R. Koenker Spring 2012

Economics 574
Problem Set 3

1. (*How to get rich quick*)

   "Assume that we are hardened and unscrupulous types with an infinitely wealthy friend." Breiman (1961). The friend agrees, *à la* Samuelson at lunch, to flip a biased coin on which you may bet any amount $B > 0$. The payoff of the bet will be

$$\begin{cases} B & \text{with probability } p > \frac{1}{2} \\ -B & \text{with probability } q = (1-p) \end{cases}$$

   Suppose, perhaps because we have little imagination, we adopt a strategy of betting a constant fraction, θ, of our wealth W_n in each period. Thus, if we have initial wealth W_0,

$$W_n = W_0(1+\theta)^{S_n}(1-\theta)^{F_n}$$

   where S_n is the number of "successes" in the flipping, and F_n is the number of "failures." A measure of our average rate of increase in wealth is

$$\log((W_n/W_0)^{1/n}) = \frac{S_n}{n}\log(1+\theta) + \frac{n-S_n}{n}\log(1-\theta)$$

   Following Bernoulli, we might consider maximizing the expectation of this quantity:

$$E\log(W_n/W_0)^{1/n} = p\log(1+\theta) + q\log(1-\theta).$$

   (a) Show $\theta^* = 2p - 1$ is the maximizer.

   (b) Show that there is a $\tilde{\theta} \in (\theta^*, 1)$ such that for $\theta > \tilde{\theta}$ we almost surely are ruined, that is we experience the humiliating event $W_n \to 0$, while for $\theta < \tilde{\theta}$ we will have W_n eventually exceeding any fixed bound.

   (c) Taking $p = .6$ do some simulations to support your findings in parts (a) and (b). In the process find an approximate value for $\tilde{\theta}$ in part (b).

   (d) Generalize the foregoing to a situation in which you are offered several coins with differing p_i's to bet on in each period and you can allocate your bets among these coins and the safe option.

2. The Rayleigh distribution is commonly used to model failure time and other waiting-time phenomena. The density is

$$f(z|\theta) = (z/\theta^2)\exp\{-z^2/2\theta^2\}, \qquad z > 0, \quad \theta > 0.$$

   (a) Express the likelihood of a random sample $\{z_1, \cdots, z_n\}$ from a Rayleigh distribution and show that it is a one parameter exponential family.

(b) Express the likelihood in natural form.

(c) Find the maximum likelihood estimator of the natural parameter and using the rule $\hat{\theta}^2 = (c^{-1}(\eta))^2$ find the mle of θ^2.

(d) Compute the Cramer-Rao lower bound for unbiased estimates of θ^2 and the variance the mle and compare.

3. Suppose Z is $N(\mu, \sigma^2)$ and $Z \equiv g(X) = \log(X - \alpha)$, then X is said to have a 3-parameter log-normal distribution.

(a) Show that the density of X is

$$f_X(x|\alpha, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-(\log(x-\alpha) - \mu)^2/2\sigma^2\}[x-\alpha]^{-1}$$

(b) Note that for any fixed α we have a standard normal mle problem for the parameters μ and σ^2 we may concentrate the likelihood to get

$$L(\alpha|x) = K\hat{\sigma}^{-n} \prod_{i=1}^{n} [x_i - \alpha]^{-1}.$$

Show that for α sufficiently small

$$\hat{\sigma}^2(\alpha) = n^{-1} \sum (\log(x_i - \hat{\alpha}) - \hat{\mu})^2 \leq (\log(x_{(1)} - \alpha))^2$$

so

$$L(\alpha|x) \geq K|\log(x_{(1)} - \alpha)|^{-n} \prod [x_i - \alpha]^{-1}.$$

and since,

$$\lim_{u \to 0} \frac{1}{|\log u|^n u} = \infty.$$

the likelihood becomes unbounded as $\alpha \to x_{(1)}$.

(c) Even though the mle is obviously silly, if literally interpreted in this case, the likelihood can still play a useful role in drawing inferences about the parameter α. In S draw a sample of 100 realizations from the 3 parameter lognormal with parameter $(\alpha, \mu, \sigma^2) = (1, 0, 1)$ and plot the concentrated likelihood as a function of α. Use the asymptotic theory of the the likelihood at the local maximum below the first order statistic to construct a confidence interval for α.

4. A common model of income (and other size-related) distributions is the Pareto with density,

$$f(z|\beta) = \beta\alpha^\beta z^{-(\beta+1)} \qquad\qquad z \geq \alpha.$$

Put the likelihood for a random sample of size n in exponential family form and find the maximum likelihood estimator for β, assuming α is known. What is the asymptotic mean and variance of this estimator? What is the mle of α if β is known? Method of moments has problems here. show that for the mean, if $\beta \geq 1$ then $\hat{\beta} = \frac{(\bar{x}-1)}{\bar{x}}$ is consistent, but fails when $\beta \geq 1$. Method of moments based on the median is better. Show that $\hat{\beta} = \log(2)/\log(\hat{\xi})$ is consistent, where $\hat{\xi}$ is the sample median. Compare the asymptotic performance of the 3 estimators.

5. An estimator may be called an efficient likelihood estimator (ELE) if it asymptotically achieves the CRLB. The following result is central to the theory of "Hausman-type" tests in econometrics and describes the relationship between ELE's and competing estimators.

   Theorem: Suppose $\hat{\theta}_n$ is an ELE, $\tilde{\theta}_n$ is a $\sqrt{n}$-consistent estimator of θ_0, and

$$\hat{Z}_n = \sqrt{n}(\hat{\theta}_n - \theta_0) \rightsquigarrow \hat{Z}_0$$

$$\tilde{Z}_n = \sqrt{n}(\tilde{\theta}_n - \theta_0) \rightsquigarrow \tilde{Z}_0$$

   where $(\hat{Z}_0, \tilde{Z}_0)$ have a joint normal distribution with mean zero and covariance matrix $\Sigma = (\sigma_{ij})$. Then the asymptotic relative efficiency of $\tilde{\theta}_n$ with respect to $\hat{\theta}_n$, i.e. the ratio of their limiting variances σ_{11}/σ_{22}, is given by $e = \rho^2$ where $\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ is the correlation coefficient of $\hat{Z}_n$ and $\tilde{Z}_n$.

   Prove the result by considering $\mathrm{Var}((1-\alpha)\hat{Z}_n + \alpha\tilde{Z}_n)$ and showing that its limit is minimized at $\alpha = 0$ which implies $\sigma_{11} = \sigma_{12}$.

   Explain the connection of this result to the Hausman test.

6. Suppose $\{y_1, ..., y_n\}$ are iid random variables, each normally distributed with mean μ and variance μ^2. Find the mle of μ and argue its consistency. Compare the asymptotic efficiency of the mle in this problem with that of the sample mean. This problem is related to estimating models of heteroscedasticity in linear regression which have parameters in common with the model for the conditional mean.

7. In the problem section of *(Econometric Theory, 1999, v. 15, p. 151)* Oliver Linton asks readers to consider the limiting behavior of the following estimator:

$$\hat{\theta}_n = \arg\min_{\theta\in\Re} \sum_{j=1}^{p} |\hat{\alpha}_n - \theta|$$

   where $\sqrt{n}(\hat{\alpha}_n - \theta_0 e_p) \rightsquigarrow \mathcal{N}(0, I_p)$, and e_p denotes a p-vector of ones. Observing that $\hat{\theta}_n$ is the median of the $\hat{\alpha}_n$ components, he notes that for $p = 3$ the limiting distribution of $\hat{\theta}_n$ is

$$(*) \qquad P(\sqrt{n}(\hat{\theta}_n - \theta) < x) \to 6\int_{-\infty}^{x} \Phi(t)(1-\Phi(t))\phi(t)dt$$

   where Φ and ϕ denote the *df* and density functions of a $\mathcal{N}(0,1)$ random variable, respectively.

   (a) Explain briefly where $(*)$ comes from.

   (b) Reconcile the limiting normality of $\hat{\theta}_n$ with results in Newey (1988). *Econometric Theory*, 4, 1998 336-340.

   (c) Find the normal density closest to the $\hat{\theta}_n$ density $f = 6\Phi(1-\Phi)\phi$. In the sense of $KLIC$, find σ^2 to minimize $KLIC(f, \phi(x/\sigma)/\sigma)$.

   (d) Evaluate in a small Monte Carlo experiment the Power of the LR-test to distinguish f from the closest Normal computed in Part c.

8. (Irregularity of the MLE for the $U[0, \theta]$ problem) In an effort to explore Fisher information in non-standard conditions, consider approximating the $U[0, 1]$ density by $g(x) = I(x > 0)(1-S(x))$ where S is the survival function for the logistic distribution with mean 1 and scale $1/\epsilon$. (It is convenient to take $\theta = 1$ for current purposes.) We'd like to see what happens to the Fisher information as ϵ tends to zero. Evaluate numerically and draw some pictures to illustrate.