# Censored Quantile Regression and Survival Models

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# Quantile Regression for Duration (Survival) Models

A wide variety of survival analysis models, following Doksum and Gasko (1990), may be written as,

$$h(T_i) = x_i^\top \beta + u_i$$

where $h$ is a monotone transformation, and

- $T_i$ is an observed survival time,
- $x_i$ is a vector of covariates,
- $\beta$ is an unknown parameter vector
- $\{u_i\}$ are iid with df $F$.

# The Cox Model

For the proportional hazard model with

$$\log \lambda(t|x) = \log \lambda_0(t) - x^\top \beta$$

the conditional survival function in terms of the integrated baseline hazard $\Lambda_0(t) = \int_0^t \lambda_0(s)ds$ as,

$$\log(-\log(S(t|x))) = \log \Lambda_0(t) - x^\top \beta$$

so, evaluating at $t = T_i$, we have the model,

$$\log \Lambda_0(T) = x^\top \beta + u$$

for $u_i$ iid with df $F_0(u) = 1 - e^{-e^u}$.

# The Bennett (Proportional-Odds) Model

For the proportional odds model, where the conditional odds of death $\Gamma(t|x) = F(t|x)/(1 - F(t|x))$ are written as,

$$\log \Gamma(t|x) = \log \Gamma_0(t) - x^\top \beta,$$

we have, similarly,

$$\log \Gamma_0(T) = x^\top \beta + u$$

for $u$ iid logistic with $F_0(u) = (1 + e^{-u})^{-1}$.

# Accelerated Failure Time Model

In the accelerated failure time model we have

$$\log(T_i) = x_i^\top \beta + u_i$$

so

$$\begin{aligned} P(T > t) &= P(e^u > te^{-x\beta}) \\ &= 1 - F_0(te^{-x\beta}) \end{aligned}$$

where $F_0(\cdot)$ denotes the df of $e^u$, and thus,

$$\lambda(t|x) = \lambda_0(te^{-x\beta})e^{-x\beta}$$

where $\lambda_0(\cdot)$ denotes the hazard function corresponding to $F_0$. In effect, the covariates act to rescale time in the baseline hazard.

# Beyond the Transformation Model

The common feature of all these models is that after transformation of the observed survival times we have:

- a pure location-shift, iid-error regression model
- covariate effects shift the center of the distribution of $h(T)$, but
- covariates cannot affect scale, or shape of this distribution

# An Application: Longevity of Mediterrean Fruit Flies

In the early 1990's there were a series of experiments designed to study the survival distribution of lower animals. One of the most influential of these was:

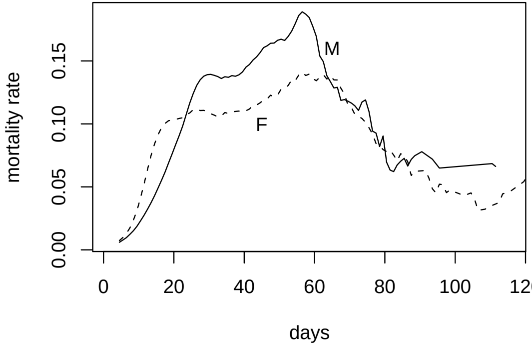
> Carey, J.R., Liedo, P., Orozco, D. and Vaupel, J.W. *(1992) Slowing of mortality rates at older ages in large Medfly cohorts, Science,* **258***, 457-61.*

- 1,203,646 medflies survival times recorded in days
- Sex was recorded on day of death
- Pupae were initially sorted into one of five size classes
- 167 aluminum mesh cages containing roughly 7200 flies
- Adults were given a diet of sugar and water *ad libitum*

# Major Conclusions of the Medfly Experiment

- Mortality rates declined at the oldest observed ages. contradicting the traditional view that aging is an inevitable, monotone process of senescence.
- The right tail of the survival distribution was, at least by human standards, remarkably long.
- There was strong evidence for a crossover in gender specific mortality rates.

# Lifetable Hazard Estimates by Gender




Smoothed mortality rates for males and females.

## Medfly Survival Prospects

| Lifespan (in days) | Percentage Surviving | Number Surviving |
|---|---|---|
| 40 | 5 | 60,000 |
| 50 | 1 | 12,000 |
| 86 | .01 | 120 |
| 146 | .001 | 12 |

Initial Population of 1,203,646

**Medfly Survival Prospects**

| Lifespan (in days) | Percentage Surviving | Number Surviving |
|---|---|---|
| 40 | 5 | 60,000 |
| 50 | 1 | 12,000 |
| 86 | .01 | 120 |
| 146 | .001 | 12 |

Initial Population of 1,203,646

**Human Survival Prospects***

| Lifespan (in years) | Percentage Surviving | Number Surviving |
|---|---|---|
| 50 | 98 | 591,000 |
| 75 | 69 | 413,000 |
| 85 | 33 | 200,000 |
| 95 | 5 | 30,000 |
| 105 | .08 | 526 |
| 115 | .0001 | 1 |

* Estimated Thatcher (1999) Model

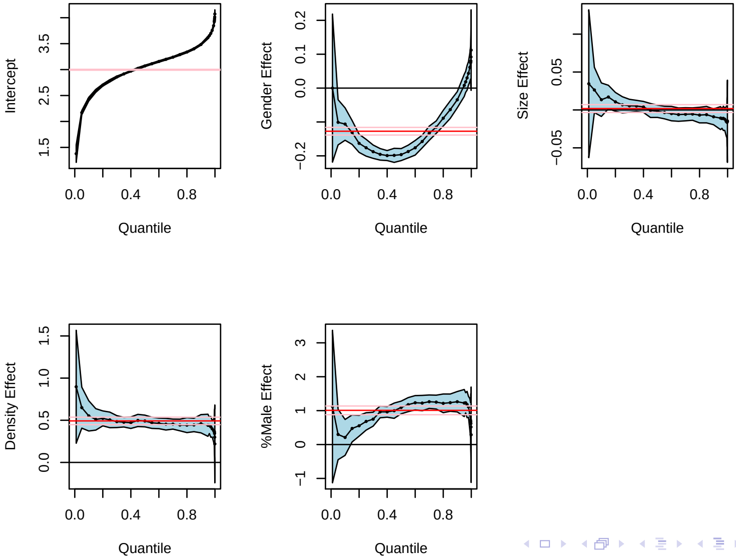
# Quantile Regression Model (Geling and K (JASA,2001))

Criticism of the Carey et al paper revolved around whether declining hazard rates were a result of confounding factors of cage density and initial pupal size. Our basic QR model included the following covariates:

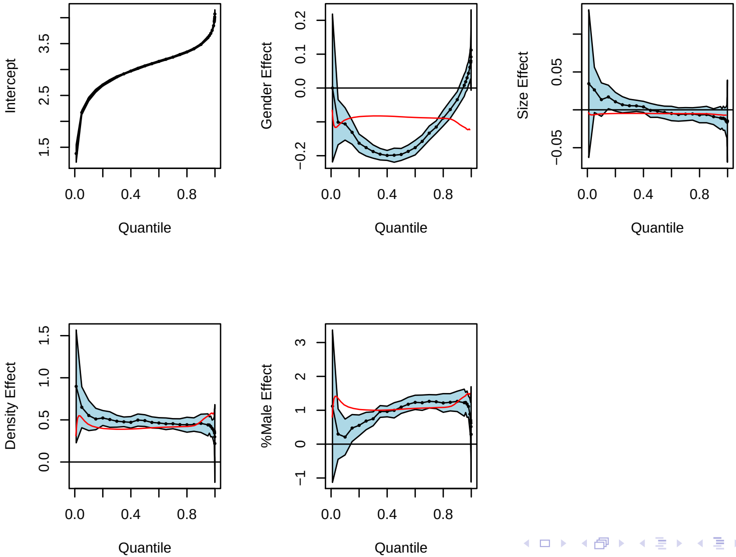
$$\begin{aligned} Q_{\log(T_i)}(\tau|x_i) &= \beta_0(\tau) + \beta_1(\tau)\text{SEX} + \beta_2(\tau)\text{SIZE} \\ &+ \beta_3(\tau)\text{DENSITY} + \beta_4(\tau)\%\text{MALE} \end{aligned}$$

- SEX Gender
- SIZE Pupal Size in mm
- DENSITY Initial Density of Cage
- %MALE Initial Proportion of Males

# Base Model Results with AFT Fit



# Base Model Results with Cox PH Fit



# What About Censoring?

There are currently 3 approaches to handling censored survival data within the quantile regression framework:

- Powell (1986) Fixed Censoring
- Portnoy (2003) Random Censoring, Kaplan-Meier Analogue
- Peng/Huang (2008) Random Censoring, Nelson-Aalen Analogue

Available for R in the package `quantreg`.

# Powell's Approach for Fixed Censoring

Rationale *Quantiles are equivariant to monotone transformation:*

$$Q_{h(Y)}(\tau) = h(Q_Y(\tau)) \textit{ for } h \nearrow$$

Model $Y_i = T_i \wedge C_i \equiv \min\{T_i, C_i\}$

$$Q_{Y_i|x_i}(\tau|x_i) = x_i^\top \beta(\tau) \wedge C_i$$

Data *Censoring times are known for all observations*

$$\{Y_i, C_i, x_i : i = 1, \cdots, n\}$$

Estimator *Conditional quantile functions are nonlinear in parameters:*

$$\hat{\beta}(\tau) = \textit{argmin} \sum \rho_\tau(Y_i - x_i^\top \beta \wedge C_i)$$

# Portnoy's Approach for Random Censoring I

**Rationale** *Efron's (1967) interpretation of Kaplan-Meier as shifting mass of censored observations to the right:*

**Algorithm** *Until we "encounter" a censored observation KM quantiles can be computed by solving, starting at* $\tau = 0$,

$$\hat{\xi}(\tau) = \mathit{argmin}_{\xi} \sum_{i=1}^{n} \rho_\tau(Y_i - \xi)$$

*Once we "encounter" a censored observation, i.e. when* $\hat{\xi}(\tau_i) = y_i$ *for some* $y_i$ *with* $\delta_i = 0$, *we split* $y_i$ *into two parts:*

- $y_i^{(1)} = y_i$ *with weight* $w_i = (\tau - \tau_i)/(1 - \tau_i)$
- $y_i^{(2)} = y_\infty = \infty$ *with weight* $1 - w_i$.

*Then denoting the index set of censored observations "encountered" up to* $\tau$ *by* $K(\tau)$ *we can solve*

$$\min \sum_{i \notin K(\tau)} \rho_\tau(Y_i - \xi) + \sum_{i \in K(\tau)} [w_i(\tau)\rho_\tau(Y_i - \xi) + (1 - w_i(\tau))\rho_\tau(y_\infty - \xi)].$$

# Portnoy's Approach for Random Censoring II

When we have covariates we can replace $\xi$ by the inner product $x_i^\top \beta$ and solve:

$$\min \sum_{i \notin K(\tau)} \rho_\tau(Y_i - x_i^\top \beta) + \sum_{i \in K(\tau)} [w_i(\tau)\rho_\tau(Y_i - x_i^\top \beta) + (1 - w_i(\tau))\rho_\tau(y_\infty - x_i^\top \beta)].$$

At each $\tau$ this is a simple, weighted linear quantile regression problem.

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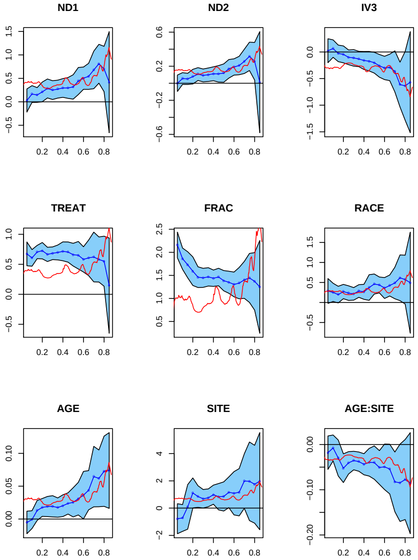
At each $\tau$ this is a simple, weighted linear quantile regression problem. The following R code fragment replicates an analysis in Portnoy (2003):

```
require(quantreg)
data(uis)
fit <- crq(Surv(log(TIME), CENSOR) ~  ND1 + ND2 + IV3 +
      TREAT + FRAC + RACE + AGE * SITE, data = uis, method = "Por")
Sfit <- summary(fit,1:19/20)
PHit <- coxph(Surv(TIME, CENSOR) ~  ND1 + ND2 + IV3 +
      TREAT + FRAC + RACE + AGE * SITE, data = uis)
plot(Sfit, CoxPHit = PHit)
```

# Reanalysis of the Hosmer-Lemeshow Drug Relapse Data



# Peng and Huang's Approach for Random Censoring I

Rationale *Extend the martingale representation of the Nelson-Aalen estimator of the cumulative hazard function to produce an "estimating equation" for conditional quantiles.*

Model *AFT form of the quantile regression model:*

$$Prob(\log T_i \leqslant x_i^\top \beta(\tau)) = \tau$$

Data $\{(Y_i, \delta_i) : i = 1, \cdots, n\}$ $Y_i = T_i \wedge C_i,\ \delta_i = I(T_i < C_i)$

Martingale *We have* $EM_i(t) = 0$ *for* $t \geqslant 0$*, where:*

$$\begin{aligned} M_i(t) &= N_i(t) - \Lambda_i(t \wedge Y_i | x_i) \\ N_i(t) &= I(\{Y_i \leqslant t\}, \{\delta_i = 1\}) \\ \Lambda_i(t) &= -\log(1 - F_i(t|x_i)) \\ F_i(t) &= Prob(T_i \leqslant t | x_i) \end{aligned}$$

# Peng and Huang's Approach for Random Censoring II

The estimating equation becomes,

$$En^{-1/2} \sum x_i [N_i(\exp(x_i^\top \beta(\tau))) - \int_0^\tau I(Y_i \geqslant \exp(x_i^\top \beta(u))) dH(u) = 0.$$

where $H(u) = -\log(1-u)$ for $u \in [0,1)$, after rewriting:

$$\begin{aligned} \Lambda_i(\exp(x_i^\top \beta(\tau)) \wedge Y_i | x_i)) &= H(\tau) \wedge H(F_i(Y_i | x_i)) \\ &= \int_0^\tau I(Y_i \geqslant \exp(x_i^\top \beta(u))) dH(u), \end{aligned}$$

# Peng and Huang's Approach for Random Censoring III

Approximating the integral on a grid, $0 = \tau_0 < \tau_1 < \cdots < \tau_J < 1$ yields a simple linear programming formulation to be solved at the gridpoints,

$$\alpha_i(\tau_j) = \sum_{k=0}^{j-1} I(Y_i \geqslant \exp(x_i^\top \hat{\beta}(\tau_k)))(H(\tau_{k+1}) - H(\tau_k)),$$

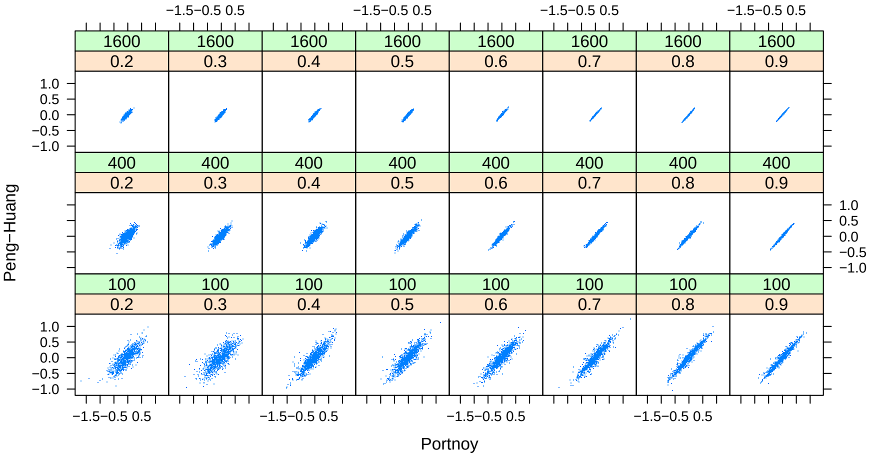
yielding Peng and Huang's final estimating equation,

$$n^{-1/2} \sum x_i [N_i(\exp(x_i^\top \beta(\tau))) - \alpha_i(\tau)] = 0.$$

Setting $r_i(b) = \log(Y_i) - x_i^\top b$, this convex function for the Peng and Huang problem takes the form

$$R(b, \tau_j) = \sum_{i=1}^{n} r_i(b)(\alpha_i(\tau_j) - I(r_i(b) < 0)\delta_i) = \min!$$

# Portnoy vs. Peng-Huang



# Some One Sample Asymptotics

Suppose that we have a random sample of pairs, $\{(T_i, C_i) : i = 1, \cdots, n\}$ with $T_i \sim F$, $C_i \sim G$, and $T_i$ and $C_i$ independent. Let $Y_i = \min\{T_i, C_i\}$, as usual, and $\delta_i = I(T_i < C_i)$. In this setting the Powell estimator of $\theta = F^{-1}(\tau)$,

$$\hat{\theta}_P = \text{argmin}_\theta \sum_{i=1}^{n} \rho_\tau(Y_i - \min\{\theta, C_i\}).$$

is asymptotically normal,

$$\sqrt{n}(\hat{\theta}_P - \theta) \rightsquigarrow \mathcal{N}(0, \tau(1-\tau)/(f^2(\theta)(1 - G(\theta)))).$$

# One Sample Asymptotics

In contrast, the asymptotic theory of the quantiles of the Kaplan-Meier estimator is slightly more complicated. Using the $\delta$-method one can show,

$$\sqrt{n}(\hat{\theta}_{KM} - \theta) \rightsquigarrow \mathcal{N}(0, \text{Avar}(\hat{S}(\theta))/f^2(\theta))$$

where, see e.g. Anderson et al,

$$\text{Avar}(\hat{S}(t)) = S^2(t) \int_0^t (1 - H(u))^{-2} d\tilde{F}(u)$$

and $1 - H(u) = (1 - F(u))(1 - G(u))$ and $\tilde{F}(u) = \int_0^t (1 - G(u)) dF(u)$. Since the Powell estimator makes use of more sample information than does the Kaplan Meier estimator it might be thought that it would be more efficient. But this isn't true.

# Kaplan Meier vs Powell

## Proposition

$$\text{Avar}(\hat{\theta}_{KM}) \leqslant \text{Avar}(\hat{\theta}_P).$$

**Proof:**

$$\begin{aligned}
f^2(\theta)\text{Avar}(\hat{\theta}_{KM}) &= S(\theta)^2 \int_0^\theta (1 - H(s))^{-2} d\tilde{F}(s) \\
&= S(\theta)^2 \int_0^\theta (1 - G(s))^{-1} (1 - F(s))^{-2} dF(s) \\
&\leqslant \frac{S(\theta)^2}{1 - G(\theta)} \int_0^\theta (1 - F(s))^{-2} dF(s) \\
&= \frac{S(\theta)^2}{1 - G(\theta)} \cdot \frac{1}{1 - F(s)} \Big|_0^\theta \\
&= \frac{S(\theta)^2}{1 - G(\theta)} \cdot \frac{F(\theta)}{1 - F(\theta)} \\
&= \frac{F(\theta)(1 - F(\theta))}{(1 - G(\theta))} \\
&= \frac{\tau(1 - \tau)}{(1 - G(\theta))}.
\end{aligned}$$

# Alice in Asymptopia

Leurgans (1987) considered the weighted estimator of the censored survival function,

$$\hat{S}_L(t) = \frac{\sum I(Y_i > t) I(C_i > t)}{\sum I(C_i > t)},$$

that uses all the $C_i$'s. Conditioning on the $C_i$'s, it can be shown that $\mathbb{E}(\hat{S}_L(t)|C) = S(t)$, and that the conditional variance is

$$\text{Var}(\hat{S}_L(t)|C) = \frac{F(t)(1 - F(t))}{1 - \hat{G}(t)}.$$

Averaging this expression gives the unconditional variance which converges to

$$\text{Avar}(\hat{S}_L(t)|C) = \frac{F(t)(1 - F(t))}{1 - G(t)},$$

and consequently quantiles based on Leurgan's estimator behave (asymptotically) just like those produced by the Powell estimator.

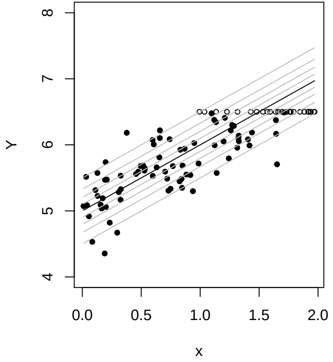
# Alice in Asymptopia

It might be thought that the Powell estimator would be more efficient than the Portnoy and Peng-Huang estimators given that it imposes more stringent data requirements. Comparing asymptotic behavior and finite sample performance in the simplest one-sample setting indicates otherwise.

| | median | Kaplan-Meier | Nelson-Aalen | Powell | Leurgans $\hat{G}$ | Leurgans G |
|---|---|---|---|---|---|---|
| n= 50 | 1.602 | 1.972 | 2.040 | 2.037 | 2.234 | 2.945 |
| n= 200 | 1.581 | 1.924 | 1.930 | 2.110 | 2.136 | 2.507 |
| n= 500 | 1.666 | 2.016 | 2.023 | 2.187 | 2.215 | 2.742 |
| n= 1000 | 1.556 | 1.813 | 1.816 | 2.001 | 2.018 | 2.569 |
| n= $\infty$ | 1.571 | 1.839 | 1.839 | 2.017 | 2.017 | 2.463 |

Scaled MSE for Several Estimators of the Median: Mean squared error estimates are scaled by sample size to conform to asymptotic variance computations. Here, $T_i$ is standard lognormal, and $C_i$ is exponential with rate parameter .25, so the proportion of censored observations is roughly 30 percent. 1000 replications.

# Simulation Settings I



# Simulations I-A

| | Intercept | | | Slope | | |
|---|---|---|---|---|---|---|
| | Bias | MAE | RMSE | Bias | MAE | RMSE |
| **Portnoy** | | | | | | |
| n = 100 | -0.0032 | 0.0638 | 0.0988 | 0.0025 | 0.0702 | 0.1063 |
| n = 400 | -0.0066 | 0.0406 | 0.0578 | 0.0036 | 0.0391 | 0.0588 |
| n = 1000 | -0.0022 | 0.0219 | 0.0321 | 0.0006 | 0.0228 | 0.0344 |
| **Peng-Huang** | | | | | | |
| n = 100 | 0.0005 | 0.0631 | 0.0986 | 0.0092 | 0.0727 | 0.1073 |
| n = 400 | -0.0007 | 0.0393 | 0.0575 | 0.0074 | 0.0389 | 0.0598 |
| n = 1000 | 0.0014 | 0.0215 | 0.0324 | 0.0019 | 0.0226 | 0.0347 |
| **Powell** | | | | | | |
| n = 100 | -0.0014 | 0.0694 | 0.1039 | 0.0068 | 0.0827 | 0.1252 |
| n = 400 | -0.0066 | 0.0429 | 0.0622 | 0.0098 | 0.0475 | 0.0734 |
| n = 1000 | -0.0008 | 0.0224 | 0.0339 | 0.0013 | 0.0264 | 0.0396 |
| **GMLE** | | | | | | |
| n = 100 | 0.0013 | 0.0528 | 0.0784 | -0.0001 | 0.0517 | 0.0780 |
| n = 400 | -0.0039 | 0.0307 | 0.0442 | 0.0031 | 0.0264 | 0.0417 |
| n = 1000 | 0.0003 | 0.0172 | 0.0248 | -0.0001 | 0.0165 | 0.0242 |

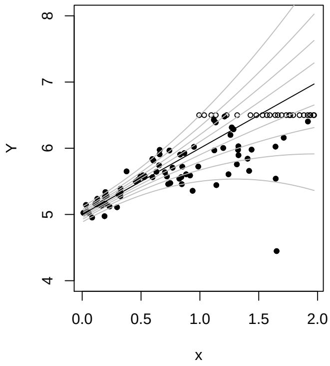
Comparison of Performance for the iid Error, Constant Censoring Configuration

# Simulations I-B

| | Intercept | | | Slope | | |
|---|---|---|---|---|---|---|
| | Bias | MAE | RMSE | Bias | MAE | RMSE |
| **Portnoy** | | | | | | |
| n = 100 | -0.0042 | 0.0646 | 0.0942 | 0.0024 | 0.0586 | 0.0874 |
| n = 400 | -0.0025 | 0.0373 | 0.0542 | -0.0009 | 0.0322 | 0.0471 |
| n = 1000 | -0.0025 | 0.0208 | 0.0311 | 0.0006 | 0.0191 | 0.0283 |
| **Peng-Huang** | | | | | | |
| n = 100 | 0.0026 | 0.0639 | 0.0944 | 0.0045 | 0.0607 | 0.0888 |
| n = 400 | 0.0056 | 0.0389 | 0.0547 | -0.0002 | 0.0320 | 0.0476 |
| n = 1000 | 0.0019 | 0.0212 | 0.0311 | 0.0009 | 0.0187 | 0.0283 |
| **Powell** | | | | | | |
| n = 100 | -0.0025 | 0.0669 | 0.1017 | 0.0083 | 0.0656 | 0.1012 |
| n = 400 | 0.0014 | 0.0398 | 0.0581 | -0.0006 | 0.0364 | 0.0531 |
| n = 1000 | -0.0013 | 0.0210 | 0.0319 | 0.0016 | 0.0203 | 0.0304 |
| **GMLE** | | | | | | |
| n = 100 | 0.0007 | 0.0540 | 0.0781 | 0.0009 | 0.0470 | 0.0721 |
| n = 400 | 0.0008 | 0.0285 | 0.0444 | -0.0008 | 0.0253 | 0.0383 |
| n = 1000 | -0.0004 | 0.0169 | 0.0248 | 0.0002 | 0.0150 | 0.0224 |

Comparison of Performance for the iid Error, Variable Censoring Configuration

# Simulation Settings II



# Simulations II-A

| | Intercept | | | Slope | | |
|---|---|---|---|---|---|---|
| | Bias | MAE | RMSE | Bias | MAE | RMSE |
| **Portnoy L** | | | | | | |
| n = 100 | 0.0084 | 0.0316 | 0.0396 | -0.0251 | 0.0763 | 0.0964 |
| n = 400 | 0.0076 | 0.0194 | 0.0243 | -0.0247 | 0.0429 | 0.0533 |
| n = 1000 | 0.0081 | 0.0121 | 0.0149 | -0.0241 | 0.0309 | 0.0376 |
| **Portnoy Q** | | | | | | |
| n = 100 | 0.0018 | 0.0418 | 0.0527 | 0.0144 | 0.1576 | 0.2093 |
| n = 400 | -0.0010 | 0.0228 | 0.0290 | 0.0047 | 0.0708 | 0.0909 |
| n = 1000 | -0.0006 | 0.0122 | 0.0154 | -0.0027 | 0.0463 | 0.0587 |
| **Peng-Huang L** | | | | | | |
| n = 100 | 0.0077 | 0.0313 | 0.0392 | -0.0145 | 0.0749 | 0.0949 |
| n = 400 | 0.0064 | 0.0193 | 0.0240 | -0.0125 | 0.0392 | 0.0493 |
| n = 1000 | 0.0077 | 0.0120 | 0.0147 | -0.0181 | 0.0279 | 0.0342 |
| **Peng-Huang Q** | | | | | | |
| n = 100 | 0.0078 | 0.0425 | 0.0538 | 0.0483 | 0.1707 | 0.2328 |
| n = 400 | 0.0035 | 0.0228 | 0.0291 | 0.0302 | 0.0775 | 0.1008 |
| n = 1000 | 0.0015 | 0.0123 | 0.0155 | 0.0101 | 0.0483 | 0.0611 |
| **Powell** | | | | | | |
| n = 100 | 0.0021 | 0.0304 | 0.0385 | -0.0034 | 0.0790 | 0.0993 |
| n = 400 | -0.0017 | 0.0191 | 0.0239 | 0.0028 | 0.0431 | 0.0544 |
| n = 1000 | -0.0001 | 0.0099 | 0.0125 | 0.0003 | 0.0257 | 0.0316 |
| **GMLE** | | | | | | |
| n = 100 | 0.1080 | 0.1082 | 0.1201 | -0.2040 | 0.2042 | 0.2210 |
| n = 400 | 0.1209 | 0.1209 | 0.1241 | -0.2134 | 0.2134 | 0.2173 |
| n = 1000 | 0.1118 | 0.1118 | 0.1130 | -0.2075 | 0.2075 | 0.2091 |

Comparison of Performance for the Constant Censoring, Heteroscedastic Configuration

# Simulations II-B

| | Intercept | | | Slope | | |
|---|---|---|---|---|---|---|
| | Bias | MAE | RMSE | Bias | MAE | RMSE |
| **Portnoy L** | | | | | | |
| n = 100 | 0.0024 | 0.0278 | 0.0417 | -0.0067 | 0.0690 | 0.1007 |
| n = 400 | 0.0019 | 0.0145 | 0.0213 | -0.0080 | 0.0333 | 0.0493 |
| n = 1000 | 0.0016 | 0.0097 | 0.0139 | -0.0062 | 0.0210 | 0.0312 |
| **Portnoy Q** | | | | | | |
| n = 100 | 0.0011 | 0.0352 | 0.0540 | 0.0094 | 0.1121 | 0.1902 |
| n = 400 | 0.0002 | 0.0185 | 0.0270 | -0.0012 | 0.0510 | 0.0774 |
| n = 1000 | -0.0005 | 0.0116 | 0.0169 | -0.0011 | 0.0337 | 0.0511 |
| **Peng-Huang L** | | | | | | |
| n = 100 | 0.0018 | 0.0281 | 0.0417 | 0.0041 | 0.0694 | 0.1017 |
| n = 400 | 0.0013 | 0.0142 | 0.0212 | 0.0035 | 0.0333 | 0.0490 |
| n = 1000 | 0.0012 | 0.0096 | 0.0139 | 0.0002 | 0.0208 | 0.0310 |
| **Peng-Huang Q** | | | | | | |
| n = 100 | 0.0044 | 0.0364 | 0.0550 | 0.0322 | 0.1183 | 0.2105 |
| n = 400 | 0.0026 | 0.0188 | 0.0275 | 0.0154 | 0.0504 | 0.0813 |
| n = 1000 | 0.0007 | 0.0113 | 0.0169 | 0.0077 | 0.0333 | 0.0520 |
| **Powell** | | | | | | |
| n = 100 | -0.0001 | 0.0288 | 0.0430 | 0.0055 | 0.0733 | 0.1105 |
| n = 400 | 0.0000 | 0.0147 | 0.0226 | 0.0001 | 0.0379 | 0.0561 |
| n = 1000 | -0.0008 | 0.0095 | 0.0146 | 0.0013 | 0.0237 | 0.0350 |
| **GMLE** | | | | | | |
| n = 100 | 0.1078 | 0.1038 | 0.1272 | -0.1576 | 0.1582 | 0.1862 |
| n = 400 | 0.1123 | 0.1116 | 0.1168 | -0.1581 | 0.1578 | 0.1647 |
| n = 1000 | 0.1153 | 0.1138 | 0.1174 | -0.1609 | 0.1601 | 0.1639 |

Comparison of Performance for the Variable Censoring, Heteroscedastic Configuration

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