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Economics 478

Lecture 2

1. Stochastic Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T = [0, \infty)$. The family

$$X = (\xi_t) \qquad t \in T$$

of r.v.'s $\xi_t(\omega)$ is called a (real) stochastic process in continuous time $t \in T$. If we restrict $t \in \mathbb{N} = \{0, 1, \ldots\}$, then we have a discrete time stochastic process. For fixed $\omega \in \Omega, \xi_t(\omega)$ is called a trajectory, or realization.

The σ-algebras $\mathcal{F}_t^\xi = \sigma[\xi_s : s \leq t]$ are minimal wrt the r.v.'s $\xi_s : s \leq t$, and may be viewed as representing the history of the process up to and including time t.

A stochastic process $X = (\xi_t)$ is called measurable if for all Borel sets $B \in \mathcal{B}$ of $\mathbb{R}$

$$\{(\omega, t) : \xi_t(\omega) \in \mathcal{B}\} \in \mathcal{F} \times \mathcal{B}(T)$$

where $\mathcal{B}(T)$ is $\sigma[T]$.

Thm. (Fubini) Let $X = (\xi_t)\ \ t \in T$ be a measurable random process. Then
(i) Almost all trajectories are measurable (relative to $\mathcal{B}(T)$).
(ii) If $\mathbb{E}\xi_t$ exists for all $t \in T$, then $m_t = \mathbb{E}\xi_t$ is a measurable function of $t \in T$.
(iii) If S is a measurable set in T and $\int_S \mathbb{E}|\xi_t| dt < \infty$, then

$$\int_S |\xi_t| dt < \infty \qquad (P - a.s)$$

and

$$\int_S \mathbb{E}\xi_t dt = \mathbb{E} \int_S \xi_t dt.$$

Def. Let $\mathcal{F} = (\mathcal{F}_t)\ \ t \in T$ be $\nearrow$, we say, $X = (\xi_t)\ \ t \in T$ *is adapted to* $\mathcal{F}$ if for any $t \in T$ the r.v.'s ξ_t are $\mathcal{F}_t$ measurable.

Remark Sometimes this is abbreviated $\mathcal{F}$-adapted, or called nonanticipative.

2. Two Classes of Random Processes

(1) Stationary Processes. A process $X = (\xi_t)$ $t \in T = [0, \infty)$ is *stationary* if for any $\Delta > 0$, and points $t_1, \ldots, t_n \in T$,

$$P(\xi_t \in A_1, \ldots, \xi_{t_n} \in A_n) = P(\xi_{t_1+\Delta} \in A_1, \ldots, \xi_{t_n+\Delta} \in A_n).$$

This implies a weaker form of stationarity requiring only that $\mathbb{E}\xi_t^2 < \infty$, and

$$\mathbb{E}\xi_t = \mathbb{E}\xi_{t+\Delta}$$
$$\mathbb{E}\xi_t\xi_s = \mathbb{E}\xi_{t+\Delta}\xi_{s+\Delta}.$$

(2) Markov Processes. A process $X = (\xi_t, \mathcal{F}_t)$ $t \in T$ on $(\Omega, \mathcal{F}, P)$ is *Markov* if

$$(*) \qquad P(A \cap B|\xi_t) = P(A|\xi_t)P(B|\xi_t) \qquad P - a.s.$$

for any $t \in T, A \in \mathcal{F}_t$ and $B \in \mathcal{F}^{\xi}_{(t,\infty)} = \sigma(\xi_s : s > t)$.

Remark Note A "occurs" before t and B after and the force of $(*)$ is to say that for any feature of $\mathcal{F}_t$ – the prior history represented by A, the conditional probabilities multiply meaning that everything about the past needed to judge the likelihood of future events B is congealed in ξ_t. A criterion for recognizing a Markov process is given by the following result.

Thm. $X \in (\xi_t)$ $t \in T$ is Markov iff for any measurable function $f(x)$ such that $\sup|f(x)| < \infty$ and any collection $\{t_i : i = 1, \ldots, n\}$ such that $0 \leq t_1 \leq \ldots, \leq t_n \leq t$

$$\mathbb{E}(f(\xi_t)|\xi_{t_1}, \ldots, \xi_{t_n}) = \mathbb{E}(f(\xi_t)|\xi_{t_n})$$

A special class of Markov processes where the foregoing conditions are quite apparent are processes with independent increments, such that for ordered t_i

$$\xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}} \text{ are } \perp\!\!\!\perp$$

Def. A random process $X = (\xi_t, \mathcal{F}_t)$ $t \in T$ is called a martingale if $\mathbb{E}|\xi_t| < \infty$ $t \in T$, and for $t > s$,

$$\mathbb{E}(\xi_t|\mathcal{F}_s) = \xi_s \qquad (P - a.s.)$$

If

$$\mathbb{E}(\xi_t|\mathcal{F}_s) \geq \xi_s \qquad (P - a.s.)$$

we call ξ_t a submartingale and if

$$\mathbb{E}(\xi_t|\mathcal{F}_s) \leq \xi_s \qquad (P - a.s.)$$

a supermartingale. Sometimes it is useful to reverse the direction of time and then one can speak about reversed martingale's and submartingales.

Thm. Suppose $X = (\xi_t, \mathcal{F}_t)$ $t \in T$ is a martingale, and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex with $\mathbb{E}|\varphi(\xi_t)| < \infty$ for $t \in T$ then $(\varphi(\xi_s), \mathcal{F}_t)$ is a submartingale.

Proof By Jensen's inequality

$$\mathbb{E}(\varphi(\xi_t)|\mathcal{F}_s) \geq \varphi(E|\xi_t|\mathcal{F}_s) = \varphi(\xi_s)$$

Examples For $(\xi_t, \mathcal{F}_t)$ a martingale

(i) $(|\xi_t|^r, \mathcal{F}_t)$ is a submartingale provided $r > 1$ and $\mathbb{E}|\xi_t|^r < \infty$.

(ii) $(\xi_t^-, \mathcal{F}_t)$ is a submartingale

(iii) $(\xi_t^+, \mathcal{F}_t)$ is a submartingale

More Examples

(i) Let $X_1, \ldots, X_n$ be iid with $\mathbb{E}X_i = 0$ so $\mathbb{E}|X_i| < \infty$ and let $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n \equiv \sigma[S_1, \ldots, S_n]$. Then $\mathbb{E}|S_n| \leq \sum \mathbb{E}|X_i| < \infty$ and $(S_n, \mathcal{F}_n)$ is a martingale.

(ii) As in (i), but also assume $\mathbb{E}X_i^2 \equiv \sigma^2 < \infty$. Set $Y_n = S_n^2 - n\sigma^2$, then $(Y_n, \mathcal{F}_n)$ is a martingale, S_n is a submartingale and we have the decomposition,

$$\underset{\text{submartingale}}{S_n^2} = \underbrace{S_n^2 - n\sigma^2}_{\text{Martingale}} + \underbrace{n\sigma^2}_{\text{increasing process}}$$

This may be viewed as our first baby step toward the Doob-Meyer decomposition.