Quantile Regression for Longitudinal Data

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Classical Linear Fixed/Random Effects Model

Consider the model,

$$y_{ij} = x_{ij}^\top \beta + \alpha_i + u_{ij} \quad j = 1, ...m_i, \quad i = 1, ..., n,$$

or

$$y = X\beta + Z\alpha + u.$$

The matrix Z represents an incidence matrix that identifies the n distinct individuals in the sample. If u and α are independent Gaussian vectors with $u \sim \mathcal{N}(0, R)$ and $\alpha \sim \mathcal{N}(0, Q)$. Observing that $v = Z\alpha + u$ has covariance matrix $Evv^\top = R + ZQZ^\top$, we can immediately deduce that the minimum variance unbiased estimator of β is,

$$\hat{\beta} = (X^\top (R + ZQZ^\top)^{-1} X)^{-1} X^\top (R + ZQZ^\top)^{-1} y.$$

A Penalty Interpretation of $\hat{\beta}$

Proposition. $\hat{\beta}$ solves $\min_{(\alpha,\beta)} \|y - X\beta - Z\alpha\|^2_{R^{-1}} + \|\alpha\|^2_{Q^{-1}}$, where $\|x\|^2_A = x^\top A x$.

Proof.

Differentiating we obtain the normal equations,

$$X^\top R^{-1} X\hat{\beta} + X^\top R^{-1} Z\hat{\alpha} = X^\top R^{-1} y$$

$$Z^\top R^{-1} X\hat{\beta} + (Z^\top R^{-1} Z + Q^{-1})\hat{\alpha} = Z^\top R^{-1} y$$

Solving, we have $\hat{\beta} = (X^\top \Omega^{-1} X)^{-1} X^\top \Omega^{-1} y$ where

$$\Omega^{-1} = R^{-1} - R^{-1} Z (Z^\top R^{-1} Z + Q^{-1})^{-1} Z^\top R^{-1}.$$

But $\Omega = R + ZQZ^\top$, see e.g. Rao(1973, p 33.). □

This result has a long history: Henderson(1950), Goldberger(1962), Lindley and Smith (1972), etc.

Quantile Regression with Fixed Effects

Suppose that the conditional quantile functions of the response of the jth observation on the ith individual y_{ij} takes the form:

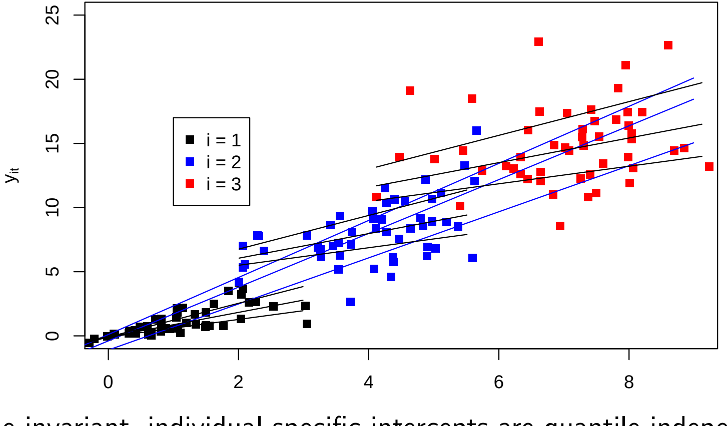
$$Q_{y_{ij}}(\tau|x_{ij}) = \alpha_i + x_{ij}^\top \beta(\tau) \quad j = 1, \ldots m_i, \quad i = 1, \ldots, n.$$

In this formulation the α's have a pure location shift effect on the conditional quantiles of the response. The effects of the covariates, x_{ij} are permitted to depend upon the quantile, τ, of interest, but the α's do not. To estimate the model for several quantiles simultaneously, we propose solving,

$$\min_{(\alpha,\beta)} \sum_{k=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m_i} w_k \rho_{\tau_k}(y_{ij} - \alpha_i - x_{ij}^\top \beta(\tau_k))$$

Note that the usual between/within transformations are not permitted.

Penalized Quantile Regression with Fixed Effects



Time invariant, individual specific intercepts are quantile independent; slopes are quantile dependent.

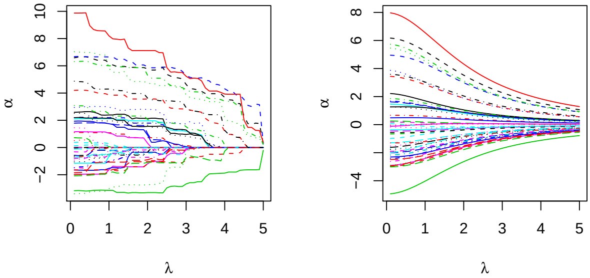
Penalized Quantile Regression with Fixed Effects

When n is large relative to the m_i's shrinkage may be advantageous in controlling the variability introduced by the large number of estimated α parameters. We will consider estimators solving the penalized version,

$$\min_{(\alpha,\beta)} \sum_{k=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m_i} w_k \rho_{\tau_k}(y_{ij} - \alpha_i - x_{ij}^\top \beta(\tau_k)) + \lambda \sum_{i=1}^{n} |\alpha_i|.$$

For $\lambda \to 0$ we obtain the fixed effects estimator described above, while as $\lambda \to \infty$ the $\hat{\alpha}_i \to 0$ for all $i = 1, 2, ..., n$ and we obtain an estimate of the model purged of the fixed effects. In moderately large samples this requires sparse linear algebra. Example R code is available from my webpages.

Shrinkage of the Fixed Effects



Shrinkage of the fixed effect parameter estimates, $\hat{\alpha}_i$. The left panel illustrates an example of the ℓ_1 shrinkage effect. The right panel illustrates an example of the ℓ_2 shrinkage effect.

Dynamic Panel Models and IV Estimation

Galvao (2010) considers dynamic panel models of the form:

$$Q_{y_{it}}(\tau | y_{i,t-1}, x_{it}) = \alpha_i + \gamma(\tau) y_{i,t-1} + x_{it}^\top \beta(\tau) \; t = 1, ...T_i, \; i = 1, ..., n.$$

In "short" panels estimation suffers from the same bias problems as seen in least squares estimators Nickel (1981) Hsiao and Anderson (1981); using the IV estimation approach of Chernozhukov and Hansen (2004) this bias can be reduced.