiv x_t^\top \alpha(u_t). \end{aligned}$$

Now, all $p+1$ random coefficients are comonotonic, functionally dependent on the same uniform random variable.

Vector QAR(1) representation of the QAR(p) Model

$$Y_t = \mu + A_t Y_{t-1} + V_t$$

where

$$\mu = \begin{bmatrix} \mu_0 \\ 0_{p-1} \end{bmatrix}, \; A_t = \begin{bmatrix} a_t & \alpha_p(u_t) \\ I_{p-1} & 0_{p-1} \end{bmatrix}, \; V_t = \begin{bmatrix} \nu_t \\ 0_{p-1} \end{bmatrix}$$

$$a_t = [\alpha_1(u_t), \dots, \alpha_{p-1}(u_t)],$$

$$Y_t = [y_t, \cdots, y_{t-p+1}]^\top,$$

$$\nu_t = \alpha_0(u_t) - \mu_0.$$

It all looks rather complex and multivariate, but it is really still nicely univariate and very tractable.

Stationarity

Theorem 1: Under assumptions A.1 and A.2, the QAR(p) process y_t is covariance stationary and satisfies a central limit theorem

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - \mu_y) \Rightarrow N\left(0, \omega_y^2\right),$$

with

$$\begin{aligned}
\mu_y &= \frac{\mu_0}{1 - \sum_{j=1}^{p} \mu_p}, \\
\mu_j &= E(\alpha_j(u_t)), \quad j = 0, \dots, p, \\
\omega_y^2 &= \lim \frac{1}{n} E[\sum_{t=1}^{n} (y_t - \mu_y)]^2.
\end{aligned}$$

Example: The QAR(1) Model

For the QAR(1) model,

$$Q_{y_t}(\tau | y_{t-1}) \quad = \quad \alpha_0(\tau) + \alpha_1(\tau) y_{t-1},$$

or with u_t iid $U[0,1]$.

$$y_t = \alpha_0(u_t) + \alpha_1(u_t) y_{t-1},$$

if $\omega^2 = E(\alpha_1^2(u_t)) < 1$, then y_t is covariance stationary and

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - \mu_y) \Rightarrow N\left(0, \omega_y^2\right),$$

where $\mu_0 = E\alpha_0(u_t)$, $\mu_1 = E(\alpha_1(u_t)$, $\sigma^2 = V(\alpha_0(u_t))$, and

$$\mu_y = \frac{\mu_0}{(1-\mu_1)}, \quad \omega_y^2 = \frac{(1+\mu_1)\sigma^2}{(1-\mu_1)(1-\omega^2)},$$

Qualitative Behavior of QAR(p) Processes

- The model can exhibit unit-root-like tendencies, even temporarily explosive behavior, but episodes of mean reversion are sufficient to insure stationarity.

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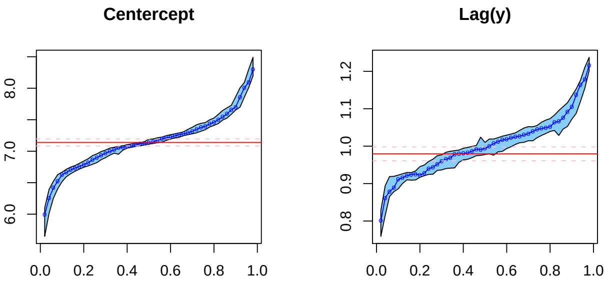
- Under certain conditions,the QAR(p) process is a semi-strong ARCH(p) process in the sense of Drost and Nijman (1993).
- The impulse response of y_{t+s} to a shock u_t is stochastic but converges (to zero) in mean square as $s \to \infty$.

Estimated QAR(1) v. AR(1) Models of U.S. Interest Rates



Data: Seasonally adjusted monthly: April, 1971 to June, 2002.

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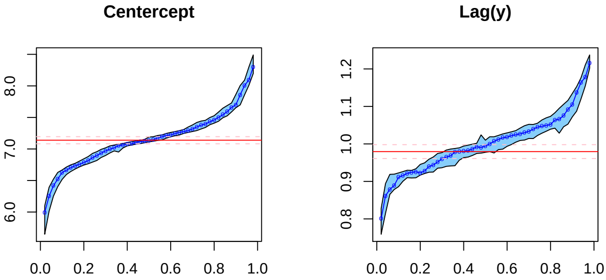

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Do 3-month T-bills really have a unit root?

Estimation of the QAR model

Estimation of the QAR models involves solving,

$$\hat{\alpha}(\tau) = \text{argmin}_{\alpha} \sum_{t=1}^{n} \rho_\tau(y_t - x_t^\top \alpha),$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$, the $\surd$-function.

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Fitted conditional quantile functions of y_t, are given by,

$$\hat{Q}_t(\tau | x_t) = x_t^\top \hat{\alpha}(\tau),$$

and conditional densities by the difference quotients,

$$\hat{f}_t(\tau | x_{t-1}) = \frac{2h}{\hat{Q}_t(\tau + h | x_{t-1}) - \hat{Q}_t(\tau - h | x_{t-1})},$$

The QAR Process

Theorem 2: Under our regularity conditions,

$$\sqrt{n}\Omega^{-1/2}(\hat{\alpha}(\tau) - \alpha(\tau)) \Rightarrow B_{p+1}(\tau),$$

a $(p+1)$-dimensional standard Brownian Bridge, with

$$\begin{aligned}
\Omega &= \Omega_1^{-1}\Omega_0\Omega_1^{-1}. \\
\Omega_0 &= E(x_t x_t^\top) = \lim n^{-1}\sum_{t=1}^{n} x_t x_t^\top, \\
\Omega_1 &= \lim n^{-1}\sum_{t=1}^{n} f_{t-1}(F_{t-1}^{-1}(\tau))x_t x_t^\top.
\end{aligned}$$

Forecasting with QAR Models

Given an estimated QAR model,

$$\hat{Q}_{y_t}(\tau|\mathcal{F}_{t-1}) = x_t^\top \hat{\alpha}(\tau)$$

based on data: $y_t : \ t = 1, 2, \cdots, T$, we can forecast

$$\hat{y}_{T+s} = \tilde{x}_{T+s}^\top \hat{\alpha}(U_s), \ s = 1, \cdots, S,$$

where $\tilde{x}_{T+s} = [1, \tilde{y}_{T+s-1}, \cdots, \tilde{y}_{T+s-p}]^\top$, $U_s \sim U[0, 1]$, and

$$\tilde{y}_t = \begin{cases} y_t & \text{if} \quad t \leqslant T, \\ \hat{y}_t & \text{if} \quad t > T. \end{cases}$$

Conditional density forecasts can be made based on an ensemble of such forecast paths.

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- Lines with distinct slopes eventually intersect. [Euclid: P5]

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- Quantile functions, $Q_Y(\tau|x)$ should be monotone in τ for all x, intersections imply point masses – or even worse.
- What is to be done?
  - Constrained QAR: Quantiles can be estimated simultaneously subject to linear inequality restrictions.
  - Nonlinear QAR: Abandon linearity in the lagged y_t's, as in the Melbourne temperature example, both parametric and nonparametric options are available.

Nonlinear QAR Models via Copulas

An interesting class of stationary, Markovian models can be expressed in terms of their copula functions:

$$G(y_t, y_{t-1}, \cdots, y_{y-p}) = C(F(y_t), F(y_{t-1}), \cdots, F(y_{y-p}))$$

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- Differentiating, $C(u, v)$, with respect to u, gives the conditional df,

$$H(y_t | y_{t-1}) = \frac{\partial}{\partial u} C(u, v)|_{(u=F(y_t), v=F(y_{t-1}))}$$

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- Inverting we have the conditional quantile functions,

$$Q_{y_t}(\tau|y_{t-1}) = h(y_{t-1}, \theta(\tau))$$

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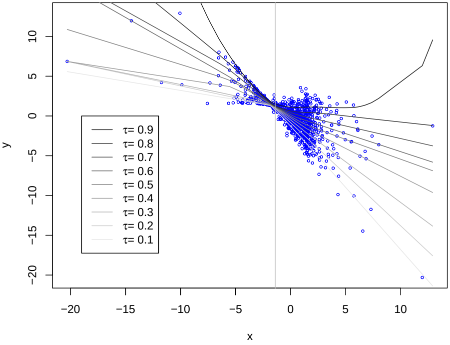
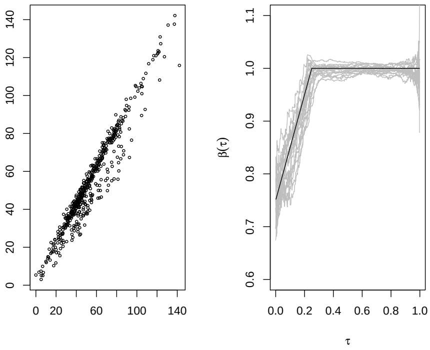
Example 1 (Fan and Fan)



Model: $Q_{y_t}(\tau|y_{t-1}) = -(1.7 - 1.8\tau)y_{t-1} + \Phi^{-1}(\tau)$.

Example 2 (Near Unit Root)



Model: $Q_{y_t}(\tau|y_{t-1}) = 2 + \min\{\frac{3}{4} + \tau, 1\}y_{t-1} + 3\Phi^{-1}(\tau)$.

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- Wald-type inference is feasible for a large class of hypotheses; rank based inference is also an attractive option.

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- Forecasting conditional densities is potentially valuable.
- Many new and challenging open problems. . . .