University of Illinois | Department of Economics
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Economics 472

Lecture 10

Introduction to Dynamic Simultaneous Equation Models

In this lecture we will introduce some simple dynamic simultaneous equation models. Problem Set 4 will deal with two classical examples of this class of models which are typically used in studying partial equilibrium models of a single market.

Let's begin by considering a rather general class of models of this form,

$$, y_t = A(L)y_{t-1} + B(L)x_t + u_t \tag{1}$$

where we will refer to y_t as a m-vector of endogenous variables, x_t as a q-vector of exogenous variables, and $A(L)$ and $B(L)$ denote matrix polynomials in the lag operator L, as usual. We will assume that given vectors y_{t-1}, and x_t, a realization of the vector u_t determines a unique realization of the response vector y_t, given the exogenous variables x_t and the past. This uniqueness requires that the matrix , be *invertible*. We may then "solve" the structural form of the model, (1), to obtain the reduced form,

$$y_t = \Psi(L)y_{t-1} + \Pi(L)x_t + v_t \tag{2}$$

where $\Psi(L) = \, ,^{-1}A(L), \Pi(L) = \, ,^{-1}B(L)$ and $v_t = \, ,^{-1}u_t$. We can think of the structural form as representing an idealized version of how the model "really works", while the reduced form is a cruder version of the model which could be used for forecasting, for example.

It is helpful at this stage to have a more concrete example so let's consider the cobweb model from part one of PS 4.

$$\begin{aligned} Q_t &= \alpha_1 + \alpha_2 P_{t-1} + \alpha_3 z_t + u_t \\ P_t &= \beta_1 + \beta_2 Q_t + \beta_3 w_t + v_t \end{aligned} \tag{3}$$

To connect this model with the notation of (1) we may write,

$$x_t = \begin{pmatrix} 1 \\ z_t \\ w_t \end{pmatrix} \qquad \text{exogenous variables}$$

$$y_t \quad = \quad \begin{pmatrix} Q_t \\ P_t \end{pmatrix} \qquad \text{endogenous variables}$$

$$, = \begin{pmatrix} 1 & 0 \\ -\beta_2 & 1 \end{pmatrix}, \quad A(L) = \begin{pmatrix} 0 & \alpha_2 \\ 0 & 0 \end{pmatrix} \quad ,^{-1} = \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix}, \quad \Psi(L) = \begin{pmatrix} 0 & \alpha_2 \\ 0 & \alpha_2\beta_2 \end{pmatrix}$$

$$B(L) = \begin{pmatrix} \alpha_1 & \alpha_3 & 0 \\ \beta_1 & 0 & \beta_3 \end{pmatrix}, \quad \Pi(L) = \begin{pmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_1\beta_2 + \beta_1 & \beta_2\beta_3 & \beta_3 \end{pmatrix}$$

Now consider solving the model for an equilibrium form. This would yield, in the general case

$$y_e = (I - \Psi(1))^{-1}\Pi(1)x_e \tag{4}$$

In the specific model (3) form, we would have

$$I - \Psi(1) = \begin{pmatrix} 1 & -\alpha_2 \\ 0 & 1 - \alpha_2\beta_2 \end{pmatrix}$$

so

$$(I - \Psi(1))^{-1} = \begin{pmatrix} 1 - \alpha_2\beta_2 & \alpha_2 \\ 0 & 1 \end{pmatrix} / (1 - \alpha_2\beta_2)$$

In this case we must check to be sure that

$$(I - \Psi)^{-1} = I + \Psi + \Psi^2 + \cdots$$

converges in order to assure stability.

Having derived the equilibrium form of the model it is now an opportune moment to discuss forecasting. This is again conveniently done using the reduced form. The one-step ahead forecast in the simple case where $\Psi(L)$ and $\Pi(L)$ are matrices independent of L, yields

$$\hat{y}_{n+1} = \Psi y_n + \Pi x_{n+1}$$

and thus two-steps ahead,

$$\begin{aligned} \hat{y}_{n+2} &= \Psi(\Psi y_n + \Pi x_{n+1}) + \Pi x_{n+2} \\ &= \Psi^2 y_n + \Psi\Pi x_{n+1} + \Pi x_{n+2} \end{aligned}$$

and thus for s steps ahead,

$$\hat{y}_{n+s} = \Psi^s y_n + \sum_{i=0}^{s-1} \Psi^i \Pi x_{n+1+i}$$

Note that, although it might not be immediately obvious, this is completely consistent with the previous discussion of equilibrium. Suppose $x_{n+i} = x_e$ for all $i \geq 0$, then presuming that $\Psi^s \to 0$ as is obviously required for stability, we have

$$\hat{y}_{n+s} \to (\sum_{i=0}^{\infty} \Psi^i)\Pi x_e = (I - \Psi)^{-1}\Pi x_e$$

as previously discussed.

Estimation of Dynamic Simultaneous Equation Models

Estimation of simultaneous equation models pose some new problems which we will now gradually introduce. It is convenient to begin with the simple case of recursive models which bring us just to the edge of simultaneity, without quite plunging into it.

Definition Model (1) is recursive if the following conditions hold, or if the endogenous variables can be reordered so that these conditions are met: (i) The matrix , is lower triangular, and (ii) The covariance matrix $\Omega = Eu_t u_t'$ is diagonal, and $\{u_t\}$ is iid over time.

We hasten to point out that condition (i) is satisfied by the cobweb model (3), so provided we assume, in addition, in that model that $Eu_t v_t = 0$ so that it satisfies (ii),we can say that it is in recursive form.

What is so special above recursive models? Closer examination reveals that recursivity implies an explicit *causal ordering* in the model. In our cobweb model (3) this is illustrated in the following diagram.

Given an initial price p_0, supply determines the next periods quantity, Q_1. Demand in period one then determines a price that will clear the market and this, then, triggers a supply response for period two and the model continues to operate. In the figure we see that for fixed supply and demand functions (recall that in the constantly *model* as specified in (3), these functions are shifting around due both to changes in the exogenous variables x_t and w_t as well the random effects u_t and v_t), this leads to convergence to an equilibrium (P_t, Q_t) pair that looks a little like a spider web as it oscillates around the equilibrium point. At this point you might explore what happens if the demand curve is quite steep, i.e., inelastic, and supply is quite flat, i.e., elastic. Explain how this connects with the algebraic formulation of the model and the stability of equilibrium.

The crucial observation about causal ordering and estimation of recursive models concerns the orthogonality of the errors and explanatory variables in models of this form. Note that in model (3) p_{t-1} is clearly orthogonal to u_t provided the $\{u_t\}$ are iid, and in the demand equation Q_t is orthogonal to v_t provided that v_t and u_t are orthogonal. Clearly, Q_t depends upon u_t, so if u_t and v_t were

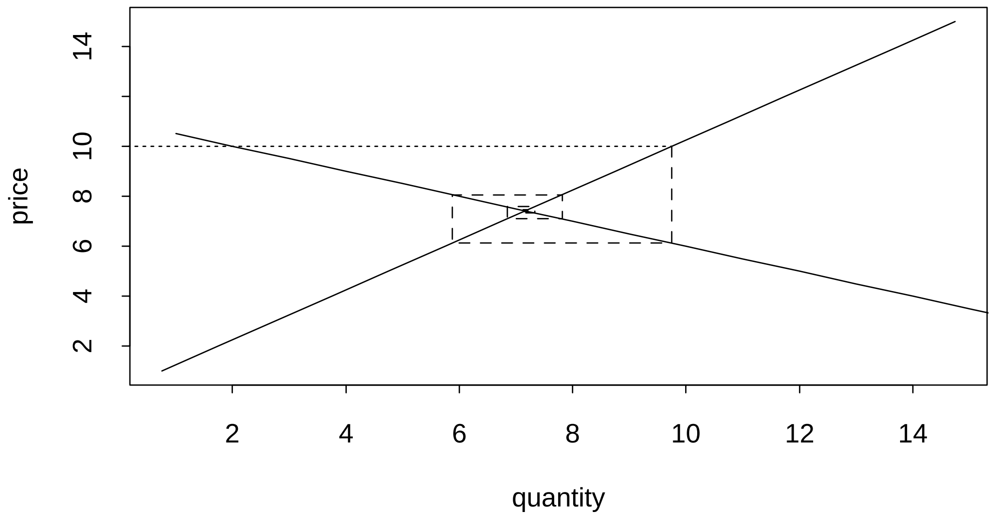

Figure 1: A simple linear cobweb model of supply and demand

correlated, then this orthogonality (i.e., $EQ_t v_t = 0$) would fail and ordinary least squares estimation would be biased. Obviously, the conditions for recursivity are rather delicate. We need, not only that the matrix , be triangular, a condition which itself is rather questionable, but we also rely on strong assumptions about the unobservable error effects in the system of equations. An illustration of how delicate things really are is provided by the effect of autocorrelation in u_t in the first equation of model (3). Note that if we perturb the model slightly and consider the possibility that

$$u_t = \rho u_{t-1} + \varepsilon_t$$

with $\{\varepsilon_t\}$ iid, now u_{t-1} clearly influences u_t, but it also helps to determine Q_{t-1}, thus p_{t-1}, so $Eu_t p_{t-1} = 0$ fails in this case and we need a more general estimation strategy that is capable of dealing with these failures of orthogonality. We will introduce a test for this kind of effect in the next lecture.

Instrumental Variables and Two Stage Least Squares

If correlation (read: lack of orthogonality) between X and u is a potential problem lurking in the shadows of recursive models, it is fully armed and dangerous in the classical simultaneous equations model. For general , in (1), the model is truly simultaneous in that the realization of the error vector u_t jointly determines all of the endogenous variables and therefore, any contemporaneous endogenous variable appearing on the right hand side of an equation is bound to be correlated with the error term in that equation.

The remedy for this, fortunately, is rather simple. We must find, as we have already seen in discussion of estimating models with lagged dependent variables and autocorrelated errors, instrumental variables, z_i, that satisfy the conditions that (i) they are (asymptotically) uncorrelated with the errors in the equation of interest, and (ii) they *are* (asymptotically) correlated with the included endogenous variables in the equation.

Consider the problem of estimating one of a system of simultaneous equations, say the first one, which we might write as,

$$
\begin{aligned}
y_1 &= Y_1\gamma_1 + X_1\beta_1 + u_1 \\
&= Z_1\delta_1 + u_1.
\end{aligned}
\tag{3}
$$

To connect this with our apparently more general class of models (1) take the first row of the matrix , in (1) to be $,_{1\cdot} = (1, -\gamma', 0')$ where the 0 corresponds to all of the endogenous variables that do not appear in the first equation. Let's begin by considering the simplest case in which (5) is exactly identified.

Partition the full set of exogenous variables as $X = (X_1 \vdots \tilde{X}_1)$ where X_1 is the set of included exogenous variables for equation one and $\tilde{X}_1$ and the excluded exogenous variables. In this case we may view $\tilde{X}_1$ as immediately available IV's for the Y_1's, since we have exactly, the same number of $\tilde{X}_1$'s as Y_1's.

Consider two apparently different instrumental variable estimators:

$$\hat{\delta}_1 = (\hat{Z}_1' Z_1)^{-1}\hat{Z}_1' y_1$$

that we will call the two stage least squares estimator, and

$$\tilde{\delta}_1 = (\tilde{Z}_1' Z_1)^{-1}\tilde{Z}_1 y_1$$

which is usually called the "indirect least squares" estimator, where $\hat{Z}_1 = P_X Z_1 = X(X'X)^{-1}X'Z_1$, and $\tilde{Z}_1 = (X_1 \vdots \tilde{X}_1) = X$. We will show that $\hat{\delta}_1 = \tilde{\delta}_1$.

We may write the claim more explicitly as

$$(Z_1' P_X Z_1)^{-1} Z_1' P_X y = (X' Z_1)^{-1} X' y$$

Now note that by assumption $X'Z_1$ is invertible so we can rewrite the lhs as

$$\begin{aligned}(Z_1'P_XZ_1)^{-1}Z_1'P_Xy_1 &= (X'Z_1)^{-1}X'X(Z_1'X)^{-1}Z_1'X(X'X)^{-1}X'y_1 \\ &= (X'Z_1)^{-1}X'y_1\end{aligned}$$

To explore the general case in which we have "more than enough" instrumental variables we consider the two stage least squares estimator in a bit more detail. Consider the model

$$y = Y_1\gamma + X_1\beta + u_1 = Z\delta + u_1.$$

and define instrumental variables $\tilde{Z} = XA$. Then it is easy to show that the instrumental variables estimator

$$\tilde{\delta} = (\tilde{Z}'Z)^{-1}\tilde{Z}'y$$

has the asymptotic linear representation

$$\sqrt{n}(\tilde{\delta} - \delta) = (n^{-1}A'X'Z)^{-1}n^{-1/2}A'X'u$$

and therefore,

$$\sqrt{n}(\tilde{\delta} - \delta) \xrightarrow{D} \mathcal{N}(0, \sigma^2(A'MD)^{-1}AMA(A'MD)^{-1})$$

where $\sigma^2 = Eu_i^2, M = \lim n^{-1}X'X$, $D = [\Pi_1\vdots\Psi_1], X\Psi_1 = X_1$, and Π_1 is the Y_1 partition of the matrix of reduced form coefficients. The following result shows that among all possible choices of the matrix A, the two stage least squares choice has a claim to optimality.

Theorem The two stage least squares choice, $\hat{\delta}$ with $A = (X'X)^{-1}X'Z_1$ is optimal, i.e., $V(\hat{\delta}) \leq V(\tilde{\delta})$ for all A.

Proof Note $A \to D$ since, $(X'X)^{-1}X'Y_1 = \hat{\Pi}_1 \xrightarrow{P} \Pi_1$ and $(X'X)^{-1}X'Z_1 = \Psi_1$. Thus, $\mathrm{avar}(\hat{\delta}) \equiv \lim V(\sqrt{n}(\hat{\delta} - \delta)) = \sigma^2(D'MD)^{-1}$. We would like to show that for all $\alpha \in R^{p_1+q_1}$ $\alpha'(V(\hat{\delta}) - V(\tilde{\delta}))\alpha \leq 0$. It is equivalent, and slightly easier, to argue that for all $\alpha \in R^{p_1+q_1}$, $\alpha'(V(\hat{\delta})^{-1} - V(\tilde{\delta})^{-1})\alpha \geq 0$. Factor $M = NN'$(Cholesky decomposition) and set $h = N^{-1}D\alpha$ as $h'h = \alpha'D'MD\alpha$ and

$$\alpha'V(\hat{\delta})^{-1}\alpha = \alpha'D'MA(A'MA)^{-1}A'MD\alpha = h'G(G'G)^{-1}G'h$$

where $G = NA$, so we may write

$$\alpha'(V(\tilde{\delta})^{-1} - V(\hat{\delta})^{-1})\alpha = h'(I - G(G'G)^{-1}G')h \geq 0$$

as required, since the matrix in parentheses is a projection and therefore positive semidefinite.